CHAPTER 4

SOME THEOREMS ON LINEAR THEORY OF THERMOELASTICITY FOR AN ANISOTROPIC MEDIUM UNDER AN EXACT HEAT CONDUCTION MODEL WITH A DELAY

4.1 Introduction

In the present Chapter, we consider the very recently proposed model by Leseduarte and Quintanilla (2013) with a single delay term that considers the micro-structural effects in the heat transport phenomenon. We attempt to establish some important theorems in this context for an anisotropic and inhomogeneous material. A generalized thermoelasticity theory was proposed by Roychoudhuri (2007) based on the heat conduction law with three-phase-lag effects for the purpose of considering the delayed responses in time due to the micro-structural interactions in the heat transport mechanism. However, the model defines an ill-posed problem in Hadamard sense. Quintanilla (2011) and subsequently, Leseduarte and Quintanilla (2013) have proposed to reformulate this constitutive equation of heat conduction theory with a single delay term and has investigated the spatial behavior of the solutions for this theory. A Phragmen- Lindelof type alternative is obtained and it has been shown that the solutions either decay in an exponential way or blow-up at infinity in an exponential way. The obtained results are extended to a thermoelasticity theory by considering the Taylor series approximation of the equation of heat conduction to the delay term and Phragmen-Lindelof type alternative is obtained for the forward and backward in time equations. In this chapter, we consider the basic equations concerning this new theory of thermoelasticity for an

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anisotropic and inhomogeneous material and established some important theorems in this context.

The variational principle in thermoelasticity is an alternative method for determining the state of dynamics of a thermoelastic system by identifying it as an extremum of a function or a functional. In the classical elasticity, Ignaczak (1963) and Gurtin (1964) have established a variational principle by developing an alternative characterization of the solution to the initial-boundary value problem. Iesan (1966, 1974) and later on, Nickell and Sackman (1968) have established convolutional type variational principles for the linear coupled thermoelasticity theory on the basis of a similar type of alternative formulation like, Ignaczak (1963) and Gurtin (1964). Subsequently, Iesan (1967) has established the first variational theorem of Gurtin type in the theory of thermoelasticity for solids with micro-structure. The reciprocity theorem is used to deduce various methods of integrating the elasticity equations by means of Green's function and it has significant practical applications in the solution of engineering problems (see Nowacki (1975)). Betti-Maxwell reciprocity theorem has been established for the static problems in theory of thermoelasticity by Maysel (1951). The reciprocity theorem is later on extended to uncoupled thermoelasticity, coupled thermoelasticity (CTE) and CTE for anisotropic homogeneous material by Predeleanu (1959), Ionescu-Cazimir (1964) and Nowacki (1975), respectively. Iesan (1967) has derived the first reciprocal relation without using the Laplace transform. The reciprocity theorems of convolution type are also derived by Iesan (1966, 1974). Scalia (1990) has used a method to derive reciprocity relations without using the Laplace transform and without incorporation of the initial data in the field equations. An exhaustive treatment of the variational principles in thermoelasticity is available in the books by Lebon (1980), Carlson (1972), Hetnarski and Ignaczak (2004) and Hetnarski and Eslami (2010). Recently, the convolution type variational principles and reciprocal relations on different theories of thermoelasticity are given by Chirita and Ciarletta (2010) and Mukhopadhyay and Prasad (2011), Kothari and Mukhopadhyay (2013).

In the present work, the main objective is to establish a uniqueness theorem, variational principle and reciprocity theorem for the new theory proposed by Quintanilla (2011). For this, the work has been organized as follows: firstly, we summarize the basic equations in the context of this new theory and consider a mixed initial-boundary value problem that considers non-homogeneous initial conditions. Then, we have established the uniqueness theorem in this context. We formulate an alternative characterization of the mixed boundary initial value problem in the present context by incorporating the initial conditions into the field equations. On the basis of this formulation, a convolutional type variational principle and a reciprocity theorem are also established.

4.2 Basic Governing Equations : Problem Formulation

We consider $\bar{V}$ as the closure of an open, bounded, connected domain whose boundary is A enclosing a non-homogeneous and anisotropic thermoelastic material. Let V denotes the interior of $\bar{V}$ and we assume that n_i are the components of outward drawn unit normal to A. Let A_i, $(i = 1, 2, 3, 4)$ are the subsets of A such that $A_1 \cup A_2 = A_3 \cup A_4 = A$ and $A_1 \cap A_2 = A_3 \cap A_4 = \varnothing$. We consider the motion relative to an undistorted stress free reference state.

By following Quintanilla (2011) and Leseduarte and Quintanilla (2013), we consider the basic governing equations and the constitutive relations under linear theory of thermoelasticity for a non-homogeneous and anisotropic material as follows:

The equation of motion:

$$\sigma_{ij,j} + \rho h_i = \rho \ddot{u}_i \tag{4.1}$$

The equation of energy:

$$\rho \theta_0 \dot{S} = -q_{i,i} + \rho \varpi \tag{4.2}$$

The constitutive relations:

$$\sigma_{ij} = C_{ijkl} e_{kl} - \beta_{ij} \theta \tag{4.3}$$

$$\rho S = \rho c_E \frac{\theta}{\theta_0} + \beta_{ij} e_{ij} \tag{4.4}$$

$$\dot{q}_i = -\{k_{ij}\frac{\partial}{\partial t} + k^*_{ij}(1 + \tau\frac{\partial}{\partial t} + \frac{\tau^2}{2}\frac{\partial^2}{\partial t^2})\}\gamma_j \tag{4.5}$$

$$\gamma_j = \theta_{,j} \tag{4.6}$$

The geometrical relation:

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) = u_{(i,j)} \tag{4.7}$$

Here, β_{ij}, k_{ij}, k^*_{ij} are the elasticity tensor, thermoelasticity tensor, thermal conductivity tensor and conductivity rate tensor respectively. γ_j is the components of the temperature gradient vector and τ is the delay term in the context of new model given by Quintanilla (2011). In the above system of equations, we have used a rectangular coordinate system x_k in three dimensional Euclidean space with usual indicial notations.

4.3 Mixed Initial-Boundary Value Problem

For a mixed initial boundary value problem, we consider the field equations and constitutive relations given by equations (4.1)-(4.7) defined in $V \times [0, \infty[$ together with the initial conditions

$$\left.\begin{array}{c} u_i(x,0) = d_i(x), \;\; \dot{u}_i(x,0) = v_i(x) \\ \theta(x,0) = \theta_1(x), \;\dot{\theta}(x,0) = \theta_2(x), \; q_i(x,0) = q_{i0}(x) \end{array}\right\} \text{on } \; V \tag{4.8}$$

and the boundary conditions

$$\left.\begin{aligned} u_i = \tilde{u}_i(x,t) \text{ on } \ A_1 \times [0,\infty[\\ \sigma_i = \sigma_{ij} n_j = \tilde{\sigma}_i(x,t) \text{ on } \ A_2 \times [0,\infty[\\ q = q_i n_i = \tilde{q}(x,t) \ \text{ on } \ A_3 \times [0,\infty[\\ \theta = \tilde{\theta}(x,t) \ \text{ on } \ A_4 \times [0,\infty[\end{aligned}\right\} \tag{4.9}$$

Here d_i, v_i, θ_1, $\theta_{2,}\, q_{i0}$ are the prescribed initial displacement component, velocity component, temperature, rate of temperature and heat flux, respectively. $\tilde{u}_i$, $\tilde{\sigma}_i$, $\tilde{\theta}$, $\tilde{q}$ denote the known surface displacement component, component of traction vector, temperature and normal heat flux, respectively. The smoothness requirements and other regularity assumptions on the ascribable functions are also introduced as hypotheses on data. We further assume that d_i, v_i, θ_1, θ_2, q_{i0} are continuous on $\bar{V}$, h_i and ϖ are continuously differentiable on $\bar{V} \times [0,\infty[$. $\tilde{q}$ and $\tilde{\sigma}_i$ are piecewise continuous on $A_3 \times [0,\infty[$ and $A_2 \times [0,\infty[$, respectively. $\tilde{u}_i$ and $\tilde{\theta}$ are continuous on $A_1 \times [0,\infty[$ and $A_4 \times [0,\infty[$, respectively.

We also assume that the C_{ijkl}, β_{ij}, k_{ij} and k^*_{ij} are smooth on $\bar{V}$ and satisfy

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij},\ \beta_{ij} = \beta_{ji},\, k_{ij} = k_{ji},\, k^*_{ij} = k^*_{ji} \tag{4.10}$$

$$C_{ijkl} e_{ij} e_{kl} > 0, \text{ for all } e_{ij} \text{ on } \bar{V} \times [0,\infty[\tag{4.11}$$

$$k_{ij} a_i a_j > 0,\ k^*_{ij} a_i a_j > 0 \text{ for any real } a_i \text{ on } \bar{V} \times [0,\infty[\tag{4.12}$$

The material constants and delay time parameter satisfy the following inequalities:

$$\rho > 0,\ c_E > 0,\ \theta_0 > 0,\ \tau > 0 \quad \text{on} \ \ V \tag{4.13}$$

Now, we represent an admissible state as $R = \{u_i, \theta, \gamma_i, e_{ij}, \sigma_{ij}, q_i, S\}$, which is an ordered array of functions u_i, θ, γ_i, e_{ij}, σ_{ij}, q_i, S defined on $\bar{V} \times [0,\infty[$ with the properties that $u_i \in C^{2,2}$, $\theta \in C^{1,2}$, $\gamma_i \in C^{0,2}$, $\sigma_{ij} \in C^{1,0}$, $q_i \in C^{1,1}$, $S \in C^{0,1}$ and

$e_{ij} = e_{ji},\ \sigma_{ij} = \sigma_{ji}$ on $\bar{V} \times [0, \infty[$. We define the addition of two admissible states and multiplication of an admissible state with a scalar as follows:

$R + R^{'} = \{u_i + u_i,\ \theta + \theta^{'}, ...,\ S + S^{'}\}$,

$\lambda^* R^{'} = \{\lambda^* u_i,\ \lambda^* \theta, ...,\ \lambda^* S\}$, where λ^* is any scalar. Therefore, the set of all admissible states is clearly a linear space.

We say that an admissible state is the solution of the present mixed problem if it satisfies all the field equations (4.1)-(4.7), the initial conditions (4.8) and the boundary conditions (4.9).

4.4 Uniqueness of Solution

In this section, we will establish the uniqueness theorem in the present context. For this, the specific internal energy for the present initial-boundary value problem can be taken in the form

$$E = \frac{1}{2} C_{ijkl} \hat{e}_{kl} \hat{e}_{ij} + \frac{\rho c_E}{2\theta_0} \hat{\theta}^2 \tag{4.14}$$

where, for any function f, $\hat{f}$ is defined as

$$\hat{f} = (\frac{\partial}{\partial t}) f \tag{4.15}$$

From equations (4.11) and (4.13), it is clear that the equation (4.14) is positive definite and we get from this equation

$$\dot{E} = \hat{\sigma}_{ij} \dot{\hat{e}}_{ij} + \rho \dot{\hat{S}} \hat{\theta} \tag{4.16}$$

Now, by using relations (4.2), (4.5), (4.6) and (4.7), we obtain

$$\begin{aligned}
\dot{E} &= \hat{\sigma}_{ij}\dot{\hat{u}}_{i,j} - \frac{1}{\theta_0}\hat{q}_{i,i}\hat{\theta} + \frac{\rho\hat{\varpi}}{\theta_0}\hat{\theta} \\
&= (\hat{\sigma}_{ij}\dot{\hat{u}}_i)_{,j} - \hat{\sigma}_{ij,j}\dot{\hat{u}}_i - \frac{1}{\theta_0}(\hat{q}_i\hat{\theta})_{,i} + \frac{1}{\theta_0}(\hat{q}_i\gamma_i) + \frac{\rho\hat{\varpi}}{\theta_0}\hat{\theta} \\
&= (\hat{\sigma}_{ij}\dot{\hat{u}}_i)_{,j} - \frac{1}{\theta_0}(\hat{q}_i\hat{\theta})_{,i} + \rho h_i\dot{\hat{u}}_i + \frac{\rho\hat{\varpi}}{\theta_0}\hat{\theta} - \rho\ddot{\hat{u}}_i\dot{\hat{u}}_i \\
&\quad - \frac{\hat{\gamma}_i}{\theta_0}\{k_{ij}\dot{\gamma}_j + k^*_{ij}\gamma_j + \tau k^*_{ij}\dot{\gamma}_j + \frac{\tau^2}{2}k^*_{ij}\ddot{\gamma}_j\}
\end{aligned} \tag{4.17}$$

Therefore, integrating both sides of equation (4.17) over V, using divergence theorem and by using (4.1), we get

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_V (E + \frac{\rho}{2}\dot{\hat{u}}_i\dot{\hat{u}}_i + \frac{k^*_{ij}}{2\theta_0}\gamma_i\gamma_j + \frac{\tau^2 k^*_{ij}}{4\theta_0}\dot{\gamma}_i\dot{\gamma}_j)dV + \frac{1}{\theta_0}\int_V (k_{ij} + \tau k^*_{ij})\dot{\gamma}_i\dot{\gamma}_j dV \\
&\qquad = \int_V (\rho\hat{h}_i\dot{\hat{u}}_i + \frac{\rho\hat{\varpi}\hat{\theta}}{\theta_0})dV + \int_A (\hat{\sigma}_i\dot{\hat{u}}_i - \frac{1}{\theta_0}\hat{\theta}\hat{q})dA
\end{aligned} \tag{4.18}$$

Now, we will establish the uniqueness of solution of the present mixed initial-boundary value problem by the following uniqueness theorem.

Theorem -4.1 (Uniqueness theorem):

The mixed initial-boundary value problem given by equations (4.1)-(4.7), which satisfies the initial conditions (4.8) and boundary conditions (4.9) has at most one solution.

Proof : We assume that for this mixed initial-boundary value problem, we have two sets of solutions u_i^α, θ^α, e_{ij}^α, σ_{ij}^α, q_i^α, S^α for $\alpha = 1, 2$. Then, we construct difference between these two sets of functions as

$$\tilde{u}_i = u_i^{(1)} - u_i^{(2)},\ \tilde{\theta} = \theta^{(1)} - \theta^{(2)}, \ldots\ldots\ldots\ldots\ldots\tilde{S} = S^{(1)} - S^{(2)} \tag{4.19}$$

As we know that the set of all admissible states is a linear space, so the difference functions defined by (4.19) also satisfy the equations (4.1)-(4.7) with zero body forces and heat source, the initial conditions (4.8) and the boundary conditions (4.9) in their homogeneous form and hence satisfy equation (4.18) too. Therefore, we obtain from equation (4.18)

$$\frac{\partial}{\partial t}\int\limits_V (\tilde{E} + \frac{\rho}{2}\dot{\tilde{u}}_i\dot{\tilde{u}}_i + \frac{k^*_{ij}}{2\theta_0}\tilde{\gamma}_i\tilde{\gamma}_j + \frac{\tau^2 k^*_{ij}}{4\theta_0}\dot{\tilde{\gamma}}_i\dot{\tilde{\gamma}}_j)dV + \frac{1}{\theta_0}\int\limits_V (k_{ij} + \tau k^*_{ij})\dot{\tilde{\gamma}}_i\dot{\tilde{\gamma}}_i dV = 0 \qquad (4.20)$$

Interchanging the variable t with η and integrating above equation over time interval $(0,t)$ and using the homogeneous initial conditions for difference functions, we obtain

$$\int\limits_V (\tilde{E} + \frac{\rho}{2}\dot{\tilde{u}}_i\dot{\tilde{u}}_i + \frac{k^*_{ij}}{2\theta_0}\tilde{\gamma}_i\tilde{\gamma}_j + \frac{\tau^2 k^*_{ij}}{4\theta_0}\dot{\tilde{\gamma}}_i\dot{\tilde{\gamma}}_j)dV + \frac{1}{\theta_0}\int\limits_0^t\int\limits_V (k_{ij} + \tau k^*_{ij})\dot{\tilde{\gamma}}_i\dot{\tilde{\gamma}}_i dV d\eta = 0 \qquad (4.21)$$

From equations (4.11), (4.12) and (4.13), we observe that the component in each term present on the left hand side of equation (4.21) is non-negative. Thus, we conclude that each term in equation (4.21) must be zero which implies that

$$\dot{\tilde{u}}_i = 0, \quad \dot{\tilde{\theta}} = 0 \quad \text{on } \bar{V} \times [0, \infty[\qquad (4.22)$$

In view of the initial conditions $\tilde{u}_i(x,0) = 0$ and $\tilde{\theta}(x,0) = 0$, for $\tilde{u}_i$ and $\tilde{\theta}$, respectively, we get from equation (4.22) that

$$\tilde{u}_i = 0, \quad \tilde{\theta} = 0 \quad \text{on } \bar{V} \times [0, \infty[$$

i.e.

$$u_i^{(1)} = u_i^{(2)}, \quad \theta^{(1)} = \theta^{(2)} \quad \text{on } \bar{V} \times [0, \infty[$$

This completes the proof of the uniqueness theorem.

4.5 Alternative Formulation of Mixed Problem

In this section, we formulate an alternative characterization of the above mixed initial-boundary value problem in which the initial conditions are combined into the field equations (Gurtin (1964)). For this, we proceed as follows:

Let ϕ and ψ be two functions defined on $\bar{V} \times [0, \infty[$ such that both are continuous on $[0,\infty[$ for each $x \in V$. Then the convolution $\phi * \psi$ of ϕ and ψ is defined as

$$[\phi * \psi](x, t) = \int_0^t \phi(x, t - \tau)\psi(x, \tau)d\tau, \qquad (x, t) \in \bar{V} \times [0, \infty[$$

We will use the following convolution properties:

$$(i)\ \psi{*}\phi = \phi{*}\psi \tag{4.23}$$

$$(ii)\ \psi{*}(\phi{*}\xi) = (\psi{*}\phi){*}\xi = \psi{*}\phi{*}\xi \tag{4.24}$$

$$(iii)\ \ \psi{*}(\phi{+}\xi) = (\psi{*}\phi){+}(\psi{*}\xi) \tag{4.25}$$

$$(iv)\ \ \phi{*}\psi = 0 \Rightarrow \phi = 0 \text{ or } \psi = 0 \tag{4.26}$$

Now, we introduce the functions g and l defined on $[0,\infty[$ as

$$g(t) = t,\ \ l(t) = 1 \tag{4.27}$$

Let f_i and W be the functions defined on $\bar{V} \times [0, \infty[$ as

$$f_i = g * \rho h_i + \rho(tv_i + d_i) \tag{4.28}$$

$$W = l * \frac{\rho\varpi}{\theta_0} + \rho S_0 \tag{4.29}$$

where

$$\rho S_0 = \rho c_E \frac{\theta_1}{\theta_0} + \beta_{ij} d_{i,j} \tag{4.30}$$

and let

$$N_i = l * (tq_{i0} + t\theta_{1,j}k_{ij} + t\tau\theta_{1,j}k^*_{ij} + t\theta_{2,j}\frac{\tau^2}{2}k^*_{ij} + \theta_{1,j}\frac{\tau^2}{2}k^*_{ij}) \tag{4.31}$$

Let $p(x,t)$ and $\dot{p}(x,t)$ be two functions of space and time defined on $\bar{V} \times [0, \infty[$ such that both are continuous and differentiable on [0,∞[. Then following relations hold evidently:

$$g * \ddot{p}(x,t) = p(x,t) - [t\dot{p}(x,0) + p(x,0)] \tag{4.32}$$

$$l * \dot{p}(x,t) = p(x,t) - p(x,0) \tag{4.33}$$

$$g * \dot{p}(x,t) = l * (l * \dot{p}(x,t)) = l * [p(x,t) - p(x,0)] = l * p(x,t) - tp(x,0) \tag{4.34}$$

By this formulation, we obtain the following result:

Theorem-4.2:

The functions u_i, θ, γ_i, e_{ij}, σ_{ij}, q_i, S satisfy equations (4.1), (4.2) and (4.5) and the initial conditions (4.8) if and only if

$$g * \sigma_{ij,j} + f_i = \rho u_i \tag{4.35}$$

$$\rho S = -l * \frac{q_{i,i}}{\theta_0} + W \tag{4.36}$$

$$M_1 * q_i = -M_1 * k_{ij}\gamma_j - M_2 * k^*_{ij}\gamma_j + N_i \tag{4.37}$$

where, $M_1 = l * l$ and $M_2 = l * (g + l * \tau + \frac{\tau^2}{2})$, f_i, W and N_i are given by equations (4.28), (4.29), and (4.31) respectively.

Proof :We first consider that the basic governing equations (4.1), (4.2) and (4.5) and initial conditions (4.8) hold good. Then by taking convolution of equation (4.1) with g and using the results from equations (4.32) and (4.8), we get the equation (4.35). Similarly, by taking the convolution of the equation (4.2) with l and using (4.33) and (4.8) we obtain equation (4.36). Again by taking convolution of equation (4.5) with $l * g$, and using the relation from (4.32), (4.34) and (4.8) we arrive at equation (4.37).

Similarly, we can prove the converse of the above theorem, by reverse arguments. This theorem gives an alternative characterization of the solution to the mixed problem in which the initial conditions are incorporated into basic field equations and hence, finally we get the following theorem.

Theorem-4.3:

Let $R = \{u_i, \theta, \gamma_i, e_{ij}, \sigma_{ij}, q_i, S\}$ be an admissible state. Then R is a solution of the mixed problem if and only if it satisfies the equations (4.35)-(4.37), (4.3), (4.4), (4.6), (4.7) and the boundary conditions (4.9).

4.6 Variational Theorem

The term functional identifies a real valued function whose domain is a subset of linear space. Let X be a linear space and Y be the subspace of X. We consider a functional

$\varOmega$ defined on Y.

Let for

$$y,\, y^{'} \in Y, \quad y + \lambda^{'} y^{'} \in Y, \ \text{ for } \ \text{all real } \lambda^{'} \tag{4.38}$$

and we define the variation of the functional $\varOmega\{y\}$ as

$$\delta_{y'} \varOmega\{y\} = \frac{d}{d\lambda'} \varOmega\{y + \lambda^{'} y^{'}\} \mid_{\lambda'=0} \tag{4.39}$$

The variation of $\varOmega\{.\}$ is said to be zero at y over Y and it is written as

$$\delta \varOmega\{y\} = 0, \quad \text{over } Y \tag{4.40}$$

if and only if $\delta_{y'} \varOmega\{y\}$ exists and is equal to zero for all $y^{'}$consistent with (4.38).

Now, we will establish a variational principle on linear theory of thermoelasticity for anisotropic and inhomogeneous medium under the present heat conduction model given by Quintanilla (2011) on the basis of the alternative formulation and the theorem established in the previous section.

Theorem-4.4 (Variational theorem):

Let $\varXi$ be the linear space of all admissible states endowed with addition and scalar multiplication as described in Section-4.3. If for each $t \in [0, \infty[$ and for every $\Gamma = \{u_i,\, \theta,\, \gamma_j,\, e_{ij},\, \sigma_{ij},\, q_i,\, S\} \in \varXi$, we define the functional $\varOmega_t\{\Gamma\}$ on Ξ by

$$\begin{aligned}\Omega_t\{\Gamma\} = \int_V \Big[&\frac{1}{2}M_1 * g * C_{ijkl}e_{kl} * e_{ij} - \frac{1}{2}M_1 * \rho u_i * u_i - M_1 * g * \sigma_{ij} * e_{ij} \\ &- M_1 * g * l * \frac{1}{\theta_0}q_i * \gamma_i + M_1 * u_i * (\rho u_i - g * \sigma_{ij,j} - f_i) - M_1 * g * \theta * (\rho S + l * \frac{q_{i,i}}{\theta_0} - W) \\ &+ g * l * \frac{1}{\theta_0}(-M_1 * \frac{1}{2}k_{ij}\gamma_j - M_2 * \frac{1}{2}k^*_{ij}\gamma_j + N_i) * \gamma_i \\ &+ \frac{\theta_0}{2\rho c_E}M_2 * g * (\rho S - \beta_{rs}e_{rs}) * (\rho S - \beta_{ij}e_{ij})\Big]dV + \int_{A_1} M_1 * g * \tilde{u_i} * \sigma_i dA \\ &+ \int_{A_2} M_1 * g * (\sigma_i - \tilde{\sigma_i}) * u_i dA + \frac{1}{\theta_0}\int_{A_3} M_1 * g * l * q * \tilde{\theta} dA \\ &+ \frac{1}{\theta_0}\int_{A_4} M_1 * g * l * (q - \tilde{q}) * \theta dA \end{aligned} \tag{4.41}$$

$$\text{then, } \delta\Omega_t\{\Gamma\} = 0, \quad t \in [0, \infty[\tag{4.42}$$

if and only if Γ is a solution of the mixed initial-boundary value problem given by equations (4.1)-(4.7) with the initial conditions (4.8) and the boundary conditions (4.9).

Proof: Let $\Gamma' = \{u_i', \theta', \gamma_i', e_{ij}', \sigma_{ij}', q_i', S'\} \in \Xi$, which implies that $\Gamma + \lambda'\Gamma' \in \Xi$, for every real λ'. Then equation (4.41) together with (4.23)-(4.26) and (4.39), along with the divergence theorem implies

$$\begin{aligned}\delta_{\Gamma'}\Omega_t\{\Gamma\} = \int_V &[M_1 * g * \{C_{ijkl}e_{kl} - \frac{\theta_0}{\rho c_E}(\rho S - \beta_{rs}e_{rs}) - \sigma_{ij}\} * e_{ij}' \\ &+ M_2 * g * \{\frac{\theta_0}{\rho c_E}(\rho S - \beta_{rs}e_{rs}) - \theta\} * \rho S' \\ &+ g * l * \frac{1}{\theta_0}(-M_1 * \frac{1}{2}k_{ij}\gamma_j - M_2 * \frac{1}{2}k^*_{ij}\gamma_j + N_i - M_1 * q_i) * \gamma_i']dV \\ &- \int_V [M_1 * (g * \sigma_{ij,j} + f_i - \rho u_i) * u_i' + M_1 * g * (\rho S + l * \frac{q_{i,i}}{T_0} - W) * \theta']dV \\ &- \int_V [M_1 * g * (e_{ij} - u_{(i,j)}) * \sigma_{ij}' - M_1 * g * l * \frac{1}{\theta_0}(\theta_{,i} - \gamma_i) * q_i']dV \\ &+ \int_{A_1} M_1 * g * (\tilde{u_i} - u_i) * \sigma_i' dA + \int_{A_2} M_1 * g * (\sigma_i - \tilde{\sigma_i}) * u_i' dA \\ &+ \frac{1}{\theta_0}\int_{A_3} M_1 * g * l * (\tilde{\theta} - \theta) * q' dA + \frac{1}{\theta_0}\int_{A_4} M_1 * g * l * (q - \tilde{q}) * \theta' dA \end{aligned} \tag{4.43}$$

for all $t \in [0, \infty[$.

Firstly, assume that Γ is a solution of the mixed initial-boundary value problem, then according to the theorem-43, the relations (4.35) to (4.37) and the boundary conditions (4.9) yield

$$\delta_{\Gamma'} \Omega_t\{\Gamma\} = 0,\ t \in [0, \infty[\tag{4.44}$$

for every $\Gamma' = \{u_i', \theta', \gamma_i', e_{ij}', \sigma_{ij}', q_i', S'\} \in \Xi$, and therefore we find that (4.42) holds. This completes the proof of the if-part of the theorem-4.4.

Conversely, for "only if" part, let (4.42) holds true and hence (4.44) holds for every $\Gamma' = \{u_i', \theta', \gamma_i', e_{ij}', \sigma_{ij}', q_i', S'\} \in \Xi$. Then, we have to prove that Γ is a solution of the present mixed initial-boundary value problem.

Now, since (4.44) holds for every $\Gamma' \in \Xi$, we choose $\Gamma' = \{u_i', 0, 0, 0, 0, 0, 0\}$ and let u_i', together with all the space derivatives, vanish on $A \times [0, \infty[$. Therefore, we obtain from equations (4.43) and (4.44)

$$\int_V (g * \sigma_{ij,j} + f_i - \rho u_i) * u_i' dV = 0 \qquad \text{for } t \in [0, \infty[\tag{4.45}$$

Now, by using Lemma-1 (Gurtin (1964)) and convolution properties, we find that equation (4.35) holds.

Similarly, by making suitable choices of Γ'into (4.43), we can prove with the help of three Lemmas (1-3) (Gurtin (1964)) that Γ also satisfies the equations (4.36), (4.37), (4.3), (4.4), (4.6), (4.7) and the boundary conditions (4.9). Therefore, it can be concluded from theorem-4.3 that Γ is the solution of the present mixed problem. The proof of the above theorem is therefore complete.

4.7 Reciprocity Theorem

In this section, the reciprocity theorem of convolution type (see Iesan (1974)) is being derived in this regard. We consider two different systems of thermoelastic loadings

$$L^{\beta} = (h_i^{(\beta)},\, \varpi^{(\beta)},\, \tilde{u_i}^{(\beta)},\, \tilde{\theta}^{(\beta)},\, \tilde{q}_i^{(\beta)},\, \tilde{\sigma}_i^{(\beta)},\, d_i^{(\beta)},\, v_i^{(\beta)},\, \theta_1^{(\beta)},\, \theta_2^{(\beta)},\, q_{i0}^{(\beta)}),\;\; \beta = 1,\, 2 \tag{4.46}$$

and the corresponding thermoelastic configurations

$$I^{\beta} = (u_i^{(\beta)},\, \theta^{(\beta)}) \tag{4.47}$$

that satisfy (4.35)-(4.37), (4.3), (4.4), (4.6), (4.7) and (4.9).

The reciprocity theorem states the relation between these two sets of thermoelastic loadings and thermoelastic configurations. We use the following notations:

$$f_i^{(\beta)} = \rho(g * h_i^{(\beta)} + tv_i^{(\beta)} + d_i^{(\beta)}) \tag{4.48}$$

$$W^{(\beta)} = l * \frac{\rho\varpi^{(\beta)}}{\theta_0} + \rho S_0^{(\beta)} \tag{4.49}$$

$$\rho S_0^{(\beta)} = \rho c_E \frac{\theta_1^{(\beta)}}{\theta_0} + \beta_{ij} d_{i,j}^{(\beta)} \tag{4.50}$$

$$N_i^{(\beta)} = l * (tq_{i0}^{(\beta)} + tk_{ij}\theta_{1,j}^{\beta} + t\tau k_{ij}^{*}\theta_{1,j}^{\beta} + t\frac{\tau^2}{2}k_{ij}^{*}\theta_{2,j}^{\beta} + \frac{\tau^2}{2}k_{ij}^{*}\theta_{1,j}^{\beta}) \tag{4.51}$$

for $\beta = 1, 2$.

Theorem -4.5 (Reciprocity theorem):

If a thermoelastic solid is subjected to two different systems of thermoelastic loadings, L^{β}, $(\beta = 1,\, 2)$ and I^{β}, $(\beta = 1,\, 2)$ are the corresponding thermoelastic configurations, then the following reciprocity relation holds:

$$\int_V M_1 * [f_i^{(1)} * u_i^{(2)} - g * W^{(1)} * \theta^{(2)}]dV + \int_A M_1 * g * \left[\sigma_i^{(1)} * u_i^{(2)} + \frac{1}{\theta_0} l * q^{(1)} * \theta^{(2)}\right] dA$$
$$- \int_V g * l * \left[\frac{1}{\theta_0} N_i^{(1)} * \gamma_i^{(2)}\right] dV = \int_V M_1 * \left[f_i^{(2)} * u_i^{(1)} - g * W^{(2)} * \theta^{(1)}\right] dV$$
$$+ \int_A M_1 * g * \left[\sigma_i^{(2)} * u_i^{(1)} + \frac{1}{\theta_0} l * q^{(2)} * \theta^{(1)}\right] dA - \int_V g * l * \left[\frac{1}{\theta_0} N_i^{(2)} * \gamma_i^{(1)}\right] dV \tag{4.52}$$

where, $f_i^{(\beta)},\ W^{(\beta)},\ N_i^{(\beta)}\ \ (\beta = 1,\, 2)$ associated with two systems are given by equations (4.48), (4.49), (4.51) respectively.

Proof: From equation (4.3), we have

$$\sigma_{ij} = C_{ijkl}^{(\beta)} e_{kl}^{(\beta)} - \beta_{ij}\theta^{(\beta)} \tag{4.53}$$

Now, taking convolution of equation (4.53) for $\beta = 1$ with $e_{ij}^{(2)}$ and for $\beta = 2$ with $e_{ij}^{(1)}$ and then subtracting the results, we get

$$(\sigma_{ij}^{(1)} + \beta_{ij}\theta^{(1)}) * e_{ij}^{(2)} = (\sigma_{ij}^{(2)} + \beta_{ij}\theta^{(2)}) * e_{ij}^{(1)} + C_{ijkl}(e_{kl}^{(1)} * e_{ij}^{(2)} - e_{kl}^{(2)} * e_{ij}^{(1)})$$

Hence, due to the symmetry properties of C_{ijkl}, we find

$$(\sigma_{ij}^{(1)} + \beta_{ij}\theta^{(1)}) * e_{ij}^{(2)} = (\sigma_{ij}^{(2)} + \beta_{ij}\theta^{(2)}) * e_{ij}^{(1)} \tag{4.54}$$

Again from equation (4.4), we can write

$$\rho S^{(\beta)} - \beta_{ij} e_{ij}^{(\beta)} = \rho c_E \frac{\theta^{(\beta)}}{\theta_0}, \qquad \beta = 1,\, 2 \tag{4.55}$$

Taking convolution of equation (4.55) for $\beta = 1$ with $\theta^{(2)}$ and for $\beta = 2$ with $\theta^{(1)}$ and subtracting, we get

$$(\rho S^{(1)} - \beta_{ij}e_{ij}^{(1)}) * \theta^{(2)} = (\rho S^{(2)} - \beta_{ij}e_{ij}^{(2)}) * \theta^{(1)} \tag{4.56}$$

Equations (4.54) and (4.56) yield

$$(\sigma_{ij}^{(1)} * e_{ij}^{(2)} - \rho S^{(1)} * \theta^{(2)}) = (\sigma_{ij}^{(2)} * e_{ij}^{(1)} - \rho S^{(2)} * \theta^{(1)}) \tag{4.57}$$

Now, we introduce the notation

$$L_{\alpha\beta} = \int\limits_V M_1 * g * \left[\sigma_{ij}^{(\alpha)} * e_{ij}^{(\beta)} - \rho S^{(\alpha)} * \theta^{(\beta)}\right] dV, \quad \alpha, \beta = 1,\, 2 \tag{4.58}$$

Now, from equations (4.7) and (4.35)-(4.37), we get

$$\begin{aligned}
M_1 * g * (\sigma_{ij}^{(\alpha)} * e_{ij}^{(\beta)} - \rho S^{(\alpha)} * \theta^{(\beta)}) &= M_1 * g * \sigma_{ij}^{(\alpha)} * u_{i,j}^{\beta} - M_1 * g * (-l * \frac{q_{i,i}^{(\alpha)}}{\theta_0} + W^{(\alpha)}) * \theta^{(\beta)} \\
&= M_1 * g * (\sigma_{ij}^{(\alpha)} * u_i^{\beta})_{,j} - M_1 * g * (\sigma_{ij,j}^{(\alpha)} * u_i^{\beta}) \\
&+ \frac{1}{\theta_0} M_1 * g * (l * q_i^{(\alpha)} * \theta^{(\beta)})_{,i} - \frac{1}{\theta_0} M_1 * g * l * q_i^{(\alpha)} * \gamma_i^{(\beta)} \\
&- M_1 * g * W^{(\alpha)} * \theta^{(\beta)} \\
&= M_1 * g * (\sigma_{ij}^{(\alpha)} * u_i^{\beta})_{,j} - M_1 * \rho u_i^{(\alpha)} * u_i^{(\beta)} + M_1 * f_i^{(\alpha)} * u_i^{(\beta)} \\
&+ \frac{1}{\theta_0} M_1 * g * l * (q_i^{(\alpha)} * \theta^{(\beta)})_{,i} + \frac{1}{\theta_0} g * l * (M_1 * k_{ij}\gamma_j^{(\alpha)} \\
&+ M_2 * k_{ij}^{*}\gamma_j^{(\alpha)}) * \gamma_i^{(\beta)} - \frac{1}{\theta_0} g * l * N_i^{(\alpha)} * \gamma_i^{(\beta)} \\
&- M_1 * g * W^{(\alpha)} * \theta^{(\beta)}
\end{aligned} \tag{4.59}$$

From equations (4.58) and (4.59), we therefore obtain

$$\begin{aligned}
L_{\alpha\beta} = &\int\limits_V M_1 * \left[f_i^{(\alpha)} * u_i^{(\beta)} - g * W^{(\alpha)} * \theta^{(\beta)}\right] dV + \int\limits_A M_1 * g * \left[\sigma_i^{(\alpha)} * u_i^{\beta} + \frac{1}{\theta_0} l * q_i^{(\alpha)} * \theta^{(\beta)}\right] dA \\
&- \int\limits_V \left[M_1 * \rho u_i^{(\alpha)} * u_i^{(\beta)} - \frac{1}{\theta_0} g * l * M_1 * k_{ij}\gamma_j^{(\alpha)} * \gamma_i^{(\beta)} - \frac{1}{\theta_0} g * l * M_2 * k_{ij}^{*}\gamma_j^{(\alpha)} * \gamma_i^{(\beta)}\right] dV \\
&- \int\limits_V \left[\frac{1}{\theta_0} g * l * N_i^{(\alpha)} * \gamma_i^{(\beta)}\right] dV
\end{aligned} \tag{4.60}$$

Further, from equations (4.58) and (4.60) we have

$$L_{12} = L_{21} \tag{4.61}$$

Hence, we find that equations (4.60) and (4.61) yield the reciprocity relation (4.52), which completes the proof of the theorem-4.5.