CHAPTER-1

Introduction

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1.1 Dynamical Systems

Dynamics is the study of change, and dynamical systems are the recipe for saying how a system of variables interact and changes with time. The concept is that naturally anything that evolves over time can be thought of as a dynamical system. So let us start with describing mathematical dynamical systems.

A dynamical system has two parts: a state vector which describes precisely the state of some real or hypothetical system with the current state and a function which tells us what the state of the system will be in the next instant of time.

Definition 1.1: A dynamical system is a state space S, a set of times T and a rule R for evolution given by $R: S \times T \rightarrow S$ that gives the consequents to a state $s \in S$. A dynamical system can be described to be a model describing the temporal evolution of a system.

Nonlinear dynamics started with the work of Henry Poincare in the late 1800's to solve the three body problem. Later this approach had been used in many other fields. With the invention of modern computers, Engineers and Scientists solved many nonlinear dynamical problems and E N Lorenz (1963) had numerically first discovered chaotic dynamical system in the atmospheric model. In nonlinear dynamics chaos theory flourished very rapidly in different branches of natural sciences. In order to understand a nonlinear dynamical system, let's start with their types.

1.1.1 Types of Dynamical Systems

On the basis of representation there are mainly two types of dynamical systems: Iterated maps and Differential equations. Iterated maps also known as difference equations, recursion relation or simply maps, in which time is discrete rather than continuous. The rule $x_{n+1} = \cos x_n$ is an example of one-dimensional map, because the points x_n belongs to the one-dimensional space of real numbers and the sequence x_0, x_1, x_2 ... is called the *orbit* starting from x_0.

Differential equations describe the evolution of dynamical system in continuous time. Further we will classify this in linear/nonlinear and autonomous/non-autonomous according to their nature and are discussed as follows.

1.1.1.1 Linear System

Definition 1.2: Linear systems must satisfy two properties namely *superposition* and *homogeneity*. The principle of superposition states that for two different inputs x and y in domain of the function f must satisfy $f(x+y) = f(x) + f(y)$.

The property of homogeneity states that for a given input x in the domain of the function f and for any real number k, $f(kx) = kf(x)$.

Any function that does not follow superposition and homogeneity is nonlinear in nature. It is important to note that there is no unifying characteristic of nonlinear systems, except for not satisfying the two above mentioned properties.

The dynamics of linear systems can also be written in the form

$$X' = A(t)X, \tag{1.1.1}$$

where $A(t)$ is a $n \times n$ matrix and $X$is a $n \times 1$ state vector.

1.1.1.2 Nonlinear System

Definition 1.3: A nonlinear dynamical system can usually be represented by a set of nonlinear differential equations in the form

$$X' = f(x,t), \tag{1.1.2}$$

where f is the $n \times 1$ nonlinear vector function and x is the $n \times 1$ state vector. The number of states n is called the order of the system.

1.1.1.3 Autonomous and Non-Autonomous systems

Definition 1.4: The nonlinear system (1.1.2) is said to be autonomous if f does not depend explicitly on time, i.e. if the system's state equation can be written as $X' = f(x)$ only.

Otherwise if f depends explicitly on time as in equation (1.1.2) then the system is called non-autonomous.

1.2 Stability Theory

Stability analysis of a particular dynamical system is the investigation regarding whether or not a system is stable or will be stable with perturbation. This analysis has a crucial role in a wide range of applications as most of the phenomena observed in the real world can be described using differential equations. The dynamical stability theory addresses the stability of solutions of differential equations and behaviours in trajectories of dynamical systems under little perturbations in initial conditions.

1.2.1 Equilibrium points of a system

Consider a system of ordinary differential equations in the standard form as

$$\frac{dx}{dt} = f(x, y) \text{ and } \frac{dy}{dt} = g(x, y), \tag{1.2.1}$$

where f and g are specified nonlinear functions of x and y. Since f and g do not depend explicitly on the independent variable t, the system is said to be autonomous. Otherwise, it is non-autonomous. It should be noted that the systems arising from Newton's second law in mechanics with the structure

$$\frac{d^2x}{dt^2} = F\left(x, \frac{dx}{dt}\right), \tag{1.2.2}$$

where F is the force. This equation can be put into the standard form, by setting $\frac{dx}{dt} = y$ as

$$\frac{dx}{dt} = y = f\ ,\ \frac{dy}{dt} = F = g \tag{1.2.3}$$

The equilibrium, fixed, critical or stationary, points of equation (1.2.1) correspond to the points in the x-y plane (the phase plane) where $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$. The number and locations of the fixed points in the phase plane are found by solving the simultaneous nonlinear equations

$$f(x, y) = 0,\ g(x, y) = 0. \tag{1.2.4}$$

Unlike the situation for linear ordinary differential equations, if f and g are nonlinear functions, more than one fixed point are possible.

1.2.2 Stability of the system

Considering an autonomous system of the form

$$\frac{dx}{dt} = f(x) \tag{1.2.5}$$

A critical point x_e of the system (1.2.5) is said to be **stable** if for given any $\varepsilon > 0$, there exists a $\delta > 0$ such that every solution $x = \varphi(t)$ of the system (1.2.5) which at $t = 0$ satisfies

$$\|\varphi(0) - x_e\| < \delta, \tag{1.2.6}$$

and

$$\|\varphi(t) - x_e\| < \varepsilon \tag{1.2.7}$$

for all $t \geq 0$. This is illustrated geometrically in Fig. 1.1 and Fig. 1.2 (Boyce and Diprima, 2001). These mathematical statements say that all solutions that start “sufficiently close” (that is, within the distance δ) to x_e stay “close” (within the distance ε) to x_e. Note that in Fig. 1.1 the trajectory is within the circle $\|x - x_e\| = \delta$ at $t = 0$ and while it soon passes outside of this circle, it remains within the circle $\|x - x_e\| = \varepsilon$ for all $t \geq 0$. A critical point that is not stable is said to be **unstable**.

A critical point x_e is said to be **asymptotically stable** if it is stable and if there exists a δ_0, with $0 < \delta_0 < \delta$, such that if a solution $x = \varphi(t)$ satisfies

$$\|\varphi(0) - x_e\| < \delta_0, \tag{1.2.8}$$

then

$$\lim_{t \to \infty} \varphi(t) = x_e. \tag{1.2.9}$$

Thus trajectories that start "sufficiently close" to x_e must not only stay "close" but must eventually approach x_e as $t \to \infty$. This is the case for the trajectory in Fig. 1.2 but not for the one in Fig. 1.1.

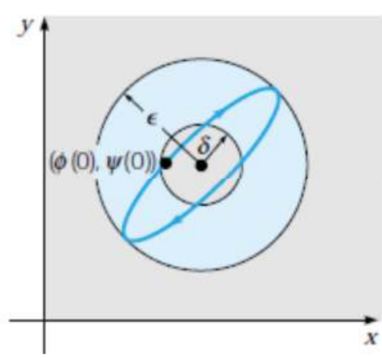

Fig. 1.1 Stability of the system

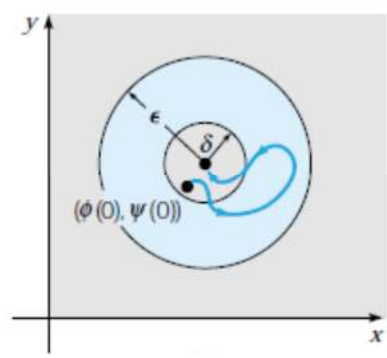

Fig. 1.2 Asymptotic stability of the system

It is important to note that asymptotic stability is a stronger property than stability, since a critical point must be stable before one can even talk about whether it might be asymptotically stable. On the other hand, the limit condition (1.2.9), which is an essential feature of asymptotic stability, does not by itself imply even ordinary stability. Indeed, examples can be constructed in which all of the trajectories approach x_e as $t \to \infty$, but for which x_e is not a stable critical point. Geometrically, all that is needed is a family of trajectories having members that start arbitrarily close to x_e, then depart an arbitrarily large distance before eventually approaching x_e as $t \to \infty$.

1.2.3 Lyapunov First Method

Theorem 1.1: Consider $x = 0$ be an equilibrium point of an autonomous nonlinear system

$$x' = f\big(x(t)\big)\,, \tag{1.2.10}$$

where $f: D \to R^n$ a continuously differentiable function with D is the neighborhood of the equilibrium point. Let λ_i $(i = 1,2,\ldots,n)$ denote the eigenvalues of the matrix $A = \frac{\partial f}{\partial x}\Big|_{x=0}$.

(i) If $Re(\lambda_i) < 0$ for all i then the equilibrium point $x = 0$ is asymptotically stable for the system (1.2.10).

(ii) If $Re(\lambda_i) > 0$ for at least one or more i then the equilibrium point $x = 0$ is unstable for the system (1.2.10).

(iii) If $Re(\lambda_i) < 0$ for all i and at least one $Re(\lambda_i) = 0$ then the equilibrium point $x = 0$ is stable, asymptotically stable or unstable for the system (1.2.10).

1.2.4 Lyapunov Second Method

Russian mathematician A.M. Lyapunov in 1892 gave the first precise definition of stability and developed the theory of stability for ordinary differential equations. The use of Lyapunov functions to prove stability has become common and is known as Lyapunov's direct method or Lyapunov's second method. This method involves determining a family of closed curves or closed surfaces in state space such that the general behaviour of nearby trajectories of a dynamical system can be investigated. This technique is relevant for examining the global stability of nonlinear systems and for deciding trapping regions for a dissipative chaotic flow.

Theorem 1.2: Consider $x = 0$ be an equilibrium point of an autonomous nonlinear system

$$x' = f\big(x(t)\big), \qquad x(0) = x_0,$$

where $x(t) \in D \subseteq R^n$ is state vector. D is an open set containing origin and $f: D \to R^n$ is continuous. Let $V: D \to R$ be a positive definite continuously differentiable function on a neighbourhood D of $x = 0$, such that $V'(x) \leq 0$ in D along the path of the system. Then, the equilibrium point $x = 0$ is stable.

Moreover, if $V'(x) < 0$ in $D - \{0\}$, then the point $x = 0$ is said to be asymptotically stable.

Theorem 1.3: Let $x = 0$ be an equilibrium point of a nonlinear system $x' = f(x(t))$. Let $V: R^n \to R$ be a positive definite continuously differentiable function, such that $V(x) \to \infty$ as $\|x\| \to \infty$ and $V'(x) \leq 0$ for all $x \neq 0$.

Then, the equilibrium point $x = 0$ is globally asymptotically stable. Note that Lyapunov functions are usually not unique for a particular system.

1.3 Chaos

The word “chaos” is evolved from the Greek word '*Χάος*' which signifies a state without order or predictability. According to ancient Greek mythology, chaos is the "primeval emptiness preceding the genesis of the universe, turbulent and disordered, mixing of all the elements".

The motions of physical systems are modelled using differential equations. When solutions of these equations are bounded, they either settle down to a fixed state or oscillate in a periodic (or quasi-periodic) state. There are some systems whose solutions do not fall in any of these categories. These solutions exhibit aperiodic (or irregular) motion for all time and never settle. Moreover, these solutions are highly sensitive to initial conditions, i.e., nearby starting trajectories separates exponentially. Thus it is difficult to predict the behaviour of the solution for a long time. Such systems are called chaotic dynamical systems.

The chaos was probably first observed by James Clerk Maxwell in 1860 while studying the motion of two colliding gas particles in a box which was unpredictable for a long duration. Henry Poincare was the first person to glimpse the possibility of chaos in 1890 while studying the famous three-body problem. In which Poincare found orbits were non-periodic and yet not forever increasing nor approaching towards a fixed point which is nowadays known as chaos. The major breakthrough in chaos theory and nonlinear dynamics was after the discovery of high-speed computers in 1950. In late 1950's, the meteorologist Edward Norton Lorenz acquired the LGP-30 computer having internal memory of 16 KB. Using computer, in 1963, He introduced the strange attractor notion and coined the term "butterfly effect". The model studied by Lorenz arising in weather prediction was consisting of an autonomous system of three ordinary differential equations containing nonlinear terms. The solutions were aperiodic and sensitive to initial conditions. A simple electronic circuit resulting chaotic attractor was given by T. Matsumoto et al. (1985). Robert May (1976) studied one-dimensional maps (difference equations) modelling population dynamics. He observed that a very simple model can generate extremely complicated dynamics. This was the pioneering work in the study of chaos in maps. Thus, the chaos can occur in (i) one-dimensional maps, (ii) nonlinear, autonomous system of differential equations of order three and higher and (iii) nonlinear, non-autonomous system of differential equations of order two and higher.

1.3.1 Definition of chaos

In nonlinear sciences fraternity, chaos is an active area of research since last few decades. Despite this there is no unified, universally accepted, rigorous definition of chaos in the current scientific literature, however, a commonly used definition confines the following nature of chaos, which most scientists agree and also mentioned by Steven H. Strogatz in his book (Strogatz (1994)).

"*Chaos is aperiodic long-term behaviour, in a deterministic system that exhibits sensitive dependence on initial conditions.*"

"Aperiodic long-term behaviour" means that there are the trajectories which do not settle down to fixed points, periodic orbits, or quasi-periodic orbits as time becomes large.

"Deterministic" means that the system has no random or noisy inputs or parameters. This irregular behaviour arises from the presence of nonlinear terms in the systems, rather than from noisy driving forces.

"Sensitive dependence on initial conditions" means that a small change in initial conditions will lead to progressively more significant changes in later states or we can say as an arbitrarily small perturbation of current trajectory may lead to significantly different future behaviour or trajectories separate exponentially fast, i.e., the system has a positive Lyapunov exponent.

1.3.2 Attractor and strange attractor

An attractor is a set to which all neighbouring trajectories converge. Stable fixed points and stable limit cycles are examples of attractors. More precisely, we define an attractor to be a closed set A with the following properties (Strogatz (1994)):

(i) A is an invariant set: i.e., for any trajectory $x(t)$ that starts in A stays in A for all time.

(ii) A attracts an open set of initial conditions: there is an open set U containing A such that if $x(0) \in U$, then the distance from $x(t)$ to A tends to zero as $t \to \infty$. This means that A attracts all trajectories that start sufficiently close to it. The largest such U is called the basin of attraction of A.

(iii) A is minimal, if there is no proper subset of A that satisfies conditions (i) and (ii).

Finally, a strange attractor is an attractor that exhibits sensitive dependence on initial conditions and is called strange because it is often fractal sets. Nowadays this geometric property is regarded as less important than the dynamical property of sensitive dependence on initial conditions.

1.3.3 The Lyapunov Exponent

The Lyapunov exponent is a quantitative evaluation of sensitive dependence on initial conditions. It gives a determination of divergence or convergence of two neighbouring trajectories. Let us consider a continuous time autonomous dynamical system represented by following first order differential equation

$$\frac{dX}{dx} = f(X), X \in R^n$$

Suppose $X(0)$ is an initial condition and $X(t)$ is its corresponding trajectory. If we consider a small displacement from the initial condition $X(0)$, and this is in the direction of the tangent vector $w(0)$, then evolution of the tangent vector $w(t)$ is found by linearizing equation

$$\frac{dw(t)}{dx} = Df(X(t), w(t)),$$

where Df is the Jacobian matrix of f. This determines the evolution of the infinitesimal displacement $w(t)$ of the trajectory from the unperturbed trajectory $X(t)$. The eigenvalues obtained from the Jacobian matrix indicate how a particular orbit expands. The exponential growth rate of $\|w(t)\|$ is a number λ such that

$$\|w(t)\| = e^{\lambda t}\|w(0)\|$$

Taking natural logarithm on both sides we have

$$\lambda = \frac{1}{t}\ln\left(\frac{\|w(t)\|}{\|w(0)\|}\right).$$

Here λ is called a Lyapunov exponent.

If $\lambda > 0$, the neighbouring trajectories separate exponentially fast i.e. it exhibits sensitive dependence on initial conditions and therefore chaotic in nature.

If $\lambda < 0$, the trajectory attracts to a fixed stable point or stable to periodic orbit.

If $\lambda = 0$, the trajectories will be a neutral fixed point or an eventually fixed point.

A system has many Lyapunov exponents as per the number of dimensions of the phase space. Any system that contains at least one positive Lyapunov exponent is called a chaotic system and if system has more than one positive Lyapunov exponent is called Hyper-chaotic system

1.3.4 Chaos in fractional order systems

Chaotic fractional order dynamical systems are obtained by replacing the derivative in the system by fractional derivative. Fractional order, in this case, works as a chaos controller, i.e., the chaotic system can be made regular by appropriate choice of fractional derivative. It is observed that there is a critical value of the fractional order below which the system is regular and for the higher values chaotic. There is one more reason to study fractional chaotic systems that these are very useful in secure communications. Secure codes can be made using fractional chaotic systems which are difficult to break. Fractional order derivative acts as additional parameter which works as a key.

1.4 Chaos Synchronization

Fujisaka and Yamada (1983a, 1983b) paved the way with their pioneering studies on chaos synchronization, but it was not until 1990 when Pecora and Carroll (1990) introduced their method of chaotic synchronization and suggested application to secure communications that the subject received considerable attention within the scientific community. L. M. Pecora and T. L. Carroll (1990) was the first to introduce a method to synchronize drive and response systems of two identical or non-identical systems with different initial conditions. They wrote that:

"*Chaotic systems would seem to be dynamical systems that defy synchronization. Two identical autonomous chaotic systems started at nearly the same initial points-in phase space have trajectories which quickly become uncorrelated, even though each maps out the same attractor in phase space. It is thus practically impossible to construct, identical, chaotic, synchronized system in laboratory*".

Chaos synchronization or we may say synchronization of chaos is a phenomenon where two or more, identical or non-identical, chaotic systems adjust a given property of their motion to a common behaviour due to coupling or forcing. It might seem that the synchronization of chaotic systems is difficult to achieve due to their extremely sensitive dependence on initial conditions and system parameters. And due to this property synchronization of the complex chaotic systems has important applications in secure communication. The synchronization scenario has been of long-standing interest for many researchers and studied extensively.

The contribution of research work in this thesis is to reveal the significant influence of time delay and fractional order derivative on chaos synchronization and has suggested some new approach to achieve synchronization of chaotic systems.

1.4.1 Types of synchronization

Motivated by the seminal works of Fujisaka and Yamada (1983a, 1983b) and of Pecora and Carroll (1990) on synchronization of chaotic systems, various types of synchronization scenario have been investigated, viz., complete synchronization, anti-synchronization, phase synchronization, hybrid synchronization, lag synchronization, generalised synchronization, projective synchronization, function projective synchronization, dual synchronization, combination synchronization, dual combination synchronization, combination-combination synchronization, etc. These different types of synchronization are described in details in following sub-sections.

1.4.1.1 Complete synchronization

Complete synchronization was first described by Pecora and Carroll (1990). It appears as the equality of the state variables while evolving in time. In this type of synchronization, the chaotic trajectories of the coupled systems remain in step with each other when time evolves. This is observed in coupled chaotic systems with identical elements, i.e., each component having the same dynamics and parameter set.

Suppose two continuous time chaotic systems as

$$\dot{x}(t) = f\left(x(t)\right), \tag{1.4.1}$$

$$\dot{y}(t) = g\left(y(t)\right) + u\left(x(t),\, y(t)\right), \tag{1.4.2}$$

where f, $g : R^n \to R^n$ are nonlinear continuous functions and $x(t)$, $y(t) \in R^n$ are the state vectors the systems (1.4.1) and (1.4.2) respectively, $u(x(t),\ y(t))$ is the control function.

The considered chaotic systems (1.4.1) and (1.4.2) will be synchronized if $\lim_{t\to\infty} \|y(t) - x(t)\| = 0$ for initial conditions $x(0)$ and $y(0)$.

Complete synchronization is also referred as convention synchronization, or identical synchronization, or simply synchronization.

1.4.1.2 Anti-synchronization

Two chaotic systems (1.4.1) and (1.4.2) are said to be anti-synchronized, when the respective states of chaotic systems $x(t)$ and $y(t)$ have the same magnitude but opposite in sign. Mathematically, anti-synchronization is achieved when $\lim_{t\to\infty} \|y(t) + x(t)\| = 0$.

1.4.1.3 Hybrid synchronization

In hybrid synchronization, synchronization and anti- synchronization co-exist together in the systems. It is an attractive case where three states are defined in such a way that first and third states of the two systems are completely synchronized, also first and second states of the systems are anti-synchronized.

1.4.1.4 Generalized synchronization

Coupled chaotic systems are said to exhibit generalized synchronization if there exits some function relation between systems, i.e., $y(t) = \varphi(x(t))$, which means that the states of the two interacting systems are functionally synchronized. This type of synchronization occurs mainly when the coupled chaotic systems are different.

1.4.1.5 Anticipating and lag synchronization

In these cases, the synchronized state is characterised by a time interval τ such that the dynamical variables of the chaotic systems are related by $y(t) = x(t+\tau)$. This means that the dynamics of one of the systems follows and anticipates the dynamics of the other.

These types of synchronization may occur in time-delayed chaotic systems, coupled in a drive-response configuration. In case of anticipating synchronization, the response anticipates the dynamics of the drive. In lag synchronization, $\tau < 0$ appears as the asymptotic boundedness of the difference between the output of one system at time t and the output of the other shifted in time of a lag time. In particular, if the time delay may become zero i.e., $\tau = 0$, the anticipating synchronization and lag synchronization are further simplified to complete synchronization.

1.4.1.6 Phase synchronization

This scenario of the synchronization occurs when the coupled chaotic systems keep their phase difference bounded by a constant while their amplitudes remain uncorrelated. This phenomenon is mostly achieved in coupled non-identical systems. In case of phase synchronization, if $\varphi_1(t)$ and $\varphi_2(t)$ denote the phases of the two coupled chaotic systems, synchronization of the phase is described by the relation $n\,\varphi_1(t) = m\,\varphi_2(t)$, with m and n whole numbers.

1.4.1.7 Projective synchronization

The projective synchronization in partially linear systems was first introduced by R. Mainieri and J. Rehacek (1999), where the responses of two identical systems synchronize up to a constant scaling factor.

Consider the drive system as (1.4.1) and response system as (1.4.2). Defining the error state as $e(t) = y(t) - \lambda\, x(t)$, where λ is the real constant, the systems (1.4.1) and (1.4.2) are said to be projective synchronized, if $\lim_{t\to\infty} \| e(t) \| = 0$.

In particular, if $\lambda = 1$ and $\lambda = -1$, the projective synchronization is further simplified to complete synchronization and anti-phase synchronization respectively.

1.4.1.8 Function projective synchronization

In function projective synchronization, drive system is synchronized with response system up to a desired scaling function. It is introduced by Y. Chen and X. Li (2007).

Defining the error state as $e(t) = y(t) - \lambda(t)\,x(t)$, where $\lambda(t)$ is the continuously differentiable function with $\lambda(t) \neq 0, \forall t$, the systems (1.4.1) and (1.4.2) are said to be function projective synchronized if there exists a scaling function $\lambda(t)$ such that $\lim_{t\to\infty} \| e(t) \| = 0$.

1.4.1.9 Modified projective synchronization

Modified projective was proposed by G. H. Li (2007). Defining the error state as $e(t) = y(t) - A\,x(t)$, where $A = diag\,[a_1\,, a_2\,, \cdots, a_n]$ is the scaling constant matrix such that a_i's are constant scaling factors $\forall i \in N$, the systems (1.4.1) and (1.4.2) are said to be modified projective synchronized, if there exists a constant matrix A such that $\lim_{t\to\infty} \| e(t) \| = 0$.

1.4.1.10 Modified function projective synchronization

Modified function projective synchronization is more general than function projective synchronization and modified projective synchronization. If we define the error state as $e(t) = y(t) - A(t)\,x(t)$ between the drive system (1.4.1) and response system (1.4.2), where $A(t) = diag\,[a_1(t)\,, a_2(t)\,, \cdots, a_n(t)]$ is the function scaling matrix such that $a_i(t) \neq 0\,, \forall i \in N$ are continuously differentiable function. Systems (1.4.1) and (1.4.2) are said to be modified function projective synchronized if there exists a function scaling matrix $A(t)$ such that $\lim_{t\to\infty} \| e(t) \| = 0$.

1.4.1.11 Dual synchronization

In the case of dual combination synchronization the first two drive systems are taken as

Drive systems-I:

$$\dot{X} = F(X)\,, \tag{1.4.3}$$

where X is state vector variable.

Drive system-II:

$$\dot{Y} = G(Y) \,, \tag{1.4.4}$$

where Y is state vector variable.

The linear combination of the drive systems I & II gives rise to

$$V_m = \sum_{i=1}^{n} a_i X_i + \sum_{i=1}^{n} b_i Y_i$$

$$= [a_1 ,\ a_2 , \cdots,\ a_n]X + [b_1 , b_2 , \cdots, b_n]Y$$

$$= A^T X + B^T Y$$

$$= [A^T\ B^T]\begin{bmatrix} X \\ Y \end{bmatrix} = C^T \xi,$$

where $A = [a_1 ,\ a_2 , \cdots,\ a_n]^T$ and $B = [b_1 , b_2 , \cdots, b_n]^T$ are known and $C = [A^T\ \ B^T]^T$.

Next two response systems are considered as

Response system-I:

$$\dot{x} = f(x) + u^{(1)} \,, \tag{1.4.5}$$

where x is state vector variable.

Response system-II:

$$\dot{y} = g(y) + u^{(2)} \,, \tag{1.4.6}$$

where y is state vector variable and $u^{(1)}(t)$, $u^{(2)}(t)$ are control functions, such that $u^{(i)}(t) = [u_1^{(i)},\ u_2^{(i)}, \cdots ,u_n^{(i)}]^T,\ \ i = 1,\ 2$.

The linear combination of the response system I & II gives rise to

$$V_s = \sum_{i=1}^{n} a_i x_i + \sum_{i=1}^{n} b_i y_i$$

$$= [a_1, a_2, \cdots, a_n]x + [b_1, b_2, \cdots, b_n]y$$

$$= A^T x + B^T y$$

$$= [A^T\ B^T]\begin{bmatrix} X_m \\ Y_m \end{bmatrix} = C^T \eta.$$

To obtain the dual synchronization among drive and response systems, Let us define the error function among the drive systems (1.4.3), (1.4.4) and response systems (1.4.5), (1.4.6) as

$$e = V_s - V_m \ ,$$

The drive systems (1.4.3), (1.4.4) and response systems (1.4.5), (1.4.6) are said to be dual synchronized if $\lim_{t\to\infty} \| e \| = 0$, where $\| \cdot \|$ denotes matrix norm.

1.4.1.12 Combination or Combined synchronization

The drive systems are considered as

$$\dot{x}_1 = f_1(x_1) \tag{1.4.7}$$

$$\dot{x}_2 = f_2(x_2) \tag{1.4.8}$$

and the response system is taken as

$$\dot{y} = f(y) + U(x_1, x_2, y), \tag{1.4.9}$$

where $x_1 = [x_1^1, x_2^1, \cdots, x_n^1]^T$, $x_2 = [x_1^2, x_2^2, \cdots, x_n^2]^T$ and $y = [y_1, y_2, \cdots, y_n]^T$ are the state vectors of the chaotic systems. f_1, f_2, $f : R^n \to R^n$ are continuous vector functions and $U(x_1, x_2, y)$ is a control function.

Two drive systems (1.4.7), (1.4.8) and one response system (1.4.9) are said to be combination synchronized, if there exists three constants matrices called scaling matrices, A_1, A_2, A_3 and $A_3 \neq 0$, such that $\lim_{t\to\infty}\| A_1x_1 + A_2x_2 - A_3y \| = 0$, where $\| \, . \, \|$ represents the matrix norm.

It is noted that if $A_1 \neq 0$, $A_2 = 0$, $A_n = I$ then this problem is reduced to the projective synchronization, where I is an $n \times n$ identity matrix. If the scaling matrix A_1 is considered as a function, then synchronization problem is reduced into function projective synchronization problem.

1.4.1.13 Dual combination synchronization

The dual combination synchronization is proposed among four drive and two response systems. First two drive systems are defined by the equations (1.4.3) and (1.4.4).

Next two drive systems are defined as

Drive systems-III:

$$\dot{X}' = f(X'), \tag{1.4.10}$$

Drive system-IV:

$$\dot{Y}' = g(Y'), \tag{1.4.11}$$

where X' and Y' are state vector variables.

The linear combination of the drive systems III & IV, gives rise to

$$V'_m = \sum_{i=1}^{n} a_i X'_i + \sum_{i=1}^{n} b_i Y'_i$$

$$= [a_1, a_2, ...a_n]X' + [b_1, b_2, ...b_n]Y'$$

$$= A^T X + B^T Y = [A^T \; B^T]\begin{bmatrix} X' \\ Y' \end{bmatrix} = C^T \xi'.$$

The corresponding two response systems with control functions are defined by systems (1.4.5) and (1.4.6). Defining the error function among four drive systems (1.4.3), (1.4.4), (1.4.10), (1.4.11) and response systems (1.4.5), (1.4.6) as $e = V_s - V_m - V'_m$.

The drive systems (1.4.3), (1.4.4), (1.4.10), (1.4.11), and the response systems (1.4.5), (1.4.6) will be dual combination synchronized if $\lim_{t\to\infty}\| e \| = 0$, where $\|\cdot\|$ denotes the matrix norm.

1.4.1.14 Combination-combination synchronization

In this section combination-combination synchronization is proposed among two drive systems (1.4.7) and (1.4.8) and two response systems

$$\dot{y}_1 = g_1(y_1) + U_1(x_1, x_2, y_1, y_2), \tag{1.4.12}$$

$$\dot{y}_2 = g_2(y_2) + U_2(x_1, x_2, y_1, y_2), \tag{1.4.13}$$

where $x_1 = [x_1^1, x_2^1, \cdots, x_n^1]^T$, $x_2 = [x_1^2, x_2^2, \cdots, x_n^2]^T$ and $y_1 = [y_1^1, y_2^1, \cdots, y_n^1]^T$, $y_2 = [y_1^2, y_2^2, \cdots, y_n^2]^T$ are the state vectors of the chaotic systems. f_1, f_2, $g_1, g_2 : R^n \to R^n$ are continuous vector functions and $U_1(x_1, x_2, y_1, y_2)$, $U_2(x_1, x_2, y_1, y_2)$ are the control functions.

Two drive systems (1.4.7), (1.4.8) and two response system (1.4.12), (1.4.13) are said to be combination-combination synchronized, if there exists four constants matrices called scaling matrices A_1, A_2, A_3, A_4 and $A_3 \neq 0, A_4 \neq 0$ such that

$\lim_{t\to\infty}\| A_1x_1 + A_2x_2 - A_3y_1 - A_4y_2 \| = 0$, where $\| . \|$ represents the matrix norm.

In this thesis, combined synchronization and combination-combination synchronization are successfully done between time-delayed chaotic systems and fractional order complex chaotic systems respectively. Also, triple compound synchronization among eight chaotic systems with disturbances is studied via non-linear approach.

1.5 Fractional Calculus

In a letter to L`Hospital in 1695 Leibniz raised the following question: "*Can the meaning of derivatives with integer order be generalized to derivatives with non-integer orders?*" L`Hospital was interested about that question and replied by another question to Leibniz: "*What if the order will be 1/2?*" Leibniz in a letter dated September 30, 1695 — the exact birthday of the fractional calculus! — replied: "*It will lead to a paradox, from which one day useful consequences will be drawn.*" The question raised by Leibniz for a fractional derivative was an on-going topic for more than 300 years. Many renowned mathematicians contributed to this notion over the years, among them J. Liouville, B. Riemann, H. Weyl, J. Fourier, N. H. Abel, S. F. Lacroix, G. Leibniz, A. K. Grunwald and A. V. Letnikov.

Further it leads to a new branch of mathematics which deals with derivatives and integrals of arbitrary order and is known as fractional calculus (Miller and Ross (1993)). Nowadays, not only fractions but also arbitrary real and even complex numbers are considered as order of differentiation (Kilbas (2006)). Still, the name "fractional calculus" is kept for the general theory.

In 1819, S. F. Lacroix was first to define the derivative of arbitrary order wherein he found

$$\frac{d^{1/2}}{dx^{1/2}}(x^m)=\frac{\Gamma(m+1)}{\Gamma(m+\frac{1}{2})}x^{m-\frac{1}{2}} \quad .$$

Further Joseph B. J. Fourier (1822) derived the following integral representation of $f(x)$, where

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\,du\int_{-\infty}^{\infty}\cos v(x-u)\,dv \; .$$

Using the representation, he obtained

$$\frac{d^q f(x)}{dx^q} = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u)\, du \int_{-\infty}^{\infty} v^q \cos\left(v(x-u) + \frac{q\pi}{2} \right) dv .$$

Joseph Fourier stated that: "*The number* q *that appears in the above will be regarded as any quantity whatsoever, positive or negative*".

B .Ross (1975) mentioned in his book that, it was Niels Henrik Abel in 1823 that used fractional derivative while solving an integral equation arising in tautochrone problem. In 1832, J. Liouville explicitly gave the first definition of fractional derivative. In 1844, G. Boole developed symbolic method for solving linear differential equations with constant coefficients using fractional calculus. In 1847, Bernhard Riemann (Ross, 1975) proposed the following definition of fractional integration

$$D^{-q} f(x) = \frac{1}{\Gamma(q)} \int_{c}^{x} (x-t)^{q-1} f(t) dt + \psi(x) , \tag{1.5.1}$$

where $\psi(x)$ is Riemann's complementary function.

Fractional derivative in electromagnetic theory was introduced by the renowned mathematician Oliver Heaviside in 1893. In 1917, H. Weyl and G. H. Hardy studied some properties of fractional derivative/integral. In 1939, A. Erdelyi and 1971, T. J. Oslar defined fractional derivatives and Leibniz rule. Though the mathematics of fractional calculus existed in the literature for more than 300 years, its utility has been realized rather recently. Nowadays many engineers and scientists are working on fractional calculus and its applications in various areas.

1.5.1 Applications

Fractional derivative operator D^q differs from the ordinary differential operator D in many respects due to its non-locality. The product rule and chain rule become a bit complicated to derive. The geometrical and physical interpretations of fractional derivatives and fractional integral operators attempted and analysed by many researchers, though some successful and meaningful results in this direction are recently done by Igor Podlubny (2002). Also, there are several definitions of fractional derivative are given by many mathematicians. One can see some of these definitions in nest sub-sections.

In last few decades, the concept of fractional calculus has been applied to almost every area of Science and Engineering. An integer order differential operator is a local operator whereas a fractional differential operator is a non-local operator in the sense that it takes into account the fact that the future states do not only depend upon the present state but also upon all of the histories of its previous states. It is now realised that the non-locality is not a drawback, but it leads to model many natural phenomena containing long memory. Examples of such systems are abundant in nature. Few of them are atmospheric diffusion of pollution, cellular diffusion processes, network traffic, dynamics of visco-elastic materials, electronics, etc.

All such systems have non-local dynamics involving long memory which cannot be modelled efficiently using classical calculus theory. Thus fractional differential equations (FDE) are useful for the modelling of many anomalous phenomena in nature and the theory of complex systems.

Diffusion-wave equation: The time fractional order diffusion-wave equation is given by

$$\frac{\partial^q u(x,t)}{\partial t^q} = k\frac{\partial^2 u(x,t)}{\partial x^2}, \; 0 < q \le 2. \tag{1.5.2}$$

This equation represents fractional diffusion equation for $0 < q \le 1$ and fractional wave equation for $1 < q \le 2$. It is useful in modelling many anomalous phenomena such as diffusion through disordered media/porous media, amorphous through fractals, percolation clusters, polymers and biological systems etc.

W. Wyss (1986) used Mellin transform to solve Cauchy problem. R. W. Schneider and W. Wyss (1989) converted the diffusion-wave equation into the integro-differential equation and derive the corresponding Green functions in form of Fox functions. Y. Fujita (1990) has produced the existence and uniqueness of the solution of the space-time fractional diffusion equation. F. Mainardi (1996) obtained the fundamental solution for the fractional diffusion-wave equation in one space-dimension.

Electrical circuits: Classical electrical circuit contains inductor, capacitor and resistor described by integer-order models however; the circuit may have non-integer properties.

A term fractance suggested by A. Le Mehaute and G. Crepy (1983), representing electrical elements with non-integer order impedance which is modelled using fractional derivative. This gives greater flexibility in this circuit modelling. It can also be used for analogue fractional differentiation and integration. T. Hartley et al. (1995) have shown the Hartley-Chua circuit of system order less than three, exhibits chaos. Ivo Petras (2010) applied fractional calculus in a nonlinear electrical circuit, which is modeled by fractional order equations. He presented the fractional-order memristor-based Chua's equations and methods for their numerical solution, simulation, and stability analysis. He shows a total order of the system less than the number of differential equations using fractional calculus. Chaotic system usually described by three equations but, he also shows a total order less than three and chaos still can be observed. In the case of a hyperchaotic system, the situation is similar. It opens a new area of applications for the proposed chaotic system.

Control theory: Nowadays in control theory fractional derivatives are widely used because of its more realistic approach then integer order derivatives. Major contribution in fractional control theory is done by A. Oustaloup (1983). He developed a CRONE (Commande Robusted' Ordre Non Entier) controller and showed that it works better than classical PID controller.

Biology: K. S. Cole (1933) gave the concept of membrane reactance. It has been used in the conductance of membranes of cells of organisms. Membrane reactance is given by

$$X(\omega) = X_0 \, \omega^{-q} \ .$$

Cole experimentally obtained values of q for different cases such as for guinea pig lever and muscle $(q = 0.45)$ and for potato $(q = 0.25)$ etc.

Bio-engineering: Bio-engineering is a branch of life science which deals with the design, manufacture and maintenance of engineering equipments used in biosynthetic processes. In medical science this branch plays an important role in the design of artificial limbs, artificial pacemaker etc. Bio-engineering strives to develop new mathematical tools for describing the complexity of cells and tissues. Fractional order operations are useful in encoding the multi-scale pattern arising in the muscle fibres and nerve fibres. The resulting dynamics of such multi-scale processes are expressed through fractional order differential equations. Various such applications of fractional calculus in bio-engineering are described in the recent book by R.Magin (2006).

Viscoelasticity: The nature of the real materials lies in between ideal solids and ideal fluids. G. W. Scott Blair (1947) proposed for these intermediate materials as, stress is proportional to the intermediate derivative of strain, i.e.

$$\sigma(t) = E\, D^{q} \varepsilon(t)\ ,\ 0 < q < 1\ , \tag{1.5.3}$$

where E and q are constants depending on material under study. The remarkable contributors in this domain are A. Gemant (1950), A. N. Gerasimov (1948), R. L. Bagley and P. J. Torvik (1984) and many others researchers. M. Caputo and F. Mainardi (1971) used Caputo fractional derivatives to give more realistic models. More detailed discussions on this topic are now available in a recent book by Mainardi (2010).

1.5.2 Fractional derivatives

Here we will see some widely accepted and applied definitions of fractional derivatives which make fractional order modelling more realistic than integer order (Podlubny, 1999).

1.5.2.1 Grunwald-Letnikov fractional derivative

Successive differentiations of function $f(t)$ are given by

$$f^{(1)}(t) = \lim_{h \to 0} \frac{f(t) - f(t-h)}{h}\ ,$$

$$f^{(2)}(t) = \lim_{h \to 0} \frac{f^{(1)}(t) - f^{(1)}(t-h)}{h}$$

$$= \lim_{h \to 0} \frac{f(t) - 2f(t-h) + f(t-2h)}{h^{2}}\ ,$$

In general,

$$f^{(n)}(t) = D^n f(t) = \lim_{h\to 0}\frac{1}{h^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k} f(t-kh)\ , \tag{1.5.4}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is a binomial coefficient. For a non-integer $q>0$, we can write $\binom{q}{k} = \frac{\Gamma(q+1)}{k!\Gamma(q-k+1)}$.

The Grunwald-Letnikov definition is the generalisation of the definition (1.5.4) to a non-integer $q>0$.

$${}^{GL}D_a^q f(t) = \lim_{h\to 0}\frac{1}{h^q}\sum_{k=0}^{\left[\frac{t-a}{h}\right]}(-1)^k\ \frac{\Gamma(q+1)}{k!\Gamma(q-k+1)}\ f(t-kh)\ . \tag{1.5.5}$$

Fractional integral of order $q>0$ is defined by

$${}^{GL}D_a^{-q} f(t) = \lim_{h\to 0} h^q \sum_{k=0}^{\left[\frac{t-a}{h}\right]}\frac{\Gamma(q+k)}{k!\Gamma(q)}\ f(t-kh)\ . \tag{1.5.6}$$

1.5.2.2 Riemann-Liouville fractional derivative

Riemann-Liouville fractional integral operator is a direct generalization of the Cauchy's formula for an n-fold integral as (Podlubny, 1999)

$$\underbrace{\int_a^x dt\int_a^{x_1} dt\cdots\int_a^{x_{n-1}} f(t)\,dt}_{n-\text{times}} = \frac{1}{(n-1)!}\int_a^x \frac{f(t)}{(x-t)^{1-n}}\,dt \tag{1.5.7}$$

Definition 1.5: If $f(x)\in C[a,b]$ and $q>0$, then

$$J_{a^+}^q f(x) := \frac{1}{\Gamma(q)}\int_a^x \frac{f(t)}{(x-t)^{1-q}}dt\ , \qquad x>a\ ,$$

$$J_{b^-}^{q} f(x) := \frac{1}{\Gamma(q)} \int_x^b \frac{f(t)}{(x-t)^{1-q}} dt \,, \qquad x < b \tag{1.5.8}$$

are called as the left sided and the right sided Riemann-Liouville fractional integral of order q, respectively.

Definition 1.6: $${}^{RL}D_a^q f(x) := \frac{1}{\Gamma(1-q)} \frac{d}{dx} \int_a^x \frac{f(t)}{(x-t)^q} dt = D\, I_a^{1-q} f(x),\ 0<q<1, \tag{1.5.9}$$

is called the left side Riemann-Liouville fractional derivative of order q whenever the RHS exists (Podlubny, 1999).

The definitions and properties of the Riemann-Liouville fractional derivative for arbitrary value of $q>0$ are as follows.

Definition 1.7: Let $n-1<q\le n$, then the left sided and right sided Riemann-Liouville fractional derivatives of order q are defined as (Podlubny, 1999)

$${}^{RL}D_{a^+}^q f(x) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dx^n} \int_a^x \frac{f(t)}{(x-t)^{q+1-n}} dt = D^n\, I_{a^+}^{n-q} f(x) \,,\ x>a \,,$$

$${}^{RL}D_{b^-}^q f(x) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dx^n} \int_x^b \frac{f(t)}{(x-t)^{q+1-n}} dt = D^n\, I_{b^-}^{n-q} f(x) \,,\ x<b, \tag{1.5.10}$$

respectively, whenever the RHSs exist.

In further discussion, unless mentioned otherwise, we denote ${}^{RL}D_{a^+}^q f(x)$ by ${}^{RL}D_a^q f(x)$ and $J_{a^+}^q f(x)$ by $J_a^q f(x)$, respectively. Also ${}^{RL}D^q f(x)$ and $J^q f(x)$, refer to ${}^{RL}D_{0^+}^q f(x)$ and $J_{0^+}^q f(x)$, respectively.

Properties: (i) The Riemann-Liouville fractional derivative of constant is not zero.

$${}^{RL}D^q C = \frac{C\,t^{-q}}{\Gamma(1-q)} \neq 0 \,. \tag{1.5.11}$$

(ii) Initial value problem (IVP) containing Riemann-Liouville fractional derivative requires initial conditions of the form ${}^{RL}D^{q-j} f(0)$ i.e.,

$$J^q\left({}^{RL}D^q f(x)\right) = f(t) - \sum_{j=1}^{n} {}^{RL}D^{q-j} f(0) \frac{t^{q-j}}{\Gamma(q-j+1)}, \quad n-1 \le q < n, \tag{1.5.12}$$

which is not useful in real phenomena. To overcome these drawbacks, M. Caputo and F Mainardi (1971) proposed a new definition of derivatives which allows the formulation of initial conditions for fractional IVPs in a form involving only the limit values of integer order derivatives at the lower terminal.

1.5.2.3 Caputo fractional derivative

The definition and properties of the Caputo fractional derivative are given as follows;

Definition 1.8: Let $f \in C^n[a,b]$ and $n-1 < q < n$ then

$${}_a^C D_x^q f(x) = \frac{1}{\Gamma(n-q)} \int_a^x \frac{f^{(n)}(t)}{(x-t)^{(q-n+1)}} dt, \ a < x < b. \tag{1.5.13}$$

Properties: (i) ${}_a^C D^q C = 0$, C is a constant. (1.5.14)

(ii) $\lim_{q \to n} {}_a^C D^q f(x) = f^{(n)}(x)$. (1.5.15)

Lemma 1.4: (Kilbas (2006)) Let $f(x) \in R$ be a continuous and derivable function. Then for any $x \ge x_0$,

$$\frac{1}{2} {}_{x_0}^C D_x^q f^2(x) \le f(x) {}_{x_0}^C D_x^q f(x), \quad \forall \ q \in (0,1). \tag{1.5.16}$$

1.5.2.4 Relation between Riemann-Liouville and Caputo derivatives

Theorem 1.5: Let $f \in C^n[a,b]$ and $n-1 < q < n$. Then R-L and Caputo fractional derivatives are connected by the relation

$$ {}^{RL}D_a^q f(x) = D_a^q f(x) + \sum_{k=0}^{n-1} \frac{f^{(k)}(a^+)}{\Gamma(1+k-q)} (x-a)^{k-q} \ . \qquad (1.5.17) $$

Proof: ${}^{RL}D^q f(x) = D^n J^{n-q} f(x)$

$$ = D^n \left[J^{n-q} \left(J^n f^{(n)}(x) + \sum_{k=0}^{n-1} \frac{f^{(k)}(a^+)}{k!} (x-a)^k \right) \right] $$

$$ = J^{n-q} f^{(n)}(x) + D^n J^{n-q} \sum_{k=0}^{n-1} \frac{f^{(k)}(a^+)}{k!} (x-a)^k \qquad (1.5.18) $$

$$ = D_a^q f(x) + \sum_{k=0}^{n-1} \frac{f^{(k)}(a^+)}{\Gamma(1+k-q)} (x-a)^k \ . \qquad (1.5.19) $$

From the above theorem, we get the following results:

(i) If $q = n \in N$, then ${}^{RL}D_a^q f(x) = D_a^q f(x) = D^n f(x)$.

(ii) If $f^{(k)}(a) = 0$ for $k = 0, 1, \cdots, n-1$, then ${}^{RL}D_a^q f(x) = D_a^q f(x)$.

(iii) If $0 < q < 1$, then ${}^{RL}D_a^q f(x) = D_a^q f(x) + \frac{f(a)}{\Gamma(1-q)} (x-a)^{-q}$.

Theorem 1.6: Let $f \in C^n[a, b]$ and $n-1 < q < n$, then

$$ J_a^q \, D_a^q f(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a^+)}{k!} (x-a)^k \ , \ x \geq a \ . \qquad (1.5.20) $$

Proof: If $J_a^q \, D_a^q f(x) = J_a^q \, J_a^{n-q} \, f^{(n)}(x) = J^{(n)} \, f^{(n)}(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k \ , \ x \geq a$,

then equation (1.5.20) is a particular case of the more general property

$$ J_a^q \, D_a^r f(x) = J_a^q \, J_a^{m-r} \, f^{(m)}(x) = J_a^{(q-r)} \left(J^{(n)} \, f^{(n)}(x) \right) \ q > r \ , \qquad (1.5.21) $$

$$ = J_a^{q-r} f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(q-r+k+1)} (x-a)^{q-r+k} \ , \ x \geq a \ , \ m-1 < r < m \ . $$

1.5.3 Leibniz rule

(Podlubny (2002)) If $f(x)$ and $g(x) \in C^{\infty}[a, t]$ then the Leibniz rule for fractional derivative is given by

$$ {}_aD_t^q\left(g(t)\,f(t)\right)=\sum_{k=0}^{\infty}\binom{q}{k}g^{(k)}(t)\,D^{q-k}f(t)-R_n^q(t)\ , \tag{1.5.22}$$

where $R_n^q(t)=\dfrac{1}{n!\Gamma(-q)}\displaystyle\int_a^t\frac{f(\tau)}{(t-\tau)^{q+1}}\,d\tau\int_\tau^t g^{(n+1)}(\xi)(\tau-\xi)^n\,d\xi$.

1.5.4 Fractional differential equation

In this section, some results are discussed on existence and uniqueness of fractional differential equations involving Riemann-Liouville derivative ${}^{RL}D$ and Caputo derivative D.

Theorem 1.7: (Daftardar-Gejji and Babakhani (2004)) the unique solution of the initial value problem (IVP) is

$$ {}^{RL}D^q[\bar{x}(t)-\bar{x}(0)]=A\bar{x}(t)\ ,\ \bar{x}(0)=\bar{x}_0,\ 0<q<1,\ t\in[0,\ \chi]\ ,\ \chi>0\ , \tag{1.5.23}$$

where A is an $n\times n$ matrix, is $E_q(t^qA)\bar{x}_0$.

Theorem 1.8: (Daftardar-Gejji and Babakhani (2004)) Let $f_i : W \to R$ be continuous, $i=1,\ 2,\ \cdots,\ n$, where

$$W=[0,\ \chi^*]\times\Pi_{j=1}^n[x_j(0)-l_j\ ,\ x_j(0)+l_j]\ ,\ \chi^*>0\ ,\ l_j>0\ ,\ \forall j\ ,$$

and $\bar{f}=(f_1\ ,\ f_2\ ,\cdots,f_n)$. Then the non-autonomous IVP

$$ {}^{RL}D^q[\bar{x}(t)-\bar{x}(0)]=\bar{f}(t,\ \bar{x})\ ,\ \bar{x}(0)=\bar{x}_0\ ,\ 0<q<1\ , \tag{1.5.24}$$

has a solution $\bar{x}(t):[0,\ \chi]\to R^n$, where

$$\chi = \min\left\{ \chi^* , \left(\frac{l\,\Gamma(q+1)}{\|\bar{f}\|_\infty} \right)^{1/q} \right\} , \; l = \min\{ l_1 , l_2 , \cdots , l_n \} .$$

Theorem 1.9: (Daftardar-Gejji and Jafari (2007)) Let $f = (f_1 , f_2 , \cdots , f_n) : W \to R^n$ be in C^1, where

$$W = [0,\, \chi^*] \times \Pi_{j=1}^{n} [x_j(0) - l_j , x_j(0) + l_j] , \;\; \chi^* > 0 , \;\; l_j > 0 , \;\; \forall j ,$$

then the system of non-autonomous equations

$$D^q x_i(t) = f_i(t , x_1 , x_2 , \cdots , x_n) , \; x_i^{(k)}(0) = C_k^i , \; 1 \le i \le n , \; 1 \le k \le m_i , \qquad (1.5.25)$$

where $m_i < q_i < m_i + 1$ has a unique solution $\bar{x}(t) : [0 , \chi] \to R^n$, where

$$\chi = \min\left\{ \chi^* , \left(\frac{l\,\Gamma(q_i+1)}{[1+q]\|f\|} \right)^{1/q_i} , \left(\frac{lk!}{[1+q]\left|C_k^i\right|} \right)^{1/k} \right\} ,$$

$$i = 1 , 2 , \cdots , n , \; k = 1 , 2 , \cdots , m_i ,$$

$$l = \min\{ l_1 , l_2 , \cdots , l_n \} ,$$

$$q = \max\{ q_1 , q_2 , \cdots , q_n \} ,$$

and $[1+q]$ denotes integral part of $1+q$.

Lemma 1.10: (Norelys et al. (2014)) Let $f(t) \in R$ be a continuous and derivable function. Then for any time instant $t \ge t_0$,

$$\frac{1}{2} D^q f(t) \le f(t)\, D^q f(t), \; \forall \; q \in (0,\, 1) . \qquad (1.5.26)$$

1.6 Delay Differential Equations

In many applications, it is considered that the systems under study satisfy the *principle of causality* that is the rate of change of the state of system is independent of the past and is determined solely by the present inputs. But one could realize that this is only a first approximation to the true situation.

Realistic modelling which represents the rate of variation in the system's state should depend not only on its current value but also on the past events. Ordinary or partial differential equations are used to model those systems which are governed by an equation that is not associated with and dependent on its past history. However in most of the phenomena, time delays are not neglectable, such models assimilating past history generally modelled with delay differential equations or functional differential equations (FDEs).

Delay differential equations (DDE) are a class of functional differential equations where the highest order derivative of the unknown function at a certain time depends on the solution of the function at previous times. DDEs are also referred as hereditary differential equations, retarded functional differential equations, in control theory as time-delay systems and equations with after effect or dead-time, differential- difference equations.

Mathematically, the delay differential equations can be expressed in the form

$$x'(t) = f(t, x(t), x_t),\ t \geq t_0 \tag{1.6.1}$$

with initial history

$$x(t) = \phi(t),\ t_0 - \tau \leq t < t_0\ , \tag{1.6.2}$$

where $x(t) \in R^n$, $x_t = \{x(t-\tau): \tau \leq t\}$ are the solution state of the system at time t and in the past respectively, τ is lag or time delay. Instead of a usual initial condition, an initial history function $\phi(t)$ needs to be specified on the whole interval $[t_0 - \tau, t_0]$. This initial function is generally taken to be continuous, which is an infinite set of values that makes the DDE problem inherently infinite-dimensional. This infinite dimensional nature of DDE is apparent in the area of dynamical system.

There might be more general delay equations could be considered that contain constant time delays (τ_j are positive constant), time-dependent delays $\left(\tau_j = \tau_j(t)\right)$, state-dependent delays $\left(\tau_j = \tau_j\left(t, x(t)\right)\right)$, continuously distributed delays and higher derivatives all occur in applications and lead to more complicated evolution equations.

However equations of the form (1.6.1) and (1.6.2) constitute a sufficiently a broad class of system arise in practice for a variety of reasons, and provide an important category of dynamical systems called as time-delay dynamical systems.

ODEs and DDEs are not very different in theatrical point of view. It's quite intuitive to define the ideas of linear, nonlinear and homogenous equations for DDEs as defined for ODEs. The analytical and numerical techniques for solutions developed for ODEs could be extend for DDEs as well. The phase space for an ODE is always finite dimensional whereas DDEs show an infinite dimensional dynamical system because the fact that instead of an initial value, an initial function is necessary to determine the solution.

From the literature survey it can be found that there are lot of applications of DDE in the following areas.

Population dynamics: In 1845, the well-known logistic equation given by Verhulst and Pierre-Francois

$$\dot{N}(t) = r\,N(t)\left(1 - \frac{N(t)}{K}\right) \tag{1.6.3}$$

describes the growth of population, where $N(t)$ is population at time t and $r > 0$ is Malthusian parameter describing growth rate, and K is carrying capacity. The model assumes that the population density negatively affects the per capita growth rate due to environmental degradation. G. Hutchinson (1948) introduced a delay into the logistic equation to account for hatching and maturation periods, which is given by

$$\dot{N}(t) = r\,N(t)\left(1 - \frac{N(t-\tau)}{K}\right) . \tag{1.6.4}$$

Virology: R. Culshaw and S. Ruan (2000) proposed a HIV model to include a time delay between virus-cell contact and subsequent infection of the $CD4+T-$cell. The model is

$$\dot{T}(t) = s - \mu_T T(t) + rT(t)\left(1 - \frac{T(t)+I(t)}{T_{\max}}\right) - k_1 T(t)V(t) ,$$

$$\dot{I}(t) = k_1' T(t-\tau)V(t-\tau) - \mu_1 I(t) ,$$

$$\dot{V}(t) = N\mu_b I(t) - k_1 T(t)V(t) - \mu_V V(t) , \tag{1.6.5}$$

where T denotes healthy $T-$cells in the blood, I is the HIV infected $T-$cells and V is the HIV virus level in the blood.

Nonlinear optics: Ikeda et al. (1980) considered a nonlinear absorbing medium containing two-level atoms placed in a ring cavity and subject to a constant input of light. The optical system undergoes a time-delayed feedback that destabilizes its steady-state output. The DDE formulated by Ikeda et al. (1980) is

$$\tau\dot{\phi}(t) = -\phi(t) + A^2\left[1 + 2B\cos\left(\phi(t-t_D) - \phi_0\right)\right] . \tag{1.6.6}$$

1.6.1 Existence and uniqueness of a solution: Method of steps

Theorem 1.11: Let $f(t, x, y)$ and $f_x(t, x, y)$ be continuous on R^3 and let $\phi: [-\tau, 0] \to R$ be continuous. Then for $\sigma > 0$ there exists a unique solution of the IVP

$$\dot{x}(t) = f\left(t, x(t), x(t-\tau)\right) , \qquad t \geq 0 ,$$

$$x(t) = \phi(t) , \qquad -\tau \leq t \leq 0 \tag{1.6.7}$$

on $[-\tau, \sigma]$.

One need a solution $u(t)$ such that $u(t)=\phi(t)$, $-\tau \le t \le 0$ and satisfying (1.6.7) for $t \ge 0$. For $0 \le t \le \tau$, the function $u(t)$ must satisfy the IVP

$$\dot{x}(t) = f(t, x(t), \phi(t-\tau)) = g(t, x(t)), \qquad x(0) = \phi(0) . \tag{1.6.8}$$

Since $g(t, x(t)) = f(t, x(t), \phi(t-\tau))$ and $g_x(t, x(t))$ are continuous, the local solution of the ODE (1.6.8) is guaranteed by standard results from ODE theory. If this local solution exists for the entire interval $[0, \tau]$, then the solution $u(t)$ is defined so far on $[-\tau, \tau]$ and one may repeat the above argument to extend their solution still further to the right.

1.6.2 Numerical solution of a DDE

Consider the DDE, given by equation (1.6.7) which is equivalent to

$$x(t) = x(0) + \int_0^t f(z, x(z), x(z-\tau))dz . \tag{1.6.9}$$

Assign the step-size $h = \tau / k$ for some fixed natural number k so that $h \le \tau$. Consider the nodes $t_n = nh$, $n = 0, 1, 2, \cdots$ and denote $x_n = x(t_n)$. For $0 \le n \le k$, the term $x(t_n - \tau) =$ $x(nh - kh) =$ $x(-(k-n)h) =$ $\phi(-(k-n)h)$. For $n > k$, $x(t_n - \tau) =$ $x((n-k)h) = x_{n-k}$. Assume that we have obtained the values $x_1, x_2, \cdots , x_{n-1}$. For $n-$th step, it can be written as

$$x_n = x_{n-1} + \int_{t_{n-1}}^{t_n} f(z, x(z), x(z-\tau))dz , \tag{1.6.10}$$

using some suitable integration formulae.

1.6.3 Stability of Delay Differential Equations

The concept of stability of systems of differential equations with delays has been an active area of research in science and engineering. Stability of nonlinear functional differential equations is a quite complicated problem due to lack of complete Lyapunov functional structure whose existence is necessary for the stability of usual nonlinear time-delay systems. N. N. Krasovskii (1959) introduced the Lyapunov-Krasovskii stability criterion, and it is the generalization of the classical Lyapunov stability theory for ordinary systems to time-delayed systems (infinite dimensional systems). The choice of an appropriate Lyapunov-Krasovskii functional is crucial for deriving of stability criteria. Krasovskii extended the complete theory of Lyapunov by using functional $V: C \to R$, where $C = C([-\tau, 0], R^n)$.

Theorem 1.5: (Krasovskii and Brenner (1963)) Suppose that $u, v, w: [0, \infty) \to [0, \infty)$ are continuous nonnegative non-decreasing functions, $u(s), v(s)$ are positive for $s > 0, u(0) = v(0) = 0$. If there is a continuous function $V: C \to R$, such that

$$u(|\varphi(0)|) \leq V(\varphi) \leq v(|\varphi|), \quad \varphi \in C,$$

$$V'(\varphi) = \lim_{t \to 0} \sup \frac{1}{t}\left[V\big(x_t(.,\varphi)\big) - V(\varphi)\right] \leq -w(|\varphi(0)|),$$

Then equilibrium point of the time-delayed system $x = 0$ is stable.

If, in addition, $w(s) > 0$ for $s > 0$, then equilibrium point $x = 0$ is asymptotically stable.

1.7 Methodology

The most productively and widely studied approach named Pecora-Carroll scheme designed by L.M. Pecora and T. L. Carroll (1990), in which two identical chaotic systems with different initial conditions are synchronized. They have theoretically proven and experimentally demonstrated that it is possible to synchronize chaotic systems by appropriate couplings between these systems. In past few years, many methods are proposed for synchronization of chaotic systems viz., Pecora-Carroll method, Active control method, Adaptive control method, Tracking control method, Nonlinear control method, Backstepping method etc.

1.7.1 Active Control Method

E. W. Bai and K. E. Lonngren (1997) were first designed the active control method in 1997 and synchronize the identical Lorenz chaotic system using active control method. They showed the sequential synchronization (Bai and Lonngren (2000)) of two Lorenz systems using this method. In 2002, the active control method successfully applied for synchronization of two different chaotic systems viz., easy periodic system and Rossler system by M. C. Ho and Y. C. Hung (2002). In 2007, J. P. Yan and C. P. Li (2007) investigated chaos synchronization of fractional order Lorenz, Rossler and Chen systems taking one system as drive and other as response system. In 2008, U. E. Vincent and J. A. Laoye (2007, 2008) presented chaos synchronization between two nonlinear systems using two different techniques viz., active control and back stepping control in terms of transient analysis. In the same year, X Zhou and Cheng (2008) showed synchronization between different fractional order chaotic systems viz., Rossler & Chen systems and Chua & Chen systems. Srivastava et al. (2014) recently applied this method for anti-synchronization between identical and non-identical fractional order chaotic systems. The active control method has received huge attention during the last few years.

1.7.2 Nonlinear Control Method

The synchronization of chaotic systems using nonlinear control method was studied by J. H. Park (2005). In 2006, Dong et al. (2006) studied synchronization of the hyperchaotic Rossler system with uncertain parameters using the same method. The method was successfully used by S. Y. Li and Z. M. Ge (2011) during the study of pragmatically adaptive synchronization of different orders chaotic systems with uncertain parameters and also by Singh et al. (2014) during synchronization and anti-synchronization of chaotic systems.

To process the method for synchronization, first consider the chaotic system as the drive system as

$$\dot{x}_i = P\,x_i + Q\,f(x_i)\ ,\ i = 1,\,2,\,\cdots,\,n\,, \tag{1.7.1}$$

where $x_i = [x_1, x_2, \cdots, x_n]^T \in R^n$ is the state vector variable, P and Q are $n \times n$ matrices of the system parameters and $f : R^n \to R^n$ is a nonlinear function of the system.

Consider another chaotic system as a response system as

$$\dot{y}_i = P_1\,y_i + Q_1\,g(y_i) + u_i\ ,\ i = 1,\,2,\,\cdots,\ n\ , \tag{1.7.2}$$

where $y_i = [y_1, y_2, \cdots, y_n]^T \in R^n$, is the state vector, P_1 and Q_1 are $n \times n$ parameter matrices, $g : R^n \to R^n$ is a nonlinear function and u_i are the control function of the system.

The error states are defined as $e_i = y_i - x_i$, $i = 1, 2, \cdots, n$. Then the error system becomes

$$\dot{e}_i = P_1 e_i + Q_1 g(y_i) + (P_1 - P) x_i - Q f(x_i) + u_i \;. \tag{1.7.3}$$

While synchronization our aim is to find the appropriate feedback controller u_i so that the dynamical error system (1.7.3) can be stabilized in order to get $\lim_{t \to \infty} \| e(t) \| = 0$ for all $e(0) \in R^n$.

Defining the Lyapunov function as

$$V = \frac{1}{2} e_i^T e_i \;, \text{ with } \; e_i(t) = [e_1(t), e_2(t), \cdots, e_n(t)]^T .$$

Derivative of $V(t)$ w. r.t. t is

$$\frac{dV}{dt} = \frac{1}{2}\frac{d(e_i^T e_i)}{dt} = \frac{1}{2}\frac{d}{dt}(e_1^2 + e_2^2 + \ldots\ldots + e_n^2) = \sum_{i=1}^{n} \frac{1}{2}\frac{de_i^2}{dt} = \sum_{i=1}^{n} e_i \frac{de_i}{dt}$$

By choosing the control functions as $u_i = -(P_1 + 1)e_i - (P_1 - P)x_i - Q_1 g(y_i) + Q f(x_i)$,

we have $$\frac{dV}{dt} = -\sum_{i=1}^{n} e_i^2 , \tag{1.7.4}$$

which indicate that the Lyapunov function $V(t)$ becomes negative definite and is necessary to get the required synchronization of the systems (1.7.1) and (1.7.2).

1.7.3 Homotopy Analysis Method

The Homotopy Analysis Method (HAM) technique was introduced by S J Liao (1992) for the linear and nonlinear partial differential equations. This technique is the combination of classical perturbation technique and homotopy, a concept of topology and differential geometry. HAM is the unification of Lyapunov artificial small parameter method, Delta expansion method, and Adomian decomposition method. On theoretical background, HAM works on the concept that a nonlinear equation can be split into a number of linear sub-equations.

The difference with the other perturbation methods is that HAM is independent of small/large physical parameters. Another important advantage as compared to the other existing perturbation and non-perturbation methods lies in the freedom to choose a proper base function to get a better approximate solution of the problems (Liao 2003, 2004). Recently, S. J. Liao (2009) has claimed that the difference with the other analytical methods is that one can ensure the convergence of series solution by means of choosing a proper value of the convergence-control parameter. Recently, Das *et al*. (2011, 2013) have successfully applied the method to investigate the influences of the auxiliary parameter to find the region of convergence through h-curve analysis in solving the considered fractional diffusion equation.

The specialty of HAM is the evaluation of convergence control parameter. While evaluating the solution of nonlinear problems, the convergence region is controlled by the plot of control parameter from this region for getting convergence of the series solution. Since HAM is good mathematical tool to solve nonlinear problems if we have the idea about the structure of the solution to the problem so that a proper base function can be selected and to study the solution as it is known that any real continuous function can be represented by so many types of base functions viz., algebraic, periodic, exponential. Thus for the same physical nonlinear problems in the physical world, sometimes it is difficult to approximate the solution when there is lack of knowledge about a proper set of base functions.

1.8 Numerical Methods

In this section the numerical methods for solving delay differential equations and fractional order differential equations are discussed.

1.8.1 Runge-Kutta method for Delay Differential Equations

L. F. Shampine and S. Thompson (2001) developed a MATLAB code dde23 to solve delay differential equations (DDEs) with constant delays in the year 2001. The method was proposed based on the Runge-Kutta triple BS (2,3) used in ode23, which nicely explains how the explicit Runge-Kutta triples can be extended and used to solve DDEs.

Consider a system of the nonlinear delay differential equation as

$$y'(t) = f(t, y(t), y(t-\tau_1), y(t-\tau_2), \dots, y(t-\tau_k)), \quad a \le t \le b, \tag{1.8.1}$$

with initial history

$$y(t) = S(t), \ t \le a, \tag{1.8.2}$$

where $\tau_j (j = 1,2, \dots, k)$ are fixed delays and $\tau = \min(\tau_1, \dots, \tau_k) > 0$.

To construct numerical strategy for DDEs', let us discuss the explicit Runge-Kutta triples to solve the general ordinary differential equation

$$y'(t) = f\big(t, y(t)\big), \ a \le t \le b, \tag{1.8.3}$$

with the initial condition $y(a)$.

Assume $y_n = y(t_n)$ is the approximate value of y_i at t_i. Let $t_{n+1} = t_n + h_n$. A triple of s-stages involves three formulas. For $i = 1,2, \dots, s$, the stages $f_{ni} = f(t_{ni}, y_{ni})$ are defined in terms of $t_{ni} = t_n + c_i h_n$ and

$$y_{ni} = y_n + h_n \sum_{j=1}^{i-1} a_{ij} f_{nj}$$

Suppose $\Phi(t_n, y_n)$ as an increment function, the approximation used to carry forward the integration is

$$y_{n+1} = y_n + h_n \sum_{i=1}^{s} b_i f_{ni} = y_n + h_n \Phi(t_n, y_n).$$

Solution satisfies above formula with a residual called the local truncation error lte_n as

$$y_{n+1} = y_n + h_n \Phi(t_n, y_n) + lte_n \, ,$$

which inherits an error of order $O\left(h_n^{p+1}\right)$ for sufficiently smooth f and $y(t)$. Select the step size triple that gives rise to another formula as

$$y^*_{n+1} = y_n + h_n \sum_{i=1}^{s} b_i^* f_{nj} = y_n + h_n \Phi^*(t_n, y_n).$$

The solution satisfies this equation with a local truncation error lte_n^*, which is an error of order $O\left(h_n^p\right)$. The trird formula is given by

$$y_{n+\sigma} = y_n + h_n \sum_{i=1}^{s} b_i(\sigma) f_{ni} = y_n + h_n \Phi(t_n, y_n, \sigma),$$

where coefficients $b_i(\sigma)$ are polynomials in σ. So this represents a polynomial approximation to $y(t_n + \sigma h_n)$ for $0 \le \sigma \le 1$. The third formula is referred as a continuous extension of the first since it yields the value y_n when $\sigma = 0$ and y_{n+1} when $\sigma = 1$ assuming that the order of the continuous extension is same as that of the first formula. These assumptions hold for the BS(2,3) triple. The formula is used to advance the integration for such triples as just the special case $\sigma = 1$ of the continuous extension and is defined by

$$y(t_n + \sigma h_n) = y(t_n) + h_n \Phi(t_n, y(t_n), \sigma) + lte_n(\sigma).$$

Suppose for smooth f and $y(t)$, there exists a constant C_1 such that

$$\| lte_n(\sigma) \| \le C_1 h_n^{p+1} \quad \text{for} \quad 0 \le \sigma \le 1.$$

Main problem is to establish approximation to the delayed term $y\left(t - \tau_j\right)$ which consists of two cases $h_n \le \tau_j$ and $h_n > \tau_j$ for some j and suppose an approximation as $y(t) = S(t)$ is accessible $\forall \, x \le x_n$.

If $h_n \leq \tau$, then all $t_{ni} - \tau_i \leq t_n$ and $f_{ni} = f(t_{ni}, y_{ni}, S(t_{ni} - \tau_1), \dots, S(t_{ni} - \tau_k))$ are explicit form of the stage and thus the formulas are explicit. After the step to x_n, we use the continuous extension to characterize $S(t)$ on $[t_n, t_{n+1}]$ as $S(t_n + \sigma h_n) = y_{n+\sigma}$.

For the second case, the implicit formulas may be evaluated when the step is bigger than τ i.e., $h_n > \tau_j$ for some j, the history term $S(t)$ is evaluated in the span of the current step and the formula as defined implicitly. Defining $S(t)$ for $x \leq x_n$, when reaching x_n and extend its definition somehow to $(t_n, t_n + h_n]$ and represent the resulting function as $S^0(t)$. The simple iteration starts with the approximate solution $S^{(m)}(t)$. The following iterations are computed with explicit formula as

$$S^{(m+1)}(t_n + \sigma h_n) = y(t_n) + h_n \Phi\left(t_n, y(t_n), \sigma : S^{(m)}(t)\right).$$

1.8.2 Adams-Bashforth-Moulton Method

An algorithm for numerical solution of fractional-order differential equations with proper initial conditions were investigated and developed by K. Diethelm et al. (2004a) and K Diethelm and J Ford (2004b). This scheme is generalization of the classical one-step Adams-Bashforth-Moulton scheme for first order equations. The algorithm may be used even for nonlinear problems, and it may also be extended to multi-term equations which involve more than one differential operator. We interpret the approximate solution of nonlinear fractional-order differential equations using this algorithm in the following way.

Let us consider differential equation as

$$D_t^{\alpha} y(t) = f(t, y(t)),\ 0 \leq t \leq T\ , \tag{1.8.4}$$

with $y^{(k)}(0) = y_0^{(k)},\ k = 0, 1, \dots\dots., m-1$ (1.8.5)

where $m = \lceil \alpha \rceil$ is the smallest integer $\geq \alpha$ and the differential operator is in the sense of Caputo derivative. The initial value problem (1.8.4) is equivalent to the Volterra integral equation

$$y(t)=\sum_{k=0}^{\lceil\alpha\rceil-1} y_0^{(k)}\frac{t^k}{k!}+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,y(s))ds. \tag{1.8.6}$$

Set, $h=T/N$, $t_n=nh, n=0,1,........N\in Z^+$. Then (1.8.6) can be discredited as follows:

$$y_h(t_{n+1})=\sum_{k=0}^{\lceil\alpha\rceil-1} y_0^{(k)}\frac{t_{n+1}^k}{k!}+\frac{h^q}{\Gamma(\alpha+2)} f(t_{n+1},y_h^p(t_{n+1}))+\frac{h^q}{\Gamma(\alpha+2)}\sum_{j=0}^{n} a_{j,n+1} f(t_h,y_h(t_j)), \tag{1.8.7}$$

$$a_{j,n+1}=\begin{cases} n^{\alpha+1}-(n-\alpha)(n+1)^{\alpha}, & if \quad j=0, \\ (n-j+2)^{\alpha+1}+(n-j)^{\alpha+1}-2(n-j+1)^{\alpha+1}, & if \quad 0\le j\le n, \\ 1, & if \quad j=n+1, \end{cases} \tag{1.8.8}$$

where predicted value $y_h^p(t_{n+1})$ is determined by fractional Adams-Bashforth method

$$y_h^p(t_{n+1})=\sum_{k=0}^{\lceil\alpha\rceil-1} y_0^{(k)}\frac{t_{n+1}^k}{k!}+\frac{1}{\Gamma(\alpha)}\sum_{j=0}^{n} b_{j,n+1} f(t_j,y_h(t_j)), \tag{1.8.9}$$

$$b_{j,n+1}=\frac{h^\alpha}{\alpha}((n+1-j)^\alpha-(n-j)^\alpha). \tag{1.8.10}$$

Thus the equations (1.8.8) and (1.8.9) with the weights $a_{j,n+1}$ and $b_{j,n+1}$ describe the fractional Adams-Bashforth-Moulton scheme.

The error estimate is

$$\max_{j=0,1,.....N}\left|y(t_j)-y_h(t_j)\right|=O(h^p), \tag{1.8.11}$$

where $p=\min(2,1+q)$.

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