CHAPTER 2

DOMAIN OF INFLUENCE THEOREM

2.1 A Domain of Influence Theorem for Thermoelasticity Without Energy Dissipation

2.1.1 Introduction

Domain of influence theorem is one of the very basic results of classical isothermal elastodynamics (Gurtin (1972), Eringen and Suhubi (1975)). It asserts that for a finite time, $t > 0$, a solution of a given initial-boundary value problem, corresponding to a data which are defined in a bounded support, vanishes outside a bounded domain. Its physical interpretation is that an initial perturbation of a bounded elastic domain gives rise to an elastic disturbance which for any time $t>0$ cannot occupy the whole space, i.e it propagates with finite speed. Such a theorem cannot be proved in classical linear coupled thermoelasticity theory, asserting that, Fourier law of heat conduction implies an infinite speed of thermal disturbances. This concept was first investigated by Nunziato and Cowin (1979). Domain of influence theorems in the contexts of linear thermoelasticity and asymmetric elastodynamics were studied by Ignaczak (1978, 1979). Carbonaro and Russo (1984) proved energy inequalities and the domain of influence theorem in classical elastodynamics. The domain of influence theorems in the context of generalized thermoelasticity, like LS model and TRDTE model were reported in the review articles by Ignaczak (1991) and Hetnarski and Ignaczak (1999). A detailed discussion on this subject is also available in a recent book by Ignaczak and Ostoja-Starzewski (2010). Chandrasekharaiah (1987) obtained a uniqueness result in

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the context of elastic materials with voids by using the concept of domain of influence. Furher, Dhaliwal and Wang (1994) derived the domain of influence theorem in the theory of elastic materials with voids. Marin (2010) reported a domain of influence theorem for microstretch elastic materials. Recently, Mukhopadhyay *et al.* (2011) and Kumar and Kumar (2015) derived the domain of influence theorems in the theory of generalized thermoelasticity with dual phase-lags and three phase lags, respectively.
The present section of the thesis is concerned with the thermoelasticity theory of Green and Naghdi of type-I, II and III. We consider a mixed initial-boundary value problem for an isotropic medium in the context of all three models of type-I, II and III in a unified way and we derive an identity in terms of the temperature and potential. On the basis of this identity, we establish the domain of influence theorem for Green-Naghdi model of type-II. This theorem implies that for a given bounded support of thermo-mechanical loading, the thermoelastic disturbance generated by the pair of temperature and potential of the system vanishes outside a well-defined bounded domain. This domain is shown to depend on the support of the load, i.e., on the initial and boundary data. It is also shown that under Green-Naghdi model-II, the thermoelastic disturbance propagates with a finite speed that is dependent on the thermoelastic parameters.

2.1.2 Problem Formulation

We consider $\bar{V}$ as the closure of an open, bounded and connected domain whose boundary B encloses a homogeneous and isotropic thermoelastic material in a three dimensional Euclidean space. The interior of $\bar{V}$ is taken as V. We refer the motion of the body to a fixed system of rectangular Cartesian axes and we employ the thermoelasticity of type-I, II and III proposed by Green and Naghdi (1991). The basic governing equations in absence of body forces and heat sources and defined in $V \times [0, \infty)$ are therefore considered as follows:

Equation of motion:

$$\mu u_{i,jj} + (\lambda + \mu) u_{j,ji} - \beta \theta_{,i} = \rho \ddot{u}_i \tag{2.1.1}$$

Heat conduction equation:

$$(\xi_1 k^* + \xi_2 k \frac{\partial}{\partial t})\nabla^2\theta = \rho c_E \ddot{\theta} + \beta\theta_0 \ddot{u}_{i,i} \tag{2.1.2}$$

We assume that the material parameters ρ, μ, $3\lambda+2\mu$, k, k^*, θ_0 are all positive constant and ∇^2, is a Laplacian operator. The constant parameters ξ_1, ξ_2 are used to consider three different theories of Green and Naghdi of type- I, II and III in a unified way and they can be recovered from equation (2.1.2) as follows:

(i) Green- Naghdi (GN) Type-I theory : $\xi_1 = 0$, $\xi_2 = 1$.

(ii) Green- Naghdi (GN) Type-II theory : $\xi_1 = 1$, $\xi_2 = 0$.

(iii) Green- Naghdi (GN)Type-III theory : $\xi_1 = 1$, $\xi_2 = 1$.

Now, we assume displacement as a gradient of a scalar field, i.e., we assume $u_i = \varphi_{,i}$, where φ is a scalar field defined on $\bar{V} \times [0,\infty)$. Hence, from equations (2.1.1) and (2.1.2), we get

$$\nabla^2\varphi - \frac{\rho}{\lambda+2\mu}\ddot{\varphi} - \frac{\beta}{\lambda+2\mu}\theta = 0 \quad \text{on}\, V \times [0,\infty) \tag{2.1.3}$$

$$(\xi_1 k^* + \xi_2 k \frac{\partial}{\partial t})\nabla^2\theta = \rho c_E \ddot{\theta} + \beta\theta_0 \ddot{\varphi} \quad \text{on}\, V \times [0,\infty) \tag{2.1.4}$$

Now, for simplicity we introduce the following dimensionless variables and notations:
$x' = \frac{c_E c_1 \rho}{k}x,\ t' = \frac{c_E c_1^2 \rho}{k}t,\ \theta' = \frac{\theta}{\theta_0},\ \varphi' = \frac{(2\mu+\lambda)c_E^2 c_1^2 \rho^2}{\beta\theta_0 k^2}\varphi,\ a_0 = \frac{k^*}{(\lambda+2\mu)c_E}$
Equations (2.1.3) and (2.1.4) then transform to the dimensionless forms as follows (we omit the primes for simplicity):

$$\nabla^2\varphi - \ddot{\varphi} - \theta = 0 \qquad \text{on } V \times [0,\infty) \tag{2.1.5}$$

$$(\xi_1 a_0 + \xi_2 \frac{\partial}{\partial t})\nabla^2\theta = \ddot{\theta} + \varepsilon\nabla^2\ddot{\varphi} \quad \text{on } V \times [0,\infty) \tag{2.1.6}$$

where $c_1^2 = \frac{2\mu+\lambda}{\rho}$, $\varepsilon = \frac{\beta^2\theta_0}{\rho^2 c_1^2 c_E}$.

In addition to the above system of equations (2.1.5) and (2.1.6), we consider the following initial and boundary conditions:

$$\begin{aligned} \varphi(x,0) = \varphi_0,\ \dot{\varphi}(x,0) = \dot{\varphi}_0 \\ \theta(x,0) = \theta_*,\ \dot{\theta}(x,0) = \dot{\theta}_* \quad \text{for } x \in V \end{aligned} \tag{2.1.7}$$

and

$$\varphi_{,i}\, n_i = f(x,t),\ \theta = \theta'(x,t) \qquad B\times]0,\infty) \tag{2.1.8}$$

where n_i are the components of outward drawn unit normal to boundary surface B.

2.1.3 Some Definitions

We define the set

$$D_0(t) = \left\{ x \in \bar{V} : \begin{array}{c} (1)\,\text{if}\ x\epsilon V,\ \text{then}\,\varphi_0 \neq 0\,\text{or}\,\dot{\varphi}_0 \neq 0\,\text{or}\,\theta_* \neq 0\,\text{or}\,\dot{\theta}_* \neq 0 \\ (2)\,\text{if}\ (x,\tau) \in B \times [0,t],\,\text{then}\,f(x,\tau) \neq 0\,\text{or}\,\theta^{'}(x,\tau) \neq 0 \end{array} \right\} \tag{2.1.9}$$

as the support of thermomechanical load of this present problem defined in the previous section as given by equations (2.1.5)-(2.1.8) for a fixed time $t \in]0,\infty[$. Further, let v be any real number and $\Omega(x,vt)$ is an open ball with radius vt and centered at x. Then, the set

$$D(t) = \{x \in \bar{V} :\ D_0(t) \cap \overline{\Omega(x,vt)} \neq \emptyset\} \tag{2.1.10}$$

is termed as the domain of influence for the above thermomechanical load. The domain of influence, $D(t)$ is clearly a set of all the points of $\bar{V}$ that may be reached by the thermomechanical disturbances propagating from $D_0(t)$ with a speed not greater than v.

Now, we establish an identity in the contexts of three models of Green and Naghdi.

2.1.4 Main Results

Theorem 2.1.1: Let $p(x) \in C^1(\bar{V})$ denote a scalar field and (φ, θ) be a solution to the mixed boundary-initial value problem (2.1.5)-(2.1.8) such that the set

$$E_0 = \{x \in \bar{V} \, : \, p(x) > 0\} \tag{2.1.11}$$

is bounded there. Then

$$\frac{1}{2}\int_V \{P(x,p(x)) - P(x,0)\}dV + \int_V \int_0^{p(x)} Q(x,t)dtdV + \int_V R_i(x,p(x))p_{,i}(x)dV = \int_B \int_0^{p(x)} R_i(x,t)n_i(x)dtdB \tag{2.1.12}$$

where

$$P(x,t) = \hat{\theta}^2 + \xi_1 a_0 \theta_{,i}\theta_{,i} + \varepsilon(\nabla^2\hat{\varphi})^2 + \varepsilon\dot{\hat{\varphi}}_{,i}\dot{\hat{\varphi}}_{,i} \tag{2.1.13}$$

$$Q(x,t) = \xi_2 \dot{\theta}_{,i}\dot{\theta}_{,i} \tag{2.1.14}$$

$$R_i(x,t) = \varepsilon(\nabla^2\hat{\varphi})\dot{\hat{\varphi}}_{,i} + \hat{\theta}(\xi_1 a_0 \theta_{,i} + \xi_2 \dot{\theta}_{,i} - \varepsilon\dot{\hat{\varphi}}_{,i}) \;\; \forall (x,t) \in \bar{V} \times [0,\infty) \tag{2.1.15}$$

and the cap operator here denotes the operator $L = \partial/\partial t$, that is

$$\hat{\varphi} = L\varphi = \dot{\varphi}\,, \;\; \hat{\theta} = L\theta = \dot{\theta} \;\; \text{on} \;\; \bar{V} \times [0,\infty) \tag{2.1.16}$$

Proof: Applying the operator L, to both sides of equation (2.1.5) and taking the gradient, we obtain

$$(\nabla^2\hat{\varphi})_{,i} - \ddot{\hat{\varphi}}_{,i} - \hat{\theta}_{,i} = 0 \quad \text{on } \bar{V}\times[0,\infty) \tag{2.1.17}$$

Multiplying equation (2.1.17) by $\varepsilon\dot{\hat{\varphi}}_{,i}$ and using the following identities

$$\dot{\hat{\varphi}}_{,i}\,(\nabla^2\hat{\varphi})_{,i} = (\dot{\hat{\varphi}}_{,i}\nabla^2\hat{\varphi})_{,i} - \frac{1}{2}\frac{\partial}{\partial t}(\nabla^2\hat{\varphi})^2$$

$$\dot{\hat{\varphi}}_{,i}\ddot{\hat{\varphi}}_{,i} = \frac{1}{2}\frac{\partial}{\partial t}(\dot{\hat{\varphi}}_{,i}\dot{\hat{\varphi}}_{,i})$$

we get

$$\frac{\varepsilon}{2}\frac{\partial}{\partial t}\{(\nabla^2\hat{\varphi})^2 + \dot{\hat{\varphi}}_{,i}\,\dot{\hat{\varphi}}_{,i}\,\} + \varepsilon\hat{\theta}_{,i}\,\dot{\hat{\varphi}}_{,i} = \varepsilon(\dot{\hat{\varphi}}_{,i}\,\nabla^2\hat{\varphi})_{,i} \tag{2.1.18}$$

Now, equation (2.1.6) can be written as

$$(\xi_1 a_0 + \xi_2\frac{\partial}{\partial t})\nabla^2\theta = \dot{\hat{\theta}} + \varepsilon\nabla^2\dot{\hat{\varphi}} \tag{2.1.19}$$

Next, multiplying equation (2.1.19) by $\hat{\theta}$ and using the following relations

$$\hat{\theta}\nabla^2\theta = (\hat{\theta}\theta_{,i})_{,i} - \hat{\theta}_{,i}\theta_{,i}$$

$$\hat{\theta}\nabla^2\dot{\theta} = (\hat{\theta}\dot{\theta}_{,i})_{,i} - \hat{\theta}_{,i}\dot{\theta}_{,i}$$

$$\hat{\theta}\nabla^2\dot{\hat{\varphi}} = (\hat{\theta}\dot{\hat{\varphi}}_{,i})_{,i} - \hat{\theta}_{,i}\,\dot{\hat{\varphi}}_{,i}$$

we get

$$\frac{1}{2}\frac{\partial}{\partial t}(\hat{\theta} + \xi_1 a_0\theta_{,i}^2) + \xi_2\dot{\theta}_{,i}\dot{\theta}_{,i} - \varepsilon\hat{\theta}_{,i}\dot{\hat{\varphi}}_{,i} = [\hat{\theta}(\xi_1 a_0\theta_{,i} + \xi_2\dot{\theta}_{,i} - \varepsilon\dot{\hat{\varphi}}_{,i})]_{,i} \tag{2.1.20}$$

Adding equations (2.1.18) and (2.1.20), we get

$$\frac{1}{2}\frac{\partial}{\partial t}P(x,t)+Q(x,t)=R_{i,i}(x,t) \tag{2.1.21}$$

where P, Q and R are defined by equations (2.1.13), (2.1.14) and (2.1.15), respectively. Now, we can write

$$\int_0^{p(x)} R_{i,i}(x,t)dt=\left[\int_0^{p(x)} R_i(x,t)dt\right]_{,i}-R_i(x,p(x))p_{,i}(x) \tag{2.1.22}$$

Therefore, integrating equation (2.1.21) from $t=0$ up to $t=p(x)$, and using the result from equation (2.1.22), we get

$$\frac{1}{2}\left[P(x,p(x))-P(x,0)\right]+\int_0^{p(x)} Q(x,t)dt+R_i(x,p(x))p_{,i}(x)=\left[\int_0^{p(x)} R_i(x,t)dt\right]_{,i} \tag{2.1.23}$$

Since, we assumed that the set E_0 defined by equation (2.1.11) is bounded, hence, each term in equation (2.1.23) has a bounded support. Therefore, integrating equation (2.1.23) over V and using divergence theorem, we obtain equation (2.1.12). This proves our theorem-2.1.1.

2.1.4.1 Spacial Case

Now, we shall consider the case of GN-II model and assume that $\xi_1= 1$, $\xi_2= 0$ to formulate the theorem of domain of influence stating that the thermomechanical load restricted to the interval $[0,t]$ does not influence the points outside the domain $D(t)$. Hence, we substitute $\xi_1= 1$, $\xi_2= 0$ in equations(2.1.12) to obtain

$$\frac{1}{2}\int_V \left\{P(x,p(x))-P(x,0)\right\}dV+\int_V R_i(x,p(x))p_{,i}(x)dV=\int_B\int_0^{p(x)} R_i(x,t)n_i(x)dtdB \tag{2.1.24}$$

where the expressions for $P(x,t)$ and $R_i(x,t)$ reduce to

$$P(x,t) = \hat{\theta}^2 + a_0\theta_{,i}\theta_{,i} + \varepsilon(\nabla^2\hat{\varphi})^2 + \varepsilon\dot{\hat{\varphi}}_{,i}\dot{\hat{\varphi}}_{,i} \tag{2.1.25}$$

$$R_i(x,t) = \varepsilon(\nabla^2\hat{\varphi})\dot{\hat{\varphi}}_{,i} + \hat{\theta}(a_0\theta_{,i} - \varepsilon\dot{\hat{\varphi}}_{,i}) \ \ \forall\,(x,t) \in \bar{V} \times [0,\infty) \tag{2.1.26}$$

The domain of influence theorem for the theory of Green-Naghdi of type -II for temperature-potential problem is established in the next theorem.

Theorem-2.1.2: If the pair (φ, θ) is a smooth solution of the problem defined by (2.1.15)-(2.1.18) for GN-II model and if the set $D(t)$, defined by equation (2.1.10), is the domain of influence for thermomechanical load at time t, where v satisfies the inequality

$$v \geq \max\{2,\, a_0 + \varepsilon\}, \tag{2.1.27}$$

then

$$\varphi = \theta = 0 \text{ on } \{\bar{V} - D(t)\} \times [0,t] \tag{2.1.28}$$

Proof: Let $(z,\tau) \in \{\bar{V} - D(t)\} \times\ (0,t)$ be a fixed point and we assume

$$\Gamma = \bar{V} \cap \overline{\Omega(z, v\tau)} \tag{2.1.29}$$

We take

$$p_\tau(x) = \begin{cases} \tau - \frac{1}{v}|x-z| & \text{for } x \in \Gamma \\ 0 & \text{for } x \notin \Gamma \end{cases} \tag{2.1.30}$$

where v is a parameter represented by the inequality (2.1.27).

Now, in view of the definitions of domain $D(t)$ given by (2.1.10) and Γ by (2.1.29) and since $\tau < t$, we find that $D_0(t)$ and Γ are disjoint sets.

Therefore,

$$\Gamma \cap D_0(t) = \emptyset \tag{2.1.31}$$

Hence, we get

$$\varphi_{,i}n_i = 0, \;\; \theta = 0 \text{ on } (\Gamma \cap B) \times [0, t] \tag{2.1.32}$$

and

$$\dot{\hat{\varphi}}_{,i}n_i = 0, \; \hat{\theta} = 0 \text{ on } (\Gamma \cap B) \times [0, t] \tag{2.1.33}$$

Moreover,

$$\varphi(x, 0) = \dot{\varphi}(x, 0) = \theta(x, 0) = \dot{\theta}(x, 0) = 0 \quad \text{on } \Gamma \tag{2.1.34}$$

From equations (2.1.30), (2.1.33) and (2.1.26), we obtain

$$\int\limits_{B} \int\limits_{0}^{p(x)} R_i(x, t)n_i(x)dtdB = 0 \tag{2.1.35}$$

Also equations (2.1.34) and (2.1.17) imply

$$\nabla^2\hat{\varphi}(x, 0) = \hat{\theta}(x, 0) = 0 \text{ on } \Gamma \tag{2.1.36}$$

$$\dot{\hat{\varphi}}_{,i}(x, 0) = \theta_{,i}(x, 0) = 0 \text{ on } \Gamma \tag{2.1.37}$$

Therefore, in view of the definitions of $P(x, t)$ and $p_\tau(x)$, we get

$$P(x, p_\tau(x)) - P(x, 0) = \begin{cases} P(x, p_\tau(x)) & \text{for } x \in \Gamma \\ 0 & \text{for } x \notin \Gamma \end{cases} \tag{2.1.38}$$

Now, substituting $p_\tau(x)$ into equation (2.1.24) and using equations (2.1.35) and (2.1.38), we find

$$\frac{1}{2}\int\limits_{\Gamma} P(x, p_\tau(x))dV = -\int\limits_{\Gamma} R_i(x, p_\tau(x))p_{\tau,i}(x)dV \tag{2.1.39}$$

Hence, by using equations (2.1.30) and (2.1.39), we get the following inequality:

$$\frac{1}{2}\int_{\Gamma} P(x,p_\tau(x))dV \leq \frac{1}{v}\int_{\Gamma} |R_i(x,p_\tau(x))|dV \qquad (2.1.40)$$

Now, from equation (2.1.26), we get

$$|R_i(x,p_\tau(x))| \leq |\varepsilon(\nabla^2\hat{\varphi})||\dot{\hat{\varphi}}_{,i}| + a_0|\hat{\theta}\theta_{,i}| + \varepsilon|\hat{\theta}\dot{\hat{\varphi}}_{,i}|$$

$$\leq \frac{\varepsilon}{2}\{(\nabla^2\hat{\varphi})^2 + 2\dot{\hat{\varphi}}_{,i}\dot{\hat{\varphi}}_{,i} + (\hat{\theta})^2\} + \frac{a_0}{2}\{(\hat{\theta})^2 + \theta_{,i}\theta_{,i}\} \qquad (2.1.41)$$

Therefore, from the definition of $P(x,t)$ and in view of the inequalities (2.1.40) and (2.1.41), we arrive at

$$\left(\frac{1}{2} - \frac{a_0+\varepsilon}{2v}\right)\int_{\Gamma}(\hat{\theta})^2 dV + \frac{a_0}{2}\left(1-\frac{1}{v}\right)\int_{\Gamma}\theta_{,i}\theta_{,i}dV$$
$$+\frac{\varepsilon}{2}\left(1-\frac{1}{v}\right)\int_{\Gamma}(\nabla^2\hat{\varphi})^2 dV + \frac{\varepsilon}{2}\left(1-\frac{2}{v}\right)\int_{\Gamma}\dot{\hat{\varphi}}_{,i}\dot{\hat{\varphi}}_{,i}dV \leq 0 \qquad (2.1.42)$$

All the integrals in equation (2.1.42) have non-negative coefficients, provided $v \geq \max(2,\ a_0+\varepsilon)$. Therefore, all integrals in equation (2.1.42) being non-negative as well the equality sign must hold for equation (2.1.42). This implies the vanishing of each term of (2.1.42) in Γ. Specifically, we have

$$\nabla^2\hat{\varphi}(x,p_\tau(x)) = 0,\ \hat{\theta}(x,p_\tau(x)) = 0 \quad \text{on } \Gamma \qquad (2.1.43)$$

By the smoothness property of (φ,θ) and definition of $p_\tau(x)$ we have

$$\left.\begin{aligned} \nabla^2\hat{\varphi}(x,p_\tau(x)) &\to \nabla^2\hat{\varphi}(z,\tau) \\ \hat{\theta}(x,g_\tau(x)) &\to \hat{\theta}(z,\tau) \end{aligned}\right\} as\, x \to z \qquad (2.1.44)$$

Now taking the limit $x \to z$, in the equation (2.1.43), and from equation (2.1.30), we get

$$\nabla^2 \hat{\varphi}(z, \tau) = 0 \ , \hat{\theta}(z, \tau) = 0 \tag{2.1.45}$$

However, (φ, θ) is sufficiently smooth on $\bar{V} \times [0, \infty)$, and (z, τ) is an arbitrary point of $\{V - D(t)\} \times (0, t)$.

Hence

$$\nabla^2 \hat{\varphi}(x, \tau) = \hat{\theta}(x, \tau) = 0 \ \text{ on } \ \{\bar{V} - D(t)\} \times [0, t] \tag{2.1.46}$$

Thus, equations (2.1.46) and (2.1.5) yield

$$\ddot{\hat{\varphi}}(x, \tau) = 0, \ \hat{\theta}(x, \tau) = 0 \quad \text{on} \ \ \{\bar{V} - D(t)\} \times [0, t] \tag{2.1.47}$$

By equation (2.1.47), we obtain

$$\begin{gathered} \ddot{\varphi}(x, \tau) = \ddot{\varphi}(x, 0), \ \ \theta(x, \tau) = \theta(x, 0) \\ \text{for } (x, \tau) \in \{V - D(t)\} \times [0, t] \end{gathered} \tag{2.1.48}$$

Further, in view of the definition of domain $D(t)$ and equation (2.1.5), we have

$$\ddot{\varphi}(x, 0) = \theta(x, 0) = 0 \text{ on } \ \{V - D(t)\} \tag{2.1.49}$$

Equation (2.1.48), combining with equation (2.1.49) yields,

$$\ddot{\varphi} = 0, \ \theta = 0 \quad \text{on } \ \{V - D(t)\} \times [0, t] \tag{2.1.50}$$

Finally, recalling the definition of $D(t)$ and using equation (2.1.50), we obtain

$$\varphi = 0, \ \theta = 0 \ \text{ on } \ \{V - D(t)\} \times [0, t] \tag{2.1.51}$$

which completes the proof of theorem-2.1.2.

This theorem implies that for a finite time t and a bounded support of the thermomechanical loading (that is for a bounded set $D_0(t)$) a thermoelastic disturbance generated by the pair (φ, θ) satisfying the system defined by equations (2.1.5) to (2.1.8) in the context of GN-II theory vanishes outside the bounded set $D(t)$, which depends on the support of the given load. This theorem also shows that the thermoelastic disturbance propagates from the domain $D_0(t)$ with a finite speed equal to or less than v defined by the relation (2.1.27). Therefore, this theorem proves the hyperbolicity of GN-II model and at any time, the associated domain of influence for a given boundary load is identified with a boundary layer of the thickness vt that is influenced by thermoelastic coupling constant ϵ and the material parameters. Furthermore, it is noted that the material parameter k^*, which is a characteristic of the theory of Green and Naghdi, also influences the domain of influence.

2.2 A Domain of Influence Theorem for a Natural Stress-Heat-Flux Disturbance in Thermoelasticity of Type- II

2.2.1 Introduction

In the previous section, we established the domain of influence theorem on the theory of thermoelasticity of type-II in terms of temperature and potential. In the present section, we aim to establish the domain of influence theorem for a stress-heat-flux disturbance under Green and Naghdi thermoelasticity theory of type-II. Here we consider a mixed problem of natural type represented as stress-heat-flux disturbance in the context of Green-Naghdi thermoelasticity theory of general type. We establish a general energy identity for the problem. Then we establish the domain of influence theorem for natural stress-heat-flux disturbance in the context of Green-Naghdi model of thermoelasticity of type-II. We prove that for a finite time $t > 0$, the pair of stress and the heat-flux field generates no disturbance outside a bounded domain. The domain of influence is further shown to be dependent on the thermoelastic parameters.

2.2.2 Basic Equations and Problem formulation

In three dimensional Euclidean space, let $\bar{V}$ denote the closure of an open, bounded and connected domain whose boundary ∂V encloses a homogeneous and isotropic thermoelastic material. We assume that the interior of $\bar{V}$ is V and n_i are the components of outward drawn unit normal to ∂V. We refer the motion of the body to a fixed system of rectangular Cartesian axes. The basic governing equations and the constitutive

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relations defined in $V \times [0, \infty)$ in the context of linear theory of thermoelasticity by Green and Naghdi of type-III for a homogeneous and isotropic material are considered as follows:

Strain-displacement relation:

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

Stress equation of motion:

$$\sigma_{ij,j} + b_i = \rho \ddot{u}_i$$

Energy equation:

$$-q_{i,i} + r = C_s \dot{\theta} + \theta_0 \alpha_t \dot{\sigma}_{kk}$$

Strain-stress relation:

$$e_{ij} = \frac{1}{2\mu}\left(\sigma_{ij} - \frac{\lambda}{3\lambda + 2\mu}\sigma_{kk}\delta_{ij}\right) + \alpha_t \theta \delta_{ij}$$

The general and unified heat conduction equation for thermoelasticity under Green-Naghdi theory of type I, II and III:

$$\dot{q}_i = -(\xi_1 k^* + \xi_2 k \frac{\partial}{\partial t})\theta_{,j}$$

From the above basic equations, we consider a natural stress-heat-flux problem (NSHFP) for a homogeneous and isotropic body for linear thermoelaticity theory of type I, II and III, that involves $(\sigma_{ij},\, q_i)$ and satisfies the following field equations:

$$\rho^{-1}\sigma_{(ik,kj)} - \left[\frac{1}{2\mu}\left(\ddot{\sigma}_{ij} - \frac{\lambda}{3\lambda + 2\mu}\ddot{\sigma}_{kk}\delta_{ij}\right) - \frac{\theta_0 \alpha_t^2}{C_s}\ddot{\sigma}_{kk}\delta_{ij}\right] + \frac{\alpha_t}{C_s}\dot{q}_{k,k}\delta_{ij} = \frac{1}{C_s}\alpha_t \dot{r}\delta_{ij} - \rho^{-1} b_{(i,j)} \tag{2.2.1}$$

$$(\xi_1 k^* + \xi_2 k \frac{\partial}{\partial t})\frac{1}{C_s}(q_{k,k} + \alpha_t \theta_0 \dot{\sigma}_{k,k})_{,i} - \ddot{q}_i = -(\xi_1 k^* + \xi_2 k \frac{\partial}{\partial t})\frac{1}{C_s} r_{,i} \tag{2.2.2}$$

The material constants are assumed to satisfy the following conditions:

$$\rho > 0,\ C_s > 0,\ k > 0,\ k^* > 0,\ \mu > 0,\ 3\lambda + 2\mu > 0,\ \theta_0 > 0,\ \alpha_t > 0 \tag{2.2.3}$$

In equations (2.2.1) and (2.2.2), we set

$$\begin{aligned} F_{(ij)} &= \rho^{-1} b_{(i,j)} - \tfrac{1}{C_s}\alpha_t \dot{r}\delta_{ij} \\ g_i &= (\xi_1 k^* + \xi_2 k \tfrac{\partial}{\partial t})\tfrac{1}{C_s} r_{,i} \end{aligned} \tag{2.2.4}$$

Then, equations (2.2.1) and (2.2.2) reduce to

$$\rho^{-1}\sigma_{(ik,kj)} - \left[\frac{1}{2\mu}\left(\ddot{\sigma}_{ij} - \frac{\lambda}{3\lambda + 2\mu}\ddot{\sigma}_{kk}\delta_{ij}\right) - \frac{\theta_0 \alpha_t^2}{C_s}\ddot{\sigma}_{kk}\delta_{ij}\right] + \frac{\alpha_t}{C_s}\dot{q}_{k,k}\delta_{ij} = -F_{(ij)} \tag{2.2.5}$$

$$(\xi_1 k^* + \xi_2 k \frac{\partial}{\partial t})\frac{1}{C_s}(q_{k,k} + \alpha_t \theta_0 \dot{S}_{k,k})_{,i} - \ddot{q}_i = -g_i \tag{2.2.6}$$

The above equations (2.2.5)-(2.2.6) reduce to the system of basic governing equations under three different theories of Green and Naghdi of type-I, II and III as follows:

(i)Theory of Green-Naghdi (GN) Type-I : $\xi_1 = 0$, $\xi_2 = 1$, i.e. when $k^* << k$

(ii)Theory of Green-Naghdi (GN) Type-II : $\xi_1 = 1$, $\xi_2 = 0$, i.e. when $k^* >> k$

(iii)Theory of Green-Naghdi (GN) Type-III : $\xi_1 = 1$, $\xi_2 = 1$

Now, in addition to above equations, we adjoin the initial conditions and boundary conditions as follows:

$$\left.\begin{aligned} \sigma_{ij}(.,0) = \sigma_{ij}^0, \quad \dot{\sigma}_{ij}(.,0) = \dot{\sigma}_{ij}^0 \\ q_i(.,0) = q_i^0 \quad \dot{q}_i(.,0) = \dot{q}_i^0 \end{aligned}\right. \quad \text{on } V \tag{2.2.7}$$

$$\left.\begin{aligned} \sigma_{ij} n_j &= \sigma_i' \\ q_i n_i &= q' \end{aligned}\right. \quad \text{on } \partial V \times [0, \infty) \tag{2.2.8}$$

2.2.3 Some Definitions

Before going to our main results, we start with the definition of the support of thermomechanical load of the problem. Next, we define the domain of influence for a prescribed thermomechanical load.

Definition 1.

Let $t \in]0, \infty[$ be a fixed time. The set

$$D_0(t) = \left\{ \begin{array}{c} x \in \bar{V} \;: (1)\ \text{If}\ x \in V,\ \text{then}\ \sigma_{ij}^0 \neq 0\ \text{or}\ \dot{\sigma}_{ij}^0 \neq 0\ \text{or}\ q_i^0 \neq 0\ \text{or}\ \dot{q}_i^0 \neq 0 \\ (2)\ \text{if}\ (x,\tau) \in \partial V \times [0,t],\ \text{then}\ F_{(ij)} \neq 0\ \text{or}\ g_i \neq 0 \\ (3)\ \text{if}\ (x,\tau) \in \partial V \times [0,t],\ \text{then}\ \sigma_i^{'} \neq 0,\ \text{or}\ q^{'} \neq 0 \end{array} \right\} \qquad (2.2.9)$$

is called the support of thermomechanical loading at time t of the problem (2.2.5)-(2.2.8).

Definition 2.

Let v be a any real parameter with the dimension of velocity and let $\Omega(x, vt)$ be an open ball with radius vt and centered at x. The domain of influence for the thermomechanical load at time t for this problem constituted by equations (2.2.5)-(2.2.8) is defined by the set

$$D(t) = \{x \in \bar{V} \;:\; D_0(t) \cap \overline{\Omega(x, vt)} \neq \emptyset\} \qquad (2.2.10)$$

Therefore, $D(t)$ is a set of all the points of $\bar{V}$ that may be reached by the thermomechanical disturbances propagating from $D_0(t)$ with a speed not greater than v (2010).

2.2.4 Main Results

Here, we establish an inequality in the present context. Then on the basis of this inequality, we will establish a domain of influence theorem for the thermoelastic model of type GN-II which is valid if ν satisfies following inequality :

$$\nu \geq \max(\nu_1, \nu_2, \nu_3) \tag{2.2.11}$$

where

$$\nu_1 = \left(\frac{2\mu}{\rho}\right)^{\frac{1}{2}} \tag{2.2.12}$$

$$\nu_2 = \left\{\frac{3\lambda + 2\mu}{\rho}\frac{C_s}{C_E}\left[1 - \left(1 - \frac{C_E}{C_s}\right)^{\frac{1}{2}}\right]^{-1}\right\}^{\frac{1}{2}} \tag{2.2.13}$$

$$\nu_3 = \left\{\frac{k^*}{C_s}\left[1 + \frac{C_s}{C_E}\left(1 - \frac{C_E}{C_s}\right)^{\frac{1}{2}}\right]\right\}^{\frac{1}{2}} \tag{2.2.14}$$

where C_E is the specific heat at zero strain which is related to C_s, specific heat at zero stress by

$$C_s = C_E + 3\theta_0(3\lambda + 2\mu)\alpha^2 \tag{2.2.15}$$

Now, in view of the definitions as given above, we will establish the following theorem for deriving a general energy identity for the present problem. The energy identity in the contexts of GN-I, GN-II and GN-III can be obtained as special cases from this identity.

Theorem – 2.2.1. Let (σ_{ij}, q_i) be a smooth solution to the mixed boundary-initial value problem (2.2.5)-(2.2.8) and $p(x)\in C^1(\bar{V})$ denote a scalar field such that the set

$$E_0 = \{x \in \bar{V} : \, p(x) \, > 0\} \tag{2.2.16}$$

is bounded there. Then

$$\frac{1}{2}\int\limits_V \Big\{P(x,p(x)) - P(x,0)\Big\}dV + \int\limits_V \int\limits_0^{p(x)} Q(x,t)dtdV + \int\limits_V R_i(x,p(x))p_{,i}(x)dV$$
$$= \int\limits_V \int\limits_0^{p(x)} M(x,t)dtdV + \int\limits_{\partial V} \int\limits_0^{p(x)} R_i(x,t)n_i(x)dtdA \quad (2.2.17)$$

where

$$P(x,t) = \rho^{-1}\hat{\sigma}_{ik,k}\hat{\sigma}_{ij,j} + \frac{1}{2\mu}\left(\dot{\hat{\sigma}}_{ij}\dot{\hat{\sigma}}_{ij} - \frac{\lambda}{3\lambda+2\mu}(\dot{\hat{\sigma}}_{kk})^2\right) - \frac{\theta_0\alpha_t^2}{C_s}(\dot{\hat{\sigma}}_{kk})^2 + \frac{1}{\theta_0 C_s}(\hat{q}_{k,k})^2 + \frac{\xi_1 k^*}{\theta_0}(\dot{q}_i)^2 \quad (2.2.18)$$

$$Q(x,t) = \frac{\xi_2 k}{\theta_0}(\ddot{q}_i)^2 \quad (2.2.19)$$

$$R_i(x,t) = \rho^{-1}\dot{\hat{\sigma}}_{ij}\hat{\sigma}_{jk,k} + \frac{\xi_1 k^*}{C_s}\left(\frac{1}{\theta_0}\hat{q}_{k,k} + \alpha_t\dot{\hat{\sigma}}_{kk}\right)\dot{q}_i + \frac{\xi_2 k}{C_s}\left(\frac{1}{\theta_0}\hat{q}_{k,k} + \alpha_t\dot{\hat{\sigma}}_{kk}\right)\ddot{q}_i \quad (2.2.20)$$

$$M(x,t) = F_{(ij)}\dot{\hat{\sigma}}_{ij} + \frac{1}{\theta_0}g_i\dot{\hat{q}}_i \quad (2.2.21)$$

and we denote $\hat{f}$ for any function $f = f(x,t)$ defined on $x \in \bar{V} \times [0,\infty)$ as

$$\hat{f} = \left(\xi_1 k^* + \xi_2 k\frac{\partial}{\partial t}\right) f \quad (2.2.22)$$

Proof : First by applying the hat operator as defined by equation (2.2.22) on equation (2.2.5) and then multiplying through $\dot{\hat{\sigma}}_{ij}$ and then multiplying equation (2.2.6) through $\theta^{-1}\dot{\hat{q}}_i$ and adding the results, we get

$$\frac{1}{2}\frac{\partial}{\partial t}P(x,t) + Q(x,t) = \left[R_i(x,t)\right]_{,i} + M(x,t) \quad (2.2.23)$$

where P, Q, R_i and M are given by equations (2.2.18)-(2.2.21), respectively.

We can write

$$\int_0^{p(x)} R_{i,i}(x,t)dt = \left[\int_0^{p(x)} R_i(x,t)dt\right]_{,i} - R_i(x,\, p(x))p_{,i}(x) \tag{2.2.24}$$

Now integrating equation (2.2.23) from $t = 0$ up to $t = p(x)$, and using the result from equation (2.2.24), we get

$$\frac{1}{2}[P(x,p(x)) - P(x,0)] + \int_0^{p(x)} Q(x,t)dt + R_i(x,p(x))p_{,i}(x) = \int_0^{p(x)} M(x,t)dt + \left[\int_0^{p(x)} R_i(x,t)dt\right]_{,i} \tag{2.2.25}$$

Since the set E_0 defined by equation (2.2.16) is bounded, hence each term in equation (2.2.25) has a bounded support. Therefore integrating equation (2.2.25) over V and using divergence theorem, we obtain equation (2.2.17). This completes the proof of theorem-1.

Spacial case:

Now, we will formulate the theorem of domain of influence for GN-II model stating that the thermomechanical load restricted to the interval $[0,\, t]$ does not influence the points outside the domain $D(t)$. For this, we consider the case GN-II by using $\xi_1 = 1$, $\xi_2 = 0$, and substitute in equations (2.2.17). Hence, (2.2.17) reduces to

$$\frac{1}{2}\int_V \{P(x,p(x)) - P(x,0)\}dV + \int_V R_i(x,p(x))p_{,i}(x)dV$$
$$= \int_V \int_0^{p(x)} M(x,t)dtdV + \int_{\partial V} \int_0^{p(x)} R_i(x,t)n_i(x)dtdA \tag{2.2.26}$$

where

$$P(x,t) = (k^*)^2\rho^{-1}\sigma_{ik,k}\sigma_{ij,j} + \frac{(k^*)^2}{2\mu}\left(\dot{\sigma}_{ij}\dot{\sigma}_{ij} - \frac{\lambda}{3\lambda+2\mu}(\dot{\sigma}_{kk})^2\right) - \frac{\theta_0\alpha_t^2(k^*)^2}{C_s}(\dot{\sigma}_{kk})^2 + \frac{(k^*)^2}{\theta_0 C_s}(\hat{q}_{k,k})^2 + \frac{k^*}{\theta_0}(\dot{q}_i)^2 \tag{2.2.27}$$

$$R_i(x,t) = (k^*)^2\rho^{-1}\dot{\sigma}_{ij}\sigma_{jk,k} + \frac{(k^*)^2}{C_s}\left(\frac{1}{\theta_0}q_{k,k} + \alpha_t\dot{\sigma}_{kk}\right)\dot{q}_i \tag{2.2.28}$$

$$M(x,t) = k^* F_{(ij)}\dot{\sigma}_{ij} + \frac{k^*}{\theta_0}g_i\dot{q}_i \tag{2.2.29}$$

The domain of influence theorem for the theory of GN-II is therefore established in the next theorem.

Theorem−2.2.2. Let ν be a real number that satisfies the inequality given by equation (2.2.11), where ν_1, ν_2, ν_3 are given by equations (2.2.12)-(2.2.14). Then if the pair $(\sigma_{ij},\, q_i)$ is a smooth solution of problem (2.2.5)-(2.2.8) for GN-II model and if $D(t)$ is the domain of influence for thermomechanical load $D_0(t)$ at time t, then

$$\sigma_{ij} = 0, \;\; q_i = 0 \;\; \text{on} \;\; \left\{\bar{V} - D(t)\right\} \times [0,t] \tag{2.2.30}$$

Proof : Let $(z,\tau) \in \{V - D(t)\} \times (0,\, t)$ be a fixed point. Let

$$\Gamma = \bar{V} \cap \overline{\Omega(z, v\tau)} \tag{2.2.31}$$

and let

$$p_\tau(x) = \begin{cases} \tau - \frac{1}{v}|x-z| & \text{for}\, x \in \Gamma \\ 0 & \text{for}\, x \notin \Gamma \end{cases} \tag{2.2.32}$$

where v is a parameter defined by the inequality (2.2.11).

Using the definitions of domain $D(t)$ and Γ and the inequality $\tau < t$, we conclude that $D_0(t)$ and Γ are disjoint, that is

$$\Gamma \cap D_0(t) = \emptyset \tag{2.2.33}$$

Hence

$$\sigma_{ij} n_i = 0, \;\; q_i n_i = 0 \text{ on } (\Gamma \cap \partial V) \times [0, t] \tag{2.2.34}$$

and

$$\dot{\sigma}_{ij} n_j = 0, \;\; \dot{q}_i n_i = 0 \text{ on } (\Gamma \cap \partial V) \times [0, t] \tag{2.2.35}$$

$$F_{(ij)} = 0, \;\; g_i = 0 \;\; \text{on } \Gamma \times (0, t) \tag{2.2.36}$$

Furthermore,

$$\sigma_{ij}(x, 0) = \dot{\sigma}_{ij}(x, 0) = q_i(x, 0) = \dot{q}_i(x, 0) = 0 \text{ on } \Gamma \tag{2.2.37}$$

as well as from (2.2.28), (2.2.34) and (2.2.35), we get

$$\int\limits_{\partial V} \int\limits_{0}^{p(x)} R_i(x, t) n_i(x) dt dA = 0 \tag{2.2.38}$$

Further, from (2.2.29), (2.2.36) and (2.2.37), we get

$$\int\limits_{V} \int\limits_{0}^{p(x)} M(x, t) dt dV = 0 \tag{2.2.39}$$

Next, by using the definition of $P(x, t)$ and $p_\tau(x)$ from (2.2.27) and (2.2.32), we get

$$P(x, p_\tau(x)) - P(x, 0) = \begin{cases} P(x, p_\tau(x)) & \text{for } x \in \Gamma \\ 0 & \text{for } x \notin \Gamma \end{cases} \tag{2.2.40}$$

Now, substituting $p_\tau(x)$ into equation (2.2.26) and using equations (2.2.38) and (2.2.39), we find

$$\frac{1}{2}\int_{\Gamma} P(x,p_\tau(x))dV = -\int_{\Gamma} R_i(x,p_\tau(x))p_{\tau,i}(x) \tag{2.2.41}$$

In view of (2.2.32) and (2.2.41), we get

$$\frac{1}{2}\int_{\Gamma} P(x,p_\tau(x))dV \leq \frac{1}{v}\int_{\Gamma} |R_i(x,p_\tau(x))|dV \tag{2.2.42}$$

Now

$$\frac{1}{\nu} \mid R_i \mid \leq (k^*)^2\rho^{-1} \mid \frac{\dot{\sigma}_{ij}}{\nu} S_{jk,k} \mid + \frac{(k^*)^2}{C_s} \mid \frac{1}{\theta_0} q_{k,k} + \alpha_t \dot{\sigma}_{kk} \mid\mid \frac{\dot{q}_i}{\nu} \mid$$

$$\leq (k^*)^2\rho^{-1} \mid \frac{\dot{S}_{ij}}{\nu} \mid\mid \sigma_{jk,k} \mid + \frac{(k^*)^2}{C_s\theta_0} \mid q_{k,k} \mid\mid \frac{\dot{q}_i}{\nu} \mid + \frac{(k^*)^2}{C_s} \mid \alpha_t \mid\mid \dot{\sigma}_{kk} \mid\mid \frac{\dot{q}_i}{\nu} \mid \tag{2.2.43}$$

In order to simplify the inequality (2.2.43), we estimate the terms of the right-hand side of equation (2.2.43) by using the relation

$$\sqrt{xy} \leq (\epsilon x + \epsilon^{-1} y) \tag{2.2.44}$$

where x and y are non-negative physical fields having the same dimension, and ϵ is a dimensionless positive parameter.

By setting $\epsilon = 1$, $x = (\sigma_{jk,k})^2$ and $y = \left(\frac{\dot{\sigma}_{ij}}{\nu}\right)^2$ in equation (2.2.44) we estimate the first term of equation (2.2.43) as

$$\mid \sigma_{jk,k} \mid\mid \frac{\dot{\sigma}_{ij}}{\nu} \mid \leq \frac{1}{2}\left(\sigma_{ij,j}\sigma_{ik,k} + \frac{1}{\nu^2}\dot{\sigma}_{ij}\dot{\sigma}_{ij}\right) \tag{2.2.45}$$

Similarly, by setting $\epsilon = 1$, $x = (q_{k,k})^2$ and $y = (\frac{\dot{q}_i}{\nu})^2$, in equation (2.2.44), we obtain the estimate for the second term of (2.2.43) as

$$\mid q_{k,k} \mid\mid \frac{\dot{q}_i}{\nu} \mid \leq \frac{1}{2}\left((q_{k,k})^2 + \frac{1}{\nu^2}(\dot{q}_i)^2\right) \tag{2.2.46}$$

To estimate the last term of the right-hand side of inequality (2.2.43), we use

$$x = (\dot{\sigma}_{kk})^2, \; y = \frac{1}{\nu^2(\alpha_t\theta_0)^2}(\dot{q}_i)^2, \; \epsilon = \frac{C_E}{C_s}\left(1 - \frac{C_E}{C_s}\right)^{-\frac{1}{2}} \tag{2.2.47}$$

Hence, by using (2.2.47) in (2.2.44), we obtain

$$\mid \dot{\sigma}_{kk} \mid \frac{1}{\nu \mid \alpha_t \mid \theta_0} \mid \dot{q}_i \mid \leq \frac{1}{2}\left\{\frac{C_E}{C_s}\left(1 - \frac{C_E}{C_s}\right)^{-\frac{1}{2}}(\dot{\sigma}_{kk})^2 + \frac{C_s}{C_E}\left(1 - \frac{C_E}{C_s}\right)^{\frac{1}{2}}\frac{1}{\nu^2\alpha_t^2\theta_0^2}(\dot{q}_i)^2\right\} \tag{2.2.48}$$

Therefore, in view of equations (2.2.43), (2.2.45), (2.2.46) and equation (2.2.48), we get

$$\begin{aligned}\frac{1}{\nu} \mid R_i \mid \leq \frac{(k^*)^2}{2}&\left\{\rho^{-1}\left(\sigma_{ij,j}\sigma_{ik,k} + \frac{1}{\nu^2}\dot{\sigma}_{ij}\dot{\sigma}_{ij}\right) + \frac{1}{C_s\theta_0}\left((q_{k,k})^2 + \frac{1}{\nu^2}(\dot{q}_i)^2\right)\right\} \\ &+ \frac{\alpha_t^2\theta_0(k^*)^2}{2C_s}\left\{\frac{C_E}{C_s}\left(1 - \frac{C_E}{C_s}\right)^{-\frac{1}{2}}(\dot{\sigma}_{kk})^2 + \frac{C_s}{C_E}\left(1 - \frac{C_E}{C_s}\right)^{\frac{1}{2}}\frac{1}{\nu^2\alpha_t^2\theta_0^2}(\dot{q}_i)^2\right\}\end{aligned} \tag{2.2.49}$$

Further, from the relation (2.2.15), we have

$$\frac{\alpha_t^2\theta_0}{C_s} = \frac{1}{3(3\lambda + 2\mu)}\left(1 - \frac{C_E}{C_s}\right) \tag{2.2.50}$$

Finally, from the inequality (2.2.42) and (2.2.49) and by using the relation (2.2.50), we find

$$\begin{aligned}&\left(\frac{1}{2\mu} - \frac{1}{\rho\nu^2}\right)\int_{\Gamma}\left(\dot{\sigma}_{ij} - \frac{1}{3}\dot{\sigma}_{kk}\delta_{ij}\right)\left(\dot{\sigma}_{ij} - \frac{1}{3}\dot{\sigma}_{kk}\delta_{ij}\right)dV \\ &\quad + \frac{1}{3}\left[\frac{1}{(3\lambda + 2\mu)}\frac{C_E}{C_s}\left\{1 - \left(1 - \frac{C_E}{C_s}\right)^{\frac{1}{2}}\right\} - \frac{1}{\rho\nu^2}\right]\int_{\Gamma}(\dot{\sigma}_{kk})^2 dV \\ &\quad + \frac{1}{\theta_0}\left[\frac{1}{k^*} - \frac{1}{C_s\nu^2}\left\{1 + \frac{Cs}{C_E}\left(1 - \frac{C_E}{C_s}\right)^{\frac{1}{2}}\right\}\right]\int_{\Gamma}(\dot{q}_i)^2 dV \leq 0\end{aligned} \tag{2.2.51}$$

Now, equation (2.2.11) implies that the coefficients of all the integrals of equation (2.2.51) are non negative. Therefore, all integrals being non negative as well, in equation (2.2.51) we find that the equality sign must hold. This implies the vanishing of each term of (2.2.51) in Γ. In particular, we have

$$\dot{\sigma}_{ij}(x, p_\tau(x)) = 0,\ \dot{q}_i(x, p_\tau(x)) = 0 \quad \text{on } \Gamma \tag{2.2.52}$$

Now, definition of $p_\tau(x)$ and smoothness property of $(\dot{\sigma}_{ij}, \dot{q}_i)$ imply that

$$\left.\begin{array}{c} \dot{\sigma}_{ij}(x, p_\tau(x)) \to \dot{\sigma}_{ij}(z, \tau) \\ \dot{q}_i(x, p_\tau(x)) \to \dot{q}_i(z, \tau) \end{array}\right\} \text{as } x \to z \tag{2.2.53}$$

Hence, taking limit of the equation (2.2.52) as $x \to z$, we get from equation (2.2.32)

$$\dot{\sigma}_{ij}(z, \tau) = 0\,, \dot{q}_i(z, \tau) = 0 \ \text{ on } \{\bar{V} - D(t)\} \times [0, t] \tag{2.2.54}$$

Since (z, τ) is an arbitrary point of $\{\bar{V} - D(t)\} \times (0, t)$ and since (σ_{ij}, q_i) is sufficiently smooth in $\bar{V} \times [0, \infty)$, hence from equation (2.2.54), we conclude that

$$\dot{\sigma}_{ij} = 0, \quad \dot{q}_i = 0 \ \text{ on } \{\bar{V} - D(t)\} \times [0, t] \tag{2.2.55}$$

But, from the definition of $D(t)$, we get

$$\sigma_{ij}(x, 0) = 0,\ q_i(x, 0) = 0 \text{ on } \{\bar{V} - D(t)\} \tag{2.2.56}$$

So integrating equation (2.2.55) over the time and from equation (2.2.56), we finally obtain

$$\sigma_{ij} = 0, \quad q_i = 0 \text{ on } \{\bar{V} - D(t)\} \times [0, t] \tag{2.2.57}$$

Thus, theorem- 2.2.2 is proved.

This theorem signifies that the stress-heat-flux disturbances described by equations (2.2.5) to (2.2.8) propagate in the thermoelastic body with a speed not greater than ν represented by equation (2.2.11). Here, ν is totally dependent on the thermoelastic parameters including k^*, the material characteristic of Green-Naghdi theory. This theory further implies that the associated domain of influence of a boundary load at any time t is identified with a boundary layer of the thickness vt. From this domain of influence theorem we can estimate the order of magnitude of ν and also the upper bound of the speed of sterss-heat-flux disturbances.