CHAPTER 6

FUNDAMENTAL SOLUTIONS OF THERMOELASTICITY WITH A RECENT HEAT CONDUCTION MODEL WITH A SINGLE DELAY TERM

6.1 Introduction

Fundamental solutions or Green's functions for a boundary value problem play an important role in both applied and theoretical studies on the physics of solids. They form basic building blocks of many further works. For example, fundamental solutions are used to construct many analytical solutions of practical problems when boundary conditions are imposed. They are essential in the boundary element method as well as the study of cracks, defects and inclusions. Specially, fundamental solutions in elastodynamic theory are important in deriving the solutions of problems involving impulse responses. In the classical theory of thermodynamics, representations of solutions were presented by Nowacki (1964). The fundamental solution in the classical theory of coupled thermoelasticity was first studied by Hetnarski (1964a, 1964b). Ciarletta (1991) proposed the representation theorem of the Galerkin type in the theory of thermoelastic materials with voids. Later on, Iesan (1998) developed the fundamental solutions on theory of thermoelasticity without energy dissipation and Kothari *et al.*(2010) established fundamental solution of generalised thermoelasticity with three phase-lags.

The present work is concerned with the thermoelasticity theory based on a very recently

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proposed heat conduction model: a heat conduction model with a delay term introduced by Quintanilla (2011). Here, we aim to obtain the fundamental solutions of this theory. We derive the solution of Galerkin-type field equations for the case of homogeneous and isotropic bodies. With the help of this solution, we determine the effects of concentrated heat sources and body forces in an unbounded medium. We further obtain the fundamental solutions of the field equations in case of steady vibrations.

6.2 Basic Equations

We consider a body that at time t = t_0 occupies the regular region B of the three-dimensional Euclidean space that has piece wise smooth surface boundary ∂B. We refer motion of the continuum to a fixed system of rectangular Cartesian axes $Ox_i (i = 1, 2, 3)$.

By following Quintanilla (2011) and Leseduarte and Quintanilla (2013), we consider the following field equations under linear theory of thermoelasticity for a homogeneous and isotropic material :

$$\mu\Delta\vec{u} + (\lambda+\mu)\text{grad div}\vec{u} - \beta\text{grad}\theta + \rho\vec{b} = \rho\ddot{\vec{u}} \tag{6.1}$$

$$\left\{k\frac{\partial}{\partial t} + k^*(1+\tau\frac{\partial}{\partial t}+\frac{1}{2}\tau^2\frac{\partial^2}{\partial t^2})\right\}\Delta\theta = \beta\theta_0\text{div}\ddot{\vec{u}} + \rho c_E\ddot{\theta} - w \tag{6.2}$$

on $B \times (0, t_1)$.

Here $w = \rho\dot{r}$ is the heat source and Δ is Laplacian operator.

We suppose that

$$3\lambda + 2\mu > 0,\ \mu > 0,\ k > 0,\ k^* >,\ \theta_0 > 0 \tag{6.3}$$

Now, we introduce the notations

$$c_1 = \sqrt{\frac{\lambda + 2\mu}{\rho}},\ c_2 = \sqrt{\frac{\mu}{\rho}},\ c_3 = \sqrt{\frac{k^*}{\rho c_E}},\ a = \frac{k}{k^*},$$
$$\epsilon = \frac{m^2\theta_0}{c_1^2\rho^2 c_E},\ b = \frac{\theta_0}{k^*},\ c_1^2 - c_2^2 = c_0,\ \kappa = \frac{\rho c_E}{k^*} \tag{6.4}$$

Therefore, equations (6.1) and (6.2) can be written in the form

$$c_2^2\mathfrak{R}_2\vec{u} + c_0\text{grad div}\vec{u} - \frac{\beta}{\rho}\text{grad}\theta = -\vec{f} \tag{6.5}$$

$$\mathfrak{R}\theta - \beta b\text{div}\ddot{\vec{u}} = -\frac{w}{k^*} \tag{6.6}$$

where, we have introduced the operators

$\mathfrak{R}_i$=Δ-$\frac{1}{c_1^2}\frac{\partial^2}{\partial t^2}$, $(i = 1, 2, 3)$

$\mathfrak{R}$=$\mathfrak{R}_3$+$(a + \tau)\Delta\frac{\partial}{\partial t}$+$\frac{1}{2}\tau^2\Delta\frac{\partial^2}{\partial t^2}$.

Hence, we establish the following theorem that gives a solution of Glerkin type of the field equations given by (6.5) and (6.6):

Theorem − 1 : Let

$$\vec{u} = c_1^2\left(\mathfrak{R}_1\mathfrak{R} - \epsilon\kappa\Delta\frac{\partial^2}{\partial t^2}\right)\vec{F} - \left(c_o\mathfrak{R} - \epsilon\kappa c_1^2\frac{\partial^2}{\partial t^2}\right)\text{grad div}\vec{F} + \frac{\beta}{\rho}\text{grad}G \tag{6.7}$$

$$\theta = b\beta c_2^2\mathfrak{R}_2\frac{\partial^2}{\partial t^2}\text{div}\vec{F} + c_1^2\mathfrak{R}_1 G \tag{6.8}$$

where the fields F_i of class C^6 and G of class C^4 on $B \times (0, t_1)$ satisfy

$$\mathfrak{R}_2\left[\mathfrak{R}_1\mathfrak{R} - \epsilon\kappa\Delta\frac{\partial^2}{\partial t^2}\right]F_i = -\frac{1}{c_1^2 c_2^2}f_i \tag{6.9}$$

$$\left[\mathfrak{R}_1\mathfrak{R} - \epsilon\kappa\Delta\frac{\partial^2}{\partial t^2}\right]G = -\frac{1}{c_1^2 k^*}w \tag{6.10}$$

Then $\vec{u}$ and θ satisfy equations (6.5) and (6.6).

Proof: We find that the operators $\mathfrak{R}_1$, $\mathfrak{R}_2$ satisfy the identity

$$c_1^2\mathfrak{R}_1 - c_2^2\mathfrak{R}_2 = c_o\Delta \tag{6.11}$$

Hence, above theorem can be proved in an analogous way by following the proof of the corresponding theorem of the classical theory (see Carlson (1972)) and thermoelasticity theory without energy dissipation given by Iesan (1998).

6.3 Fundamental Solutions

In this section, we shall use the preceding theorem-1 to determine the fundamental solution of the field equations. We consider a problem of thermoelastic body that occupies the three dimensional space. Let $x = (x_1, x_2, x_3)$ be the any point of this three dimensional space. We assume the initial conditions as follows:

$$\begin{aligned} \vec{u}(\vec{x},0) = 0,\ \dot{\vec{u}}(\vec{x},0) = 0 \\ \vec{\theta}(\vec{x},0) = 0,\ \dot{\vec{\theta}}(\vec{x},0) = 0 \end{aligned} \tag{6.12}$$

We further assume the conditions at infinity as

$$\left.\begin{aligned} u_j \to 0,\ u_{j,k} \to 0 \\ \theta \to 0,\ \theta_{,j} \to 0 \end{aligned}\right. \text{ for } |\vec{x}| \to \infty \tag{6.13}$$

The initial conditions for the function F_i and G are defined as

$$\frac{\partial^p F_i}{\partial t^p}(\vec{x},0) = 0,\ \frac{\partial^q G}{\partial t^q}(\vec{x},0) = 0,\ \vec{x} \in B\,(p = 1, 2, ...5;\ q = 1, 2, ...3) \tag{6.14}$$

It follows from equation (6.7) and (6.8) that the condition (6.14) imply the initial conditions (6.12).

We use Laplace Transform of a function $f(t)$ for the solution of the problem that is defined as

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)dt,$$

where s is the Laplace transform parameter.

We use the following notations in the preceding expressions.

$\mathfrak{R}_i' = \Delta - \frac{s^2}{c_i^2}$, $(i = 1, 2, ...3)$

$\mathfrak{R}' = \mathfrak{R}_3' + (a+\tau)\Delta s + \frac{1}{2}\tau^2 \Delta s^2$.

The Laplace transform of equations (6.7) and (6.8) along with the initial conditions (6.12) gives

$$\bar{u} = c_1^2 \left(\mathfrak{R}_1' \mathfrak{R}' - \epsilon\kappa \Delta s^2 \right) \vec{F} - \left(c_o \mathfrak{R}' - \epsilon\kappa c_1^2 s^2 \right) \text{grad div} \vec{F} + \frac{\beta}{\rho} \text{grad} G \tag{6.15}$$

$$\bar{\theta} = b\beta c_2^2 \mathfrak{R}_2' s^2 div \vec{F} + c_1^2 \mathfrak{R}_1' G \tag{6.16}$$

where F and G satisfy the equations

$$\mathfrak{R}_2' \left[\mathfrak{R}_1' \mathfrak{R}' - \epsilon\kappa \Delta s^2 \right] F_i = -\frac{1}{c_1^2 c_2^2} f_i \tag{6.17}$$

$$\left[\mathfrak{R}_1' \mathfrak{R}' - \epsilon\kappa \Delta s^2 \right] G = -\frac{1}{c_1^2 k^*} w \tag{6.18}$$

Equation (6.17), (6.18) can be written as

$$\mathfrak{R}_2' \left[\mathfrak{R}_1' \left(\mathfrak{R}_3' + (a+\tau)\Delta s + \frac{1}{2}\tau^2 \Delta s^2 \right) - \epsilon\kappa \Delta s^2 \right] \bar{F}_i = -\frac{1}{c_1^2 c_2^2} \bar{f}_i \tag{(6.19)}$$

$$\left[\mathfrak{R}_1' \left(\mathfrak{R}_3' + (a+\tau)\Delta s + \frac{1}{2}\tau^2 \Delta s^2 \right) - \epsilon\kappa \Delta s^2 \right] \bar{G} = -\frac{1}{c_1^2 k^*} \bar{w} \tag{6.20}$$

Now, equation (6.19) and (6.20) can be written as

$$\mathfrak{R}_2' \left[\Delta^2 - s^2 c_1^2 \Omega(s) \left(\frac{1}{c_1^4 \Omega(s)} + \frac{1}{c_3^2} + \epsilon\kappa \right) + \frac{s^4}{c_3^2} \Omega(s) \right] \bar{F}_i = -\frac{\Omega(s)}{c_2^2} \bar{f}_i \tag{6.21}$$

$$\left[\Delta^2 - s^2 c_1^2 \Omega(s)\left(\frac{1}{c_1^4\Omega(s)} + \frac{1}{c_3^2} + \epsilon\kappa\right)\Delta + \frac{s^4}{c_3^2}\Omega(s)\right]\bar{G} = -\frac{\Omega(s)}{k^*}\bar{w} \tag{6.22}$$

where $\Omega(s) = c_1^{-2}(1 + as + \tau s + \frac{1}{2}\tau^2)^{-1}$.

We write equations (6.21) and (6.22) in the forms

$$\mathfrak{R}_2'(\Delta - m_1^2)(\Delta - m_2^2)\bar{F}_i = -\frac{\Omega(s)}{c_2^2}\bar{f}_i \tag{6.23}$$

$$(\Delta - m_1^2)(\Delta - m_2^2)\bar{G} = -\frac{\Omega(s)}{k^*}\bar{w} \tag{6.24}$$

where m_1 and m_2 are the roots of the equation

$$m^4 - s^2 c_1^2 \Omega(s)\left(\frac{1}{c_1^4\Omega(s)} + \frac{1}{c_3^2} + \epsilon\kappa\right)m^2 + \frac{s^4}{c_3^2}\Omega(s) = 0 \tag{6.25}$$

Now, we consider two different cases:

Case − I : Concentrated Body Force

We consider that there is a concentrated time dependent body force which acts at a point $\vec{y}$ in the direction of x_j-axis ($j = 1, 2, 3$), i.e. we assume that

$$f_i = \delta_{ij}\delta(\vec{x} - \vec{y})g(t), \quad w = 0 \tag{6.26}$$

where $\delta(.)$ is the dirac delta function, $\vec{y}$ is a fixed point and $g(t)$ is any function of time. We denote the solutions for displacement components and temperature for this case as $u_i^{(j)}$ and $\theta^{(j)}$. Then we take $F_i = \delta_{ij}\varphi,\ G = 0$.

Therefore, from equation (6.23) we conclude that the function $\bar{\varphi}$ satisfies the equation

$$\mathfrak{R}_2'(\Delta - m_1^2)(\Delta - m_2^2)\bar{\varphi} = -\frac{\Omega(s)}{c_2^2}\delta(\vec{x} - \vec{y})\bar{g}(s) \tag{6.27}$$

Let

$$\Gamma = -\frac{\Omega(s)}{c_2^2}\delta(\vec{x} - \vec{y})\bar{g}(s) \tag{6.28}$$

Then above equation reduces to

$$\mathfrak{R}_2'(\Delta - m_1^2)(\Delta - m_2^2)\bar{\varphi} = \Gamma \tag{6.29}$$

Now, if the functions $\bar{\varphi}_1$, $\bar{\varphi}_2$, $\bar{\varphi}_3$ satisfy the equations

$$\mathfrak{R}_2'\bar{\varphi}_1 = \Gamma,\ \ (\Delta - m_1^2)\bar{\varphi}_2 = \Gamma\ \ (\Delta - m_2^2)\bar{\varphi}_3 = \Gamma \qquad (6.30,)$$

then from equation (6.29), we get that $\bar{\varphi}$ can be expressed in the form

$$\bar{\varphi} = A_1(s)\bar{\varphi}_1 + A_2(s)\bar{\varphi}_2 + A_3(s)\bar{\varphi}_3 \tag{6.31}$$

where

$A_1^{-1}(s)$=$\left(\frac{s^2}{c_2{}^2} - m_1^2\right)\left(\frac{s^2}{c_2{}^2} - m_2^2\right)$,

$A_2^{-1}(s) = \left(m_1^2 - \frac{s^2}{c_2{}^2}\right)\left(m_1^2 - m_2^2\right)$,

$A_3^{-1}(s) = \left(m_2^2 - \frac{s^2}{c_2{}^2}\right)\left(m_2^2 - m_1^2\right)$.

Next, in view of the conditions at infinity given by equation (6.13) and the equations (6.28) and (6.30), we get

$$\bar{\varphi}_1 = \frac{\Omega(s)\bar{g}(s)}{4\pi c_2^2 r}e^{-\frac{s}{c_2}r},\quad \bar{\varphi}_2 = \frac{\Omega(s)\bar{g}(s)}{4\pi c_2^2 r}e^{-m_1 r},\quad \bar{\varphi}_3 = \frac{\Omega(s)\bar{g}(s)}{4\pi c_2^2 r}e^{-m_2 r}, \text{ where } r = |\vec{x} - \vec{y}| \tag{6.32}$$

We note that for $\overrightarrow{x} \neq \overrightarrow{y}$ we have

$$\left[c_o\mathfrak{R}' - \epsilon\kappa c_1^2 s^2\right]\bar{\varphi} = [B_1(s)\bar{\varphi}_1 + B_2(s)\bar{\varphi}_2 + B_3(s)\bar{\varphi}_3] \tag{6.33}$$

$$\mathfrak{R}_2'\bar{\varphi} = \frac{\bar{\varphi}_2 - \bar{\varphi}_3}{(m_1^2 - m_2^2)},\ \ (\Delta - m_1^2)(\Delta - m_2^2)\bar{\varphi} = \bar{\varphi}_1 \tag{6.34}$$

where

$$B_1(s) = \frac{A_1(s)}{c_1^2\Omega(s)}\left[c_o\left\{\frac{s^2}{c_2^2} - \frac{c_1^2\Omega(s)}{c_3^2}s^2\right\} - \epsilon\kappa s^2 c_1^4\Omega(s)\right]$$

$$B_2(s) = \frac{A_2(s)}{c_1^2 \Omega(s)} \left[c_o \left\{ m_1^2 - \frac{c_1^2 \Omega(s)}{c_3^2} s^2 \right\} - \epsilon \kappa s^2 c_1^4 \Omega(s) \right]$$

$$B_3(s) = \frac{A_3(s)}{c_1^2 \Omega(s)} \left[c_o \left\{ m_2^2 - \frac{c_1^2 \Omega(s)}{c_3^2} s^2 \right\} - \epsilon \kappa s^2 c_1^4 \Omega(s) \right] \tag{6.35}$$

Therefore, in view of equations (6.15), (6.16), (6.33) and (6.34), we obtain the solutions for $\bar{u}_i$ and θ as

$$\bar{u}_i^{(j)} = \frac{1}{\Omega(s)} \bar{\varphi}_1 \delta_{ij} - (B_1(s)\bar{\varphi}_1 + B_2(s)\bar{\varphi}_2 + B_3(s)\bar{\varphi}_3)_{,ij} \tag{6.36}$$

$$\bar{\theta}^{(j)} = \frac{b\beta c_2^2 s^2}{(m_1^2 - m_2^2)} (\bar{\varphi}_2 - \bar{\varphi}_3) \tag{6.37}$$

Case – II : Concentrated Heat Source

Next, we consider that $f_i = 0$ and $w = \delta(\overrightarrow{x} - \overrightarrow{y})h(t)$. Here, we denote the solutions for displacement components and temperature for this case as $u_i^{(4)}$ and $\theta^{(4)}$. In this case, we find $F_i = 0$ and $\bar{G}$ satisfies the equation

$$(\Delta - m_1^2)(\Delta - m_2^2)\bar{G} = -\frac{\Omega(s)}{k^*} \delta(\vec{x} - \vec{y})\bar{h}(s) \tag{(6.38)}$$

Let the function $\bar{\phi}_i (i = 1, 2)$ satisfy the following equations:

$$(\Delta - m_1^2)\bar{\phi}_1 = -\frac{\Omega(s)}{k^*} \delta(\vec{x} - \vec{y})\bar{h}(s) \tag{6.39}$$

$$(\Delta - m_2^2)\bar{\phi}_2 = -\frac{\Omega(s)}{k^*} \delta(\vec{x} - \vec{y})\bar{h}(s) \tag{6.40}$$

Next, we write the function $\bar{G}$ as

$$\bar{G} = \frac{1}{(m_1^2 - m_2^2)} (\bar{\phi}_1 - \bar{\phi}_2) \tag{6.41}$$

where

$$\bar{\phi}_1 = \frac{\Omega(s)\bar{h}(s)}{4\pi k^* r} e^{-m_1 r} \tag{6.42}$$

$$\bar{\phi}_2 = \frac{\Omega(s)\bar{h}(s)}{4\pi k^* r} e^{-m_2 r} \tag{6.43}$$

Therefore, from equations (6.15) and (6.16), we get the solutions for the components of displacement and temperature in the Laplace tranmsform domain as

$$\bar{u}_i^{(4)} = \frac{\beta}{\rho(m_1^2 - m_2^2)}(\bar{\phi}_1 - \bar{\phi}_2)_{,i} \tag{6.44}$$

$$\bar{\theta}^{(4)} = \frac{c_1^2}{(m_1^2 - m_2^2)}\left[\left(m_1^2 - \frac{s^2}{c_1^2}\right)\bar{\phi}_1 - \left(m_2^2 - \frac{s^2}{c_1^2}\right)\bar{\phi}_2\right] \tag{6.45}$$

6.4 Small Time Approximated Solutions

The solutions obtained in the previous section represent the solutions in Laplace transform domain and we can obtain the solutions in space time domain by inverting Laplace transforms. However, the closed form analytical solutions is a formidable task. Hence, we aim to find the small time approximated solutions. Assuming s to be very large, and neglecting higher powers of $\frac{1}{s}$, we get the roots of the equation (6.25) by Maclaurin's series expansion as

$$m_1^2 = a_1 s^2 + a_2 + a_3 \frac{1}{s}$$

$$m_2^2 = b_1 - b_2 \frac{1}{s}$$

where

$a_1 = \frac{1}{c_1^2}$ $a_2 = \frac{2\kappa\epsilon}{\tau^2}$

$b_1 = \frac{2}{c_3^2 \tau^2}$ $b_2 = \frac{4(a+\tau)}{c_3^2 \tau^4}$

Now, we expand all the quantities involving s in the equations (6.36), (6.37), (6.44), (6.45) in the power of $\frac{1}{s}$. Then after long derivations and neglecting the higher powers of $\frac{1}{s}$, we obtain the short-time approximated solutions for the distributions of displacement components and temperature in the Laplace transform domain for above two cases in the following forms:

Case- I:

$$\begin{aligned}\bar{u}_i^{(j)} &= \frac{\bar{g}(s)}{4\pi c_2^2 r} e^{-\frac{s}{c_2}r}\delta_{ij} - \frac{\partial^2}{\partial x_i \partial x_j}\left[\frac{\bar{g}(s)}{4\pi c_1^2 c_2^2 r s^2}\left\{\left(B_{11} + \frac{B_{12}}{s^2}\right)e^{-\frac{s}{c_2}r}\right.\right. \\ &\left.\left. + \left(B_{21} + \frac{B_{22}}{s^2}\right)e^{-(\alpha_1 s + \frac{\alpha_2}{s})r} + \left(\frac{B_{31}}{s^2} + \frac{B_{32}}{s^3}\right)e^{-(\beta_1 - \frac{\beta_2}{s})r}\right\}\right] \end{aligned} \tag{6.46}$$

$$\bar{\theta}^{(j)} = \frac{b\beta g(s)}{2a_1 c_1^2 \pi \tau^2}\frac{\partial}{\partial x_j}\frac{1}{r}\left[\left(\frac{1}{s^2} + \frac{C_{11}}{s^3}\right)e^{-(\alpha_1 s + \frac{\alpha_2}{s})r} - \left(\frac{1}{s^2} + \frac{C_{11}}{s^3}\right)e^{-(\beta_1 - \frac{\beta_2}{s})r}\right] \tag{6.47}$$

Case- II:

$$\bar{u}_i^{(4)} = \frac{\beta h(s)}{2a_1 c_1^2 \pi \rho \tau^2 k*}\frac{\partial}{\partial x_i}\frac{1}{r}\left[\left(\frac{1}{s^4} + \frac{C_{11}}{s^5}\right)e^{-(\alpha_1 s + \frac{\alpha_2}{s})r} - \left(\frac{1}{s^4} + \frac{C_{11}}{s^5}\right)e^{-(\beta_1 - \frac{\beta_2}{s})r}\right] \tag{6.48}$$

$$\bar{\theta}^{(4)} = \frac{h(s)}{2\pi r \tau^2 a_1 c_1^2 k^*}\left[\left(\frac{D_{11}}{s^2} + \frac{D_{12}}{s^3}\right)e^{-(\alpha_1 s + \frac{\alpha_2}{s})r} - \left(\frac{D_{21}}{s^2} + \frac{D_{22}}{s^3}\right)e^{-(\beta_1 - \frac{\beta_2}{s})r}\right] \tag{6.49}$$

where

$\alpha_1 = \frac{1}{c_1}$, $\alpha_1 = \frac{1}{c_1}$,$\beta_1 = \frac{\sqrt{2}}{c_3\tau}$, $\beta_2 = \frac{a+\tau}{\tau^2}$,

$B_{11} = \frac{c_o}{(\frac{1}{c_2^2}-a_1)}$, $B_{12} = \frac{c_o(a_2 - b_1 a_1 c_2^2 + b_1)}{(\frac{1}{c_2^2}-a_1)^2} + \frac{c_2^2(-2c_o - 2c_1^2\epsilon\kappa c_3^2)}{\tau^2 c_3^2(\frac{1}{c_2^2}-a_1)}$,

$B_{21} = \frac{c_o}{(a_1 - \frac{1}{c_2^2})}$, $B_{22} = \frac{c_o(\frac{a_2}{c_2^2} + b_1 a_1 - \frac{b_1}{c_2^2})}{a_1(a_1 - \frac{1}{c_2^2})^2} + \frac{a_2 c_o \tau^2 - 2c_o - 2c_1^2\epsilon\kappa c_3^2}{\tau^2 a_1(a_1 - \frac{1}{c_2^2})}$,

$B_{31}=\frac{c_2^2}{a_1}(b_1c_o-\frac{2c_o}{c_3^2\tau^2}-\frac{2c_1^2\epsilon\kappa}{\tau^2})$, $B_{32}=\frac{c_2^2}{a_1}(-b_2c_o+\frac{4ac_o}{c_3^2\tau^4}+\frac{4ac_1^2\epsilon\kappa}{\tau^4}+\frac{4c_o}{c_3^2\tau^3}+\frac{4c_1^2\epsilon\kappa}{\tau^3})$,

$C_{11}=D_{22}=-\frac{2(a+\tau)}{\tau^2}$, $D_{11}=a_1c_1^2-1$, $D_{12}=\frac{2(a_1c_1^2-1)}{\tau^2}$, $D_{21}=-1$, $D_{22}=\frac{2(a+\tau)}{\tau^2}$.

6.4.1 Laplace Inversion

Now, we use the following formulae for the inversion of the Laplace transforms:

$$L^{-1}[\delta(t)]=1,\ L^{-1}[e^{-\alpha sr}]=\delta(t-\alpha r)$$

$$L^{-1}\left(\frac{e^{-\alpha sr}}{s^i}\right)=X_i(t-\alpha r)=\begin{cases}0 & t\leq\alpha r\\ \frac{(t-\alpha r)^{i-1}}{(i-1)!} & t>\alpha r\end{cases}$$

$$L^{-1}\left[\frac{e^{-\frac{a}{s}}}{s^{\nu+1}}\right]=\left(\frac{t}{a}\right)^{\frac{\nu}{2}}J_\nu(2\sqrt{at}),\ \ Re(\nu)>-1,\ a>0$$

$$L^{-1}\left[\frac{e^{\frac{a}{s}}}{s^{\nu+1}}\right]=\left(\frac{t}{a}\right)^{\frac{\nu}{2}}I_\nu(2\sqrt{at}),\ \ Re(\nu)>-1,\ a>0$$

Here, J_ν, I_ν are the Bessel and modified Bessel functions of order ν.

We assume that $g(t)=h(t)=\delta(t)$, i.e, we consider the case of instantaneous body force and instantaneous source of heat at the medium.

Therefore, after using above formulae of Laplace Inversions for the equations (6.46)-(6.49), we finally get the fundamental solutions in the physical domain as follows:

Case- I:

$$\begin{aligned}u_i^{(j)}=&\frac{1}{4\pi c_2^2 r}\delta(t-\frac{r}{c_2})\delta_{ij}-\frac{\partial^2}{\partial x_i\partial x_j}\Bigg[\frac{1}{4\pi c_1^2c_2^2 r}\Bigg\{\left(B_{11}X_2(t-\frac{r}{c_2})+B_{12}X_4(t-\frac{r}{c_2})\right)\\&+\left(B_{21}\left(\frac{t-\alpha_1 r}{\alpha_2 r}\right)^{\frac{1}{2}}J_1\left(2\sqrt{\alpha_2 r(t-\alpha_1 r)}\right)+B_{22}\left(\frac{t-\alpha_1 r}{\alpha_2 r}\right)^{\frac{3}{2}}J_3\left(2\sqrt{\alpha_2 r(t-\alpha_1 r)}\right)\right)\\&+e^{-\beta_1 r}\left(B_{31}\left(\frac{t}{\beta_2 r}\right)^{\frac{3}{2}}I_3\left(2\sqrt{\beta_2 rt}\right)+B_{32}(\frac{t}{\beta_2 r})^2 I_4\left(2\sqrt{\beta_2 rt}\right)\right)\Bigg\}\Bigg]\end{aligned}\tag{6.50}$$

$$\theta^{(j)} = \frac{b\beta}{2a_1c_1^2\pi\tau^2}\frac{\partial}{\partial x_j}\frac{1}{r}\Bigg[\left(\frac{t-\alpha_1 r}{\alpha_2 r}\right)^{\frac{1}{2}} J_1\left(2\sqrt{\alpha_2 r(t-\alpha_1 r)}\right)$$
$$+ C_{11}\left(\frac{t-\alpha_1 r}{\alpha_2 r}\right) J_2\left(2\sqrt{\alpha_2 r(t-\alpha_1 r)}\right)\Bigg)$$
$$- e^{-\beta_1 r}\Bigg\{\left(\frac{t}{\beta_2 r}\right)^{\frac{1}{2}} I_1\left(2\sqrt{\beta_2 rt}\right) + C_{11}(\frac{t}{\beta_2 r})I_2\left(2\sqrt{\beta_2 rt}\right)\Bigg\}\Bigg] \tag{6.51}$$

Case- II:

$$u_i^{(4)} = \frac{\beta}{2a_1c_1^2\pi\rho\tau^2k^*}\frac{\partial}{\partial x_i}\frac{1}{r}\Bigg[\Bigg\{\left(\frac{t-\alpha_1 r}{\alpha_2 r}\right) J_3\left(2\sqrt{\alpha_2 r(t-\alpha_1 r)}\right)$$
$$+ C_{11}\left(\frac{t-\alpha_1 r}{\alpha_2 r}\right)^2 J_4\left(2\sqrt{\alpha_2 r(t-\alpha_1 r)}\right)\Bigg\}$$
$$- e^{-\beta_1 r}\Bigg\{\left(\frac{t}{\beta_2 r}\right) I_3\left(2\sqrt{\beta_2 rt}\right) + C_{11}\left(\frac{t}{\beta_2 r}\right)^2 I_4\left(2\sqrt{\beta_2 rt}\right)\Bigg\}\Bigg] \tag{6.52}$$

$$\theta^{(4)} = \frac{\beta}{2a_1c_1^2\pi\rho\tau^2k^*}\Bigg[\Bigg\{D_{11}\left(\frac{t-\alpha_1 r}{\alpha_2 r}\right)^{\frac{1}{2}} J_1\left(2\sqrt{\alpha_2 r(t-\alpha_1 r)}\right)$$
$$+ D_{12}\left(\frac{t-\alpha_1 r}{\alpha_2 r}\right) J_2\left(2\sqrt{\alpha_2 r(t-\alpha_1 r)}\right)\Bigg\}$$
$$- e^{-\beta_1 r}\Bigg\{D_{21}\left(\frac{t}{\beta_2 r}\right)^{\frac{1}{2}} I_1\left(2\sqrt{\beta_2 rt}\right) + D_{22}\left(\frac{t}{\beta_2 r}\right) I_2\left(2\sqrt{\beta_2 rt}\right)\Bigg\}\Bigg] \tag{6.53}$$

6.5 Fundamental Solutions for Steady Oscillation

To determine the fundamental solutions of the field equations in case of steady oscillations, we use the representations of the equations (6.7) and (6.8). For this, we consider that

$$\vec{f} = Re[\vec{f}^*(\vec{x})exp(-i\omega t)], \quad w = Re[w^*(\vec{x})exp(-i\omega t)] \tag{6.54}$$

where ω is the frequency of the vibration and $i = \sqrt{-1}$.

We represent

$$\vec{F} = Re[\vec{F}^*(\vec{x},\omega)exp(-i\omega t)],\ G = Re[G^*(\vec{x},\omega)exp(-i\omega t)] \tag{6.55}$$

If we consider

$$\vec{u} = Re[\vec{u}^*(\vec{x},\omega)exp(-i\omega t)],\ \theta = Re[\theta^*(\vec{x},\omega)exp(-i\omega t)], \tag{6.56}$$

then the differential equation reduce to a differential system for the amplitudes $\vec{u}^*$ and θ^*.

Therefore, from the system of equations (6.7) and (6.8), we obtain the representations as

$$\begin{aligned}\vec{u}^* = c_1^2\Big[&(\Delta+\xi^2)(\Delta+\gamma^2) - i\omega(a+\tau)(\Delta+\xi^2)\\ &- \frac{1}{2}\tau^2\omega^2\Delta(\Delta+\xi^2) + \epsilon\kappa\omega^2\Delta\Big]\vec{F}^*\\ &- \left[c_o\left\{(\Delta+\gamma^2) - i\omega(a+\tau)\Delta - \frac{1}{2}\tau^2\omega^2\Delta\right\} + \epsilon\kappa\omega^2 c_1^2\right]\mathrm{grad\,div}\vec{F}^*\\ &+ \frac{\beta}{\rho}\mathrm{grad}\,G^*\end{aligned} \tag{6.57}$$

$$\theta^* = c_1^2(\Delta+\xi^2)G^* - m\beta c_2^2\omega^2(\Delta+\eta^2)\mathrm{div}\,\vec{F}^* \tag{6.58}$$

where, we used the following notations

$$\xi = \frac{\omega}{c_1},\quad \eta = \frac{\omega}{c_2},\ \gamma = \frac{\omega}{c_3}$$

The functions $\vec{F}^*$and G^* involved in the equations (6.57) and (6.58) satisfy the equations

$$\Bigg[(\Delta+\eta^2)\left[(\Delta+\xi^2)(\Delta+\gamma^2)-i\omega(a+\tau)(\Delta+\xi^2)\right] \\ -\frac{1}{2}\tau^2\omega^2\Delta(\Delta+\xi^2)+\epsilon\kappa\omega^2\Delta\Bigg]\vec{F}_i^* = \frac{1}{c_1^2c_2^2}f_i^* \tag{6.59}$$

$$\Bigg[(\Delta+\xi^2)(\Delta+\gamma^2)-i\omega(a+\tau)(\Delta+\xi^2)\Bigg] \\ -\frac{1}{2}\tau^2\omega^2\Delta(\Delta+\xi^2)+\epsilon\kappa\omega^2\Delta\Bigg]G^* = -\frac{1}{c_1^2k^*}w^* \tag{6.60}$$

Now, we use the following notation:

$$H(\Delta)=\Delta^2+\left\{(\xi^2+\gamma^2)-i\omega(a+\tau)\xi^2-\frac{1}{2}\tau^2\omega^2\xi^2+\epsilon\kappa\omega^2\right\}Q^*\Delta+\xi^2\gamma^2Q^* \tag{6.61}$$

where, $Q^*=\left\{1-i\omega(a+\tau)-\frac{1}{2}\tau^2\omega^2\right\}^{-1}$.

It is easy to see that $H(\Delta)=(\Delta+\lambda_1^2)(\Delta+\lambda_2^2)$, where λ_1^2 and λ_2^2 are the roots of the equation, $H(-\lambda)=0$. Equations (6.59) and (6.60) can therefore be written as

$$(\Delta+\eta^2)(\Delta+\lambda_1^2)(\Delta+\lambda_2^2)F_i^* = -\frac{Q^*}{c_1^2c_2^2}f_i^* \tag{6.62}$$

$$(\Delta+\lambda_1^2)(\Delta+\lambda_2^2)G^* = -\frac{Q^*}{c_1^2k^*}w* \tag{6.63}$$

If we assume that $f_i^*=\delta(\vec{x}-\vec{y})\delta_{ij}$ and $w^*=0$, then we take $F_i^*=\varPhi\delta_{ij}$ and $G^*=0$.

Hence, from equation (6.62), we can conclude that the function $\varPhi$ satisfies the equation

$$(\Delta+l_1^2)(\Delta+l_2^2)(\Delta+l_3^2)\varPhi=-I \tag{6.64}$$

where

$$l_1 = \lambda_1,\ l_2 = \lambda_2,\ l_3 = \eta,\ I = \frac{Q^*}{c_1^2 c_2^2}\delta(\vec{x} - \vec{y}) \tag{6.65}$$

We further assume that the function $\Phi_j (j = 1, 2, 3)$ satisfy the equations

$$(\Delta + l_1^2)\Phi_1 = -I,\ (\Delta + l_1^2)\Phi_2 = -I,\ (\Delta + l_1^2)\Phi_3 = -I$$

Then, the function Φ can be expressed in the form

$$\Phi = \alpha_1\Phi_1 + \alpha_2\Phi_2 + \alpha_3\Phi_3$$

where

$$\alpha_n^{-1} = \prod_{j=1, j\neq n}^{3} (l_j^2 - l_n^2), \quad (n = 1, 2, 3)$$

Therefore, solution of the equation (6.64) is obtained as

$$\Phi = \frac{Q^*}{4\pi c_1^2 c_2^2 r} \sum_{j=1}^{3} \alpha_j \exp(i l_j r) \tag{6.66}$$

Also, in view of the equations (6.57) and (6.58), we get the functions $u_k^{*(j)} (k = 1, 2, 3)$ and $\theta^{*(j)}$ as

$$\begin{aligned} u_k^{*(j)} &= \delta_{kj} c_1^2 (\Delta + \lambda_1^2)(\Delta + \lambda_2^2)\Phi \\ &- \left[c_o \left\{ (\Delta + \gamma^2) - i\omega(a + \tau)\Delta - \frac{1}{2}\tau^2\omega^2\Delta \right\} + \epsilon\kappa\omega^2 c_1^2 \right] \Phi_{,kj} \end{aligned} \tag{6.67}$$

$$\theta^{*(j)} = -mbc_2^2\omega^2(\Delta + \eta^2)\Phi_{,j} \tag{6.68}$$

Next, we consider the case when $f_i^* = 0$ and $w^* = \delta(\vec{x} - \vec{y})$. In this case, we assume $F_i^* = 0$ and $G^* = \Psi$, where Ψ is the solution of the equation

$$(\Delta + l_1^2)(\Delta + l_2^2)\Psi = \frac{Q^*}{k^* c_1^2}\delta(\vec{x} - \vec{y})$$

We find that

$$\Psi = \frac{Q^*}{4\pi r k^* c_1^2 (l_2^2 - l_1^2)} \left[exp(il_1 r) - exp(il_2 r)\right] \tag{6.69}$$

Hence, equation (6.57) and (6.58) yield the solution as

$$u_k^{*(4)} = \frac{\beta}{\rho}\Psi_{,j} \tag{6.70}$$

$$\theta^{*(4)} = c_1^2(\Delta + \xi^2)\Psi \tag{6.71}$$

Consequently, the functions $u_k^{*(j)}, \theta^{*(j)}, (j = 1, 2, 3, 4)$ represented by equations (6.67),(6.68) and (6.70), (6.71) express the fundamental solutions in the case of steady oscillation.

6.6 Conclusions

The present work is concerned with the thermoelasticity theory based on a very recently proposed heat conduction model: a heat conduction model with a delay term introduced by Quintanilla(2011). Here we aim to obtain the fundamental solutions of this theory. We derive the solution of Galerkin-type field equations for the case of homogeneous and isotropic bodies. With the help of this solution we determine the effects of concentrated heat sources and body forces in an unbounded medium. We further obtain the fundamental solutions of the field equations in case of steady vibrations.