CHAPTER 1

INTRODUCTION AND REVIEW OF LITERATURE

1.1 Introduction

1.1.1 Basic Definitions

In order to discuss on coupled dynamical thermomechanical theory we need to describe average properties of material without reference to the detailed atomic structure. The study that takes account of the existence of gaps is called microscopic study. On the other hand, the study that ignores the gaps and treats a physical object as a continuous distribution of matter is called macroscopic study. A ***continuum*** is a medium with a geometric configuration in the Euclidean space which exhibits continuous macroscopic properties. The neighboring molecules remain neighboring under the action of any loading conditions and no geometric discontinuity may occur. A part of a continuum which is separated from the rest of the continuum is called a ***system***. A closed system does not exchange matter with its surroundings. If a system does not interact with the surroundings, it is called an isolated system. The macroscopic characteristics of the system defined by means of statistical averaging procedures are called ***state variables*** and for a system they are the density, specific volume, specific pressure, temperature, velocity, enthalpy, entropy, and internal energy. For a thermodynamic process to proceed from a given state to a final state, the functional relationship between the properties must be known. When a property of the system is expressed by a single-valued mathematical relationship to a set of other system properties, the mathematical relation is called the ***equation of state***.

There are two type of forces acting on the continuous body. They can be categorized as internal forces or external forces. Internal forces are due to interaction between the constituent particles of a continuum whereas external forces are exerted by the external agencies. Whenever there is a change in the relative position of the particle of a body, it is said that body is deformed or strained and the transformation which cause such change is called ***deformation***. A continuum solid when exhibits elastic property, i.e., when it is subjected to external loads gets deformed and returns to its original shape and size on removal of external forces, is called ***elastic body.*** The deformation in a continuum is mainly due to two external forces (a) Body forces and (b) Surface forces. A ***body force*** is a force that acts throughout the volume of a body. Forces due to gravity, electric fields and magnetic fields are examples of body forces. ***Surface force*** always acts on a surface and results from physical contact with another body. Surface force can be decomposed into two perpendicular components: normal forces and shear forces. A normal force acts normally over an area and a shear force acts tangentially over an area.

1.1.2. Stress, Strain and Hook's Law

External loads are cause of internal forces. Internal forces represent the action of one part of a material on another part of the same material across an internal surface. The force per unit area set up inside the body to resist deformation is called ***stress***. We define it as

$$\sigma = \frac{Force}{Area}$$

There are two types of stress, if the resisting area is perpendicular to the applied force, then such type of stresses are known as ***normal stress***. Forces parallel to the area resisting the force cause ***shearing stress.*** All materials have a complex molecular micro-structure and each molecule exerts a force on each of its neighbours. The complex interaction of countless molecular forces maintain a body in equilibrium in its unstressed state. If we take a plane of surface area S through the point, on which a force F is acted and shrink the plane, it shrinks in size and both S and F get smaller. The direction in which the force acts may change, but eventually the ratio $\frac{F}{S}$ will remain constant and the force will act in a particular direction. The limiting value of this ratio of force over surface area is defined as the traction vector (or stress vector).

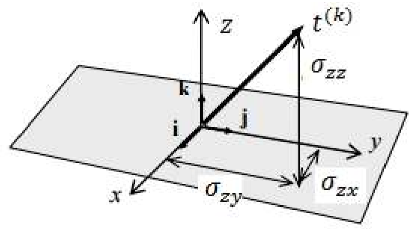
$$T^n = \lim_{\Delta s \to 0} \frac{\Delta F}{\Delta S}$$

An infinite number of traction vectors act at any single point, since an infinite number of different planes pass through a point. For this reason the plane on which the traction vector acts must be specified, and this can be done by specifying the normal, $\vec{n}$ to the surface on which the traction acts. Thus the traction vector can be decomposed into components which act normal and parallel to the surface upon which it acts. These components are called the ***stress components***, and are denoted by the symbol σ_{ij}. Subscripts are added to signify the surface on which the stresses act and the directions in which the stresses act. The first subscript denotes the direction of the normal to the plane and the second denotes the direction of the component. For example, let $\vec{k}$ direction is taken to be the normal to the plane then traction can be represented as:

$$T^{(k)} = \sigma_{zx}\vec{i} + \sigma_{zy}\vec{j} + \sigma_{zz}\vec{k}$$

The first two stresses, the components acting tangential to the surface, are shear stresses, whereas, σ_{zz} acting normal to the plane, is a normal stress. This is Cauchy's formula relating the traction vector to the stresses. Generalizing Cauchy's formula leads to the following:

$$T_i^{(n)} = \sigma_{ji} n_j$$



This relation insures that the given stress tensor at a point, fully defines the components of the traction vector on any given orientation about the point. The stress is positive when the direction of the normal and the direction of the stress component are both positive and both negative. The stress is negative when one of the direction is positive and the other is negative.

Strain is the ratio of the change in dimension caused by the applied force, to the original dimension, i.e.

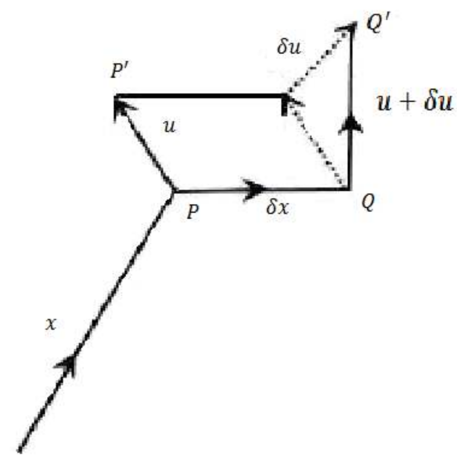
$$\epsilon = \frac{\delta}{L}$$

Here, δ is the deformation and L is the original length. Thus, strain (ϵ) is dimensionless. There are also different types of strain like tensile strain, compressive strain, shear strain, volumetric strain. Let we consider particle motion of two neighbouring points; P and Q in the medium for describing the deformation. Both points are separated in the undeformed medium by δx. The point P is deformed to P' by u. The point Q moves to Q' and the motion is $u + \delta u$. The motion u is a uniform translation and is described by the rigid body motion.

We write the first-order deformation as:

$$\delta u_i = \frac{\partial u_i}{\partial x_j} \delta x_j$$

Now, we will separate the gradient into symmetric and anti-symmetric parts, i.e.



$$\delta u_i = e_{ij}\delta x_j - \xi_{ij}\delta x_j$$

where

$$e_{ij} = \frac{1}{2}(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}) = e_{ji}$$

$$\xi_{ij} = \frac{1}{2}(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}) = -\xi_{ji}$$

The three independent (as skew symmetric), non-zero components of ξ_{ij} can be considered as a vector

$$\delta\theta = \begin{bmatrix} \xi_{23} \\ \xi_{31} \\ \xi_{12} \end{bmatrix}$$

and if $e_{ij} = 0$, the deformation can be written as

$$\delta u = \delta\theta \times \delta x$$

The rotation of Q about P given by ξ_{ij} does not deform the medium. It is described by the rigid body rotation of the medium. The total motion of the medium is described by rigid body translation and rotation, and deformation. So by default, the component e_{ij}, given above, must represent the deformation of the medium. Thus, the terms e_{ij} are the Cartesian components of the strain tensor e and the strain tensor in matrix form is given by

$$e = [e_{ij}] = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix}$$

Hook's law:

The relations that characterize the physical properties of materials are called constitutive relations. Stress and strain are related in elastic media by stress-strain constitutive relation which is also known as Hook's Law. This law states that the stress on a solid substance is directly proportional to the strain produced, provided the stress is less than the elastic limit of the substance. It is expressed in standard tensor notation as

$$\sigma_{ij} = C_{ijkl} e_{kl}$$

where, C_{ijkl} is a fourth-order tensor whose components include all the material parameter necessary to characterize the material and are called as elastic moduli. They have units of stress. In general, the fourth-order tensor C_{ijkl} has 81 components, But due to symmetry of the stress and strain tensor i.e. $C_{ijkl} = C_{jikl}$ and $C_{ijkl} = C_{ijlk}$, it reduces to 36. The number of independent elastic constants reduced further to 21 if there exists a strain energy density function and also we address further the issue of material homogeneity and isotropy. If the material is homogeneous, then the elastic property does not vary spatially. Similar to homogeneity, isotropy is another fundamental property of solid material, which deals with difference in material moduli with respect to orientation. Materials like, crystalline minerals, wood, and fiber-reinforced composites have different elastic moduli in different directions. Such materials are said to be anisotropic. For most real anisotropic materials there exists particular directions where the properties are the same. These directions indicate material symmetries. However for many engineering materials (most structural metals and many plastics), the orientation of crystalline and grain microstructure is distributed randomly so that macroscopic elastic properties are found to be essentially the same in all directions. Such materials with complete symmetry are called ***isotropic***. It can be shown that (see Chandrasekhariah and Dednath (1994)) the most general form that satisfies this isotropy condition is given by

$$C_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}$$

where α, β and γ are arbitrary constants. Using this relation, stress-strain relation reduces to

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij}$$

where we replaced particular constants using λ and μ. The elastic constants λ and μ are called as ***Lame's constants.*** μ is also referred as the ***shear modulus of rigidity***. These relations are called the generalized Hook's law for linear isotropic elastic solids.

1.1.3 Equation of Motion

We consider an elastic body on which an arbitrary traction surface force T^n and the body force X per unit volume have been acted. We assume that the body occupies the volume V and is bounded by an exterior surface A at time t. The resulting force acting on the body is

$$F_i = \int\limits_A T_i^n dA + \int\limits_V X_i dV \tag{1.1}$$

We have Cauchy's formula as

$$T_i^{(n)} = \sigma_{ji} n_j \tag{1.2}$$

Gauss theorem can be used to transform the surface integral of the traction force to the volume integral as

$$\int\limits_A T_i^n dA = \int\limits_A \sigma_{ji} n_j dA = \int\limits_V \sigma_{ji,j} dV \tag{1.3}$$

Thus, the total force acting on the body is, in components

$$F_i = \int\limits_V (X_i + \sigma_{ji,j}) dV \tag{1.4}$$

Now defining the linear momentum by

$$\mathcal{P}_i = \int\limits_V \rho \dot{u}_i dV \tag{1.5}$$

Newton Law of motion states that

$$F_i = \dot{\mathcal{P}}_i \tag{1.6}$$

Now substituting equations (1.4) and (1.5) into equation (1.6), we get

$$\int_V (X_i + \sigma_{ji,j})dV = \int_V \rho \ddot{u}_i dV \tag{1.7}$$

Since the volume V is arbitrary, equation (1.7) reduces to the following equation of motion:

$$\sigma_{ji,j} + X_i = \rho \ddot{u}_i \tag{1.8}$$

1.1.4 Heat Conduction and Fourier's Law

Heat is a form of energy which is transferred through the boundaries of a system or control volume due to the difference of the temperature with surroundings. The heat transferred to a system or control volume is considered positive, and heat transferred from a system or control volume is considered negative. Heat can only be transferred through three means: conduction, convection and radiation. ***Heat conduction*** is movement of heat from one solid to another one having different temperatures. Conduction is the most significant means of heat transfer within a solid or between solid objects in thermal contact. When several systems are in thermal equilibrium, a single-valued function which is identical between the systems must describe the thermal equilibrium and identify the quality of the equilibrium. This function may be called ***temperature*** and it is identical among systems in thermal equilibrium with each other. Now we define the term heat flux which is the basic term used in defining Fourier law. Heat flux is defined as the amount of heat transferred per unit area per unit time from or to a surface. In another way, the heat transfer (Q) per unit area (A) normal to the direction of heat transfer when area tends to zero is called ***heat flux*** i.e. $\vec{q} = \underset{A \to 0}{lim} \frac{Q}{A}$.

Fourier's law of heat conduction: In 1822, Joseph Baptiste Fourier, a French scientist, pointed out his most famous work named "Analytical Theory of Heat" and proposed the famous law of heat conduction and called as Fourier Law (Fourier (1952)).

It must also be mentioned that Fourier law of heat conduction was the first constitutive relation of heat flux which states that time rate of heat transfer through a material is proportional to the negative temperature gradient through which heat flows. i.e.

$$\vec{q}(\vec{r},t) = -k\vec{\nabla}\theta(\vec{r},t) \tag{1.9}$$

Here $\vec{q}$ is the heat flux vector, $\vec{\nabla}$ is gradient operator, θ is temperature and k is thermal conductivity of a material. This is very simple empirical law that has been widely used to explain heat transport phenomena appearing often in daily life, engineering applications and scientific research.

1.1.5 Thermoelasticity

Thermoelasticity deals with the dynamical system whose interactions with the surrounding include not only mechanical work and external work but the exchange of heat also. Changes in temperatures causes thermal effects on materials. Some of these thermal effects include thermal stress, strain, and deformation. Thermal deformation simply means that as the "thermal" energy (and temperature) of a material increases, so does the vibration of its atoms/molecules and this increased vibration results in what can be considered a stretching of the molecular bonds - which causes the material to expand. Again if the thermal energy (and temperature) of a material decreases, the material will shrink or contract. Thus, thermoelasticity is based on temperature changes induced by expansion and compression of the test part. The theory of thermoelasticity is concerned with predicting the thermomechanical behavior of elastic solids. It represents a generalization of both the theory of elasticity and theory of heat conduction in solids. Initially, the investigations in this area were based on the “uncoupled theory of thermoelasticity” with the simplifying assumption that the influence of the strain and stresses on the temperature field may be neglected. The classical “uncoupled theory of thermoelasticity” is therefore affected by the shortcoming that the elastic change have no effect on the temperature and vice versa. However the "coupled theory of thermoelasticity" includes the coupling between the thermal and strain fields.

1.1.5.1 Classical coupled theory of thermoelasticity: General Theory

According to the coupled theory of thermoelasticity, the non-uniform elevated temperature fields may be developed due to the thermomechanical coupling, external heating and the internal energy dissipation. The stress system in the solid therefore gets modified with a temperature gradient term pertaining to these effects included in the basic theory. It is worth to recall that the theory of thermoelasticity was founded in 1838 by Duhamel (1837, 1838), who derived the equations for the strain in an elastic body with temperature gradients. Neumann, also have found the same results in 1841. However, the theory was based on independence of the thermal and mechanical effects. The total strain was determined by superimposing the elastic strain and the thermal expansion caused by the temperature distribution only. The theory thus did not describe the motion associated with the thermal state, nor did it include the interaction between the strain and the temperature distributions. Hence, thermodynamic arguments were needed, and it was Thomson, in 1857 who first used the laws of thermodynamics to determine the stresses and strains in an elastic body in response to varying temperatures. The first satisfactory formulation of the dynamical version of the coupled thermoelasticity theory by taking into account the coupling between thermal and strain fields was given by Biot in the year 1956. He also presented the fundamental methods for solving the thermoelasticity equations and established variational theorem. The following basic equations represent the Biot's coupled thermoelasticity theory for isotropic medium:

Strain-displacement relation:

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{1.10}$$

Equations of compatibility:

$$e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jlik} = 0, \quad i, j, k, l = 1, 2, 3 \tag{1.11}$$

Equation of motion:

$$\sigma_{ij,j} + h_i = \rho \ddot{u}_i \tag{1.12}$$

Stress-strain-temperature relation:

$$\sigma_{ij} = 2\mu e_{ij} + (\lambda e_{kk} - \beta\theta)\delta_{ij} \tag{1.13}$$

Law of heat conduction:

$$q_i = -k\theta_{,i} \tag{1.14}$$

Energy equation:

$$q_{i,i} = \rho(R - \dot{S}\theta_0) \tag{1.15}$$

Entropy equation:

$$\rho S = \frac{\rho c_E}{\theta_0}\theta + \beta e_{kk} \tag{1.16}$$

Here, R is the strength of the internal heat source per unit mass in an arbitrary material of mass density ρ. Biot's coupled thermoelasticity theory was based on the firm ground of irreversible thermodynamics. Renowned researchers like, Chadwick(1960), Boley and Weiner(1960), Nowacki (1962,1975), Parkus(1976), Dhaliwal and Singh(1980), Chandrasekharaiah (1986) have produced extensive and elaborated analysis with amusing applications and theorems based on the Biot's theory.

1.1.6 Shortcoming of Fourier law and 'second sound effect'

According to Fourier law described by equation (1.9), we say that heat flow occurs at a position r and at time t simultaneously with the establishment of temperature gradient at the same position and at the same instant of time. It means that heat flux is the instantaneous effect of a temperature gradient. Combining equation (1.9) with the energy equation

$$\rho c_E \dot{\theta} = -\vec{\nabla}.\overrightarrow{q} + r \tag{1.17}$$

where r is the external heat source, c_E is the specific heat at constant strain and ρ is the mass density, yields the corresponding heat conduction equation as

$$k\nabla^2\theta = \rho c_E \dot{\theta} - r \tag{1.18}$$

This heat conduction law has been extensively and successfully employed to the conventional engineering heat conduction problems which involve large spatial dimension and when the focus is on long time behavior of the system. However, due to the intrinsic parabolic nature of the partial differential equation (1.18), the diffusion of heat gives rise to infinite speeds of heat propagation. This conclusion, called by some authors the paradox of instantaneous heat propagation, is not physically reasonable because it clearly violates one important principle of the Einstein's special theory of relativity: The velocity of light in vacuum is the greatest known speed and has finite value of $2.998 \times 10^8 m/s$. Fourier law of heat conduction is also in contradiction for situations involving temperature near absolute zero, extreme thermal gradients, high heat flux conduction and short time behavior. Also with the rapid development of nanotechnology, many nano-scale devices have been developed. The heat conduction of these small devices also shows many distinct phenomena such as the size effect and wave phenomena, which are not described by the conventional Fourier law. All these made a significant impact of research activity in the area of heat conduction theory, and it drew attention of researchers towards the form of Fourier law. It is worth to recall here that in 1867, while carrying out some experiment on kinetic theory of gases, Maxwell postulated for the first time that the thermal disturbance is a wave like phenomena rather than diffusion phenomena and suggested for the modification in Fourier law. The wave-type heat flow is later called as 'second Sound' effect while the first sound is the usual sound (wave). Nerst in the year 1917 found the possibility of the occurrence of temperature waves in good conductor at low temperatures. The theories involving the 'second sound' effect were also motivated by experiments and exhibited the actual occurrence of second sound at low temperatures and for small time intervals. In liquid helium, Tisza (1947) speculated the possibility of small heat propagation rates and Landau (1941) predicted the 'second sound' as the propagation of phonon density disturbance in super fluid helium. Pesshkov (1944) experimentally first detected the 'second sound' in liquid helium. Afterward many other scientists like, Maurer and Herlin (1949), Pellam and Scott (1949) Atkins and Osborne (1950) also detected this experimentally. Subsequently, second sound was also observed by Ackerman *et al.* (1966), Ackerman and Overton (1969), Bertman and Standiford (1970), Jackson and Walker (1971), Rogers (1971) etc. in other elements.

1.1.7 Generalized thermoelasticity

Biot (1956) introduced the theory of coupled thermoelasticity to overcome the shortcoming of classical thermoelasticity theory. The governing equations for this theory are coupled, and eliminated the paradox of classical theory. However, it has been realized that this theory further suffers from the another shortcoming since the heat equation for the coupled theory is also parabolic type and indicates an infinite speed for thermal signal. Generalized thermoelasticity theories have been developed with the objective of removing the paradox of infinite speeds of heat propagation inherent in the conventional coupled dynamical theory of thermoelasticity. The improved theories make use of modified Fourier law of heat conduction and lead to the governing equations of the hyperbolic type which admit finite speeds of thermoelastic disturbances. According to such theories, the heat propagation is viewed as a wave phenomenon rather than a diffusion process. The thermoelasticity theory that admits finite speed of heat propagation is called thermoelasticity theory with 'second sound' or hyperbolic thermoelasticity theory. We refer the review article by Chandrasekhariah (1998) for a detailed account in this respect. A brief history of the development of some well studied generalized thermoelasticity theories is given in the next section.

1.1.7.1 Thermoelasticity with one thermal relaxation parameter

The first pioneering contribution to the field of generalized thermoelasticity is provided by Lord and Shulman (1967) who derived a generalized thermoelasticity theory with the modification of governing equations of classical coupled theory with the introduction of one relaxation time. The heat conduction equation of Lord-Shulman (1967) theory specially inherits the effects of fast transient process of heat conduction i.e. wave-type nature which ensures finite speed of propagation for both heat and elastic waves. The heat conduction law of this theory is based on Cattaneo (1958) and Vernotte (1958, 1961) heat conduction model. Cattaneo (1958) and Vernotte (1958, 1961) proposed one model of heat conduction by incorporating the heat flux rate term into Fourier law. The heat conduction equation of this model has the form

$$\vec{q}(\vec{r},t) + \tau \frac{\partial \vec{q}(\vec{r},t)}{\partial t} = -k\vec{\nabla}\theta(\vec{r},t) \tag{1.19}$$

which is quite a modification of Fourier law and is called as CV model. Here τ is a non negative parameter and is known as thermal relaxation time. It is interpreted as the time lag needed to establish the steady state of heat conduction in a material element when a temperature gradient is suddenly imposed on that element. This model also contradicts Fourier law assumption that the heat flux is an instantaneous effect of temperature gradient. This heat conduction equation yields the hyperbolic type heat conduction equation that carries the combined diffusion and wave-like behavior of heat transport. The remaining governing equations of Lord-Shulman (1967) theory, i.e equation of motion and the constitutive relations remain same as Biot's coupled thermoelastic theory. This theory is also known as Extended thermoelasticity theory (ETE). Later on, this theory was extended by Dhaliwal and Sherif (1980) corresponding to general anisotropic media in the presence of heat sources.

1.1.7.2 Thermoelasticity with two thermal relaxation parameters

The second generalization to the coupled theory is known as the generalized theory with two relaxation times. Muller (1971) introduced the theory of generalized thermoelasticity with two relaxation times which is based on a generalized inequality of thermodynamics. A more explicit version was then introduced by Green and Laws (1972), Green and Lindsay (1972) and independently by Suhubi (1975). By the help of two non-negative constants that act as relaxation times, all the equations of this coupled thermoelasticity theory have been modified. They included a temperature-rate among the constitutive variables. This theory is thus also called as temperature- rate dependent thermoelasticity theory (TRDTE). This theory also predicts finite speeds of propagation for heat and elastic waves similar to the Lord-Shulman theory. In this theory, Fourier law of heat conduction is not violated if the body under consideration has a center of symmetry. The stress-strain-temperature relations and the generalized heat conduction equation in context of Green and Lindsay theory (1972) for homogeneous and isotropic elastic media is given as follows:

$$\sigma_{ij} = \lambda e\delta_{ij} + 2\mu e_{ij} - \beta(\theta + \tau_2\frac{\partial\theta}{\partial t})\delta_{ij} \tag{1.20}$$

$$k\nabla^2\theta = \rho c_E(\frac{\partial}{\partial t} + \tau_1\frac{\partial^2}{\partial t^2})\theta + \beta\theta_0\dot{e} \tag{1.21}$$

where, τ_1, τ_2 are two thermal relaxation parameters introduced in the theory.

1.1.7.3 Green-Naghdi thermoelasticity theory

In early 90's Green and Naghdi (1991, 1992, 1993) proposed a theory in a very alternative way based on thermodynamics principles. They incorporated the approach based on Fourier law, the theory without energy dissipation, and the theory with energy dissipation. In the theories by Green and Naghdi, the gradient of thermal displacement, $\nabla\nu$ is considered as a new constitutive variable. The thermal displacement, ν satisfies $\dot{\nu} = \theta$, where θ is the temperature. The heat conduction law for GN-III model is given by

$$\vec{q}(\vec{r}, t) = -[k\vec{\nabla}\theta(\vec{r}, t) + k^*\vec{\nabla}\nu(\vec{r}, t)] \tag{1.22}$$

Here k^*, a material parameter called as conductivity rate of the material, is also newly introduced and is considered as the characteristic of the Green-Naghdi theory.

Their theory is divided into three parts, which are subsequently referred to as thermoelasticity of types GN-I, GN- II and GN-III. When we assume k is much greater than k^*, GN-III model gives its first special case: GN-I model. The linearized form of GN-I is same as classical coupled thermoelasticity theory, and thus exhibits the paradox of infinite heat propagation. When we assume that k^* is much greater than k, GN-III model reduces into second special case: GN-II model. In model- II, the internal rate of production of entropy is considered to be identically zero, implying no dissipation of thermal energy. This model admits undamped thermoelastic waves in a thermoelastic material and is known as the theory of thermoelasticity without energy dissipation. The principal feature of this theory is that in contrast to the classical thermoelasticity, the heat flow does not involve energy dissipation. Also, the same potential function which is defined to derive the stress tensor is used to determine the constitutive equation for the entropy flux vector. In addition, the theory permits the transmission of heat as thermal waves at finite speeds. The derivation of a complete set of governing equations of linearized version of Green-Naghdi-II theory for homogeneous and isotropic materials in terms of displacement and temperature fields and a proof of the uniqueness of the solution for the corresponding initial-boundary-value problem is available in the work of Green and Nagdhi (1993). Model GN- III includes the previous two models as special cases and admits dissipation of energy.

1.1.7.4 Thermoelasticity with dual phase-lag effect

Two dual phase-lag thermoelastic models are developed by Chandrasekharaiah (1998) in the frame of the conventional thermoelasticity theory by considering the dual phase-lag heat conduction law given by Tzou [1995a, 1995b]. The dual phase-lag heat conduction has been introduced by including two phase-lags in the Fourier law of heat conduction in order to take into account the microstructural effects that arise in high rate heat transfer. It has been observed that both the Fourier model and the CV model cannot describe the fast-transient process of heat transfer in the thin film like structure (like gold) due to the micro-structural interaction effects (Brorson *et al.* (1987)). In 1995, Tzou proposed a dual phase-lag model (DPLM) in which heat transport equation relating $\vec{q}$ at a point $\vec{r}$ to ∇T at $\vec{r}$ is written as

$$\vec{q}(\vec{r}, t+\tau_q) = -\left[k\vec{\nabla}\theta(\vec{r}, t+\tau_T)\right] \tag{1.23}$$

Here the first time lag, τ_q , which is denoted as the phase-lag of the heat flux vector, captures the thermal wave behavior, a small-scale response in time. The second time lag, τ_T is termed as the phase-lag of the temperature gradient and it captures the effect of phonon-electron interactions, a micro-scale response in space. The time-lags are assumed to be positive and they are the intrinsic properties of the medium. It is to be noted that, Fourier law implies that heat flux is the instantaneous result of a temperature gradient and according to CV law (single phase-lag) the heat flux is the result (effect) of a temperature gradient (cause) in a transient process. The law given by Tzou (1995a, 1995b) allows either the temperature gradient or the heat flux to become the effect and the remaining one the cause. For the case in which $\tau_q > \tau_T$, the heat flux (effect) established across the material is a result of the temperature gradient (cause), while for $\tau_T > \tau_q$, the heat flux (cause) induces the temperature gradient (effect). The dual-phase-lag model has been shown to be admissible by the second law of extended irreversible thermodynamics by Tzou (1997) and by Xu (2011) and also by the Boltzmann transport equation by Xu and Wang (2005). Tzou referred that when $\tau_T = 0, \tau_q = \tau$, this dual phase-lag model turned to single phase-lag model. Again if $\tau_T = \tau_q (\neq 0)$, this is similar to classical Fourier law. Quintanilla and Racke (2006a) described the stability of this dual phase-lag heat conduction model. With the help of Taylor's series expansion Chandrasekharaiah (1998) extended the dual phase-lag model of heat conduction law developed by Tzau (1995b) to a parabolic as well as hyperbolic thermoelastic model.

1.1.7.5 Thermoelasticiy and three phase-lag heat conduction

Roychoudhuri (2007) developed another thermoelaticity theory with the proposition of a new heat conduction equation. He has established the three phase-lag (TPL) constitutive model by introducing three phase-lag parameters in the heat flux (q), temperature gradient ($\nabla\theta$) and thermal displacement gradient ($\nabla\nu$) in the heat conduction equation of type-III model by Green and Naghdi. The constitutive equation for the heat flux vector in the three-phase-lag theory (2007) is proposed in the form

$$\vec{q}(\vec{r}, t + \tau_q) = -\left[k\vec{\nabla}\theta(\vec{r}, t + \tau_T) + k^*\vec{\nabla}\nu(\vec{r}, t + \tau_\nu)\right] \tag{1.24}$$

Here τ_ν is the additional phase-lag known as phase-lag of thermal displacement gradient beside τ_T, and τ_q. This phase-lag is understood as a delay in terms of the micro-structure of the material. This delay time τ_ν is effective, since, in the three-phase-lag model, the thermal displacement gradient is considered as a constitutive variable whereas in the conventional thermoelasticity theory temperature gradient is considered as a constitutive variable. All the above mentioned theories are supposed to be spacial case of this theory.

1.1.7.6 An Exact Heat Conduction Model with a single delay term

In recent years, the above mentioned thermoelasticity theories have picked significant attention of researchers and we obtain distinct characteristics of various thermoelasticity models. Critical analysis of these thermoelastic models are also discussed. For example, a critical analysis on dual phase-lag and three phase-lag heat conduction model are discussed by Dreher *et al.* (2009). They have shown that when we link up these constitutive equations with energy equation

$$-\nabla.\vec{q}(\vec{r},t) = \rho c_E \dot{\theta}(\vec{r},t) \tag{1.25}$$

we consistently obtain a sequence of eigenvalues in the point spectrum such that its real parts tends to infinity (Dreher *et al.* (2009)). This indicates the ill-posedness of the problem in the Hadamard sense. It means, we can not obtain continuous dependence of the solution with respect to initial parameters (Dreher *et al.* (2009)). Since the phase-lag theories could let to parabolic-type or hyperbolic -type differential equations based on the order of Taylor series expansion of the phase lag parameters, a big interest has been developed to study the different Taylor approximations to these heat conduction equations where continuous dependence and also stability can be achieved (Horgan & Quintanilla (2005), Mukhopadhyay and Kumar (2010a, 2010b), Quintanilla (2002), Quintanilla (2003a) and Quintanilla and Racke (2006a, 2006b, 2007, 2008)). Recently, Quintanilla (2011) has reformulated the three-phase-lag model in an alternative way by defining $\tau_\nu < \tau_q = \tau_T$ and $\tau = \tau_q - \tau_\nu$, and combining equation of heat conduction equation with the above mentioned energy equation and the field equation with a single delay term is developed in the form

$$\rho c_E \ddot{\nu}(t) = k\triangle\theta(t) + k^*\triangle\nu(t-\tau) \tag{1.26}$$

Here $\triangle$ is the Laplacian operator. Subsequently, Leseduarte and Quintanilla (2013) have examined this reformulated model in a precised way. They explored the spatial behavior of the solutions for this theory. A Phragmen- Lindelof type alternative is obtained and it has been shown that the solutions either decay in an exponential way or blow-up at infinity in an exponential way. The obtained results are extended to a thermoelasticity theory by considering the Taylor series approximation of the equation of heat conduction to the delay term and Phragmen-Lindelof type alternative is obtained for the forward and backward in time equations.

1.2 Literature Review

The thermoelasticity theory has been progressed to a large extent due to considerable research during last few decades. The concept of advanced thermoelasticity covers a wide range of extensions of classical dynamical coupled thermoelasticity. Generalized Thermoelasticity theories predicting a finite speed for the propagation of thermal signals have come into existence during the last few decades. These generalized thermoelasticity theories have been the center of active research and have drawn the significant attention of scientists in recent years and they explored various problems relating thermoelastic interactions in different kind of media. For literature on generalized thermoelasticity, we refer the books by Suhubi (1975), Jou (1996), Iesan (1994), Ignaczak and Ostoja - Starzewski (2010), Hetnarski and Eslami (2010), etc. and also the review articles by Chandrasekharaiah(1986, 1998). The Ph.D. theses of Kothari (2013), Kumar (2010) and Prasad (2012) may also be mentioned in this reference. Some of the important works relevant to our present work may be mentioned as below.

Peria (1959) discussed the first coupled theremoelastic problem of a half space with a spot on the front surface suddenly raised to a new temperature under coupled thermoelasticity theory(CTE). He used Laplace transform technique to obtain the solution valid for short time. Hetnarski (1961) explored one dimensional coupled thermoelasticity problem with bounding plane and subjected to sudden heating with constant temperature. Hetnarski (1964) again studied the fundamental solution of coupled thermoelastic problem for small time.The half space problem with different types of boundary conditions had been explored by many auther like Lord and Shulman (1967), Achenbach (1968), Lord and Lapez (1970), Rama Murthy (1978, 1979), Dhaliwal and Rokne (1989), Anwar (1991), Balla (1991), Rama Murthy and Sharma (1991) under extended thermoelasticity theory. Nayfeh (1977) and Nayfeh and Nemat-Nasser (1971) considered the Lord and Shulman theory to study the effects of the thermal coupling on both plane harmonic thermoelastic waves in unbounded media and Rayliegh surface wave propagation along the free surface of a halfspace. Chen and Dargush (1995) used the boundary element method to analyze the transient and dynamic problems in generalized thermoelasticity of a half-space using the Laplace transform method. Chen and Lin (1995) employed a hybrid numerical method based on the Laplace transform and control volume method to analyze the transient coupled thermoelastic problems with relaxation times involving a nonlinear radiation boundary condition. Boundary element method is employed by Hosseini *et al.* (2000, 2003) for the analysis of coupled thermoelastic problems in a finite domain, where they studied the coupling coefficient effects on thermal and elastic waves propagation. Finite element method is considered by Bagri and Eslami (2004) to study the generalized coupled thermoelasticity of an annular disk based on the LS theory. They investigated the thermal and stress wave propagation through the radius of the disk and showed that for thermal shock problems the coupling coefficient has significant effect on the variation of thermal stresses, displacement, and temperature. Only a few authors have studied the problem of generalized thermoelasticity of heterogeneous and/or anisotropic materials. Tokuoka (1973) considered the governing equations of the

LS theory for an anisotropic and homogeneous medium and studied the plane waves using the theory of singular surfaces. Banerjee and Pao (1974) analyzed the problem of anisotropic and homogeneous medium and investigated the thermal relaxation in NaF and helium crystals based on the LS model. Mengi and Turhan (1978) have studied the isotropic inhomogeneous half-space and infinite space with cylindrical and spherical cavities under thermal shock load using the characteristic method and based on the LS model. Sherief (1986) has obtained the fundamental solutions for spherically symmetric space. Chattopadhyay *et al.* (1985) studied a problem of an infinite aelotropic medium having a cylindrical hole. Several problems of thermoelastic interactions due to heat sources in an unbounded elastic medium have been studied. Works of Roychoudhuri and Bhatta (1981), Roychoudhuri and Sain (1982), Sherief and Anwar (1986), Sharma (1986), Mishra *et al.* (1987), Chandrasekharaiah (1988), Sherief and Anwar (1992), Das *et al.* (1997) and Chakravorty and Chakravorty (1998) etc. are worth to be mentioned here. Youssef (2006) and Mukhopadhyay (2002) respectively, studied the problem of a generalized thermoelastic (isotropic for the first, and transversely isotropic for the second) infinite medium with a cylindrical cavity subjected to ramp-type heating and loading. The two problems have been solved analytically using a direct approach where the exact solution is obtained with the help of the Laplace transform technique, followed by analytical inversions of these transforms. Mukhopadhyay and Kumar (2009) investigated thermoelastic interactions under this theory in a cylindrical annulus with temperature dependent physical properties by using finite difference method. Zenkour and Abbas (2014a, 2014b) also discussed a generalized thermoelasticity problem of an annular cylinder with temperature-dependent density and material properties. Ibrahim (2017) also solved one problem of generalized thermoelastic interactions in a hollow cylinder with temperature dependent material property.

In the context of the thermoelasticity theory with two relaxation parameters, Chandrasekharaiah and Srikantiah (1984) have investigated the interfacial waves of arbitrary form in a composite medium which consists of a liquid sandwiched between two solid half spaces. Several interesting investigation like Wojnar (1986), Roychoudhuri and

Chatterjee (1989, 1990), Dhaliwal and Rokne (1989), Furukawa *et al.* (1989, 1990), Chandrasekharaiah and Srikantaiah (1987), Chatterjee and Roychoudhuri (1990), Sherief and Saleh (1998), Erbay and Suhubi (1986), Erbay *et al.* (1991), Sherief (1992, 1994), Sinha and Elsibai (1996a, 1996b), Das and Das (1984), Anwar and Sherief (1988a, 1988b,1994), El-Maghraby and Youssef (2004), Sherief, (1993), Sharma (1995), Misra *et al.* (1993) also carried out on this theory and highlighted several important characterizations regarding this theory. The fundamental solutions for cylindrical regions have discussed by Ezzat (1995). Othman (2004) investigated the effect of rotation on plane wave in GL theory. A few works are reported for the anisotropic materials based on the LS and GL models using the unified approach. Chandrasekharaiah and Keshavan (1992) considered the transversely isotropic medium and studied the plane harmonic waves. Recently Filippi *et al.* (2017) and Entezari *et al.* (2017) reported an unified finite element approach for solving a problem related to generalized coupled thermoelastic analysis of 3D beam-type structures in two parts. In the first part, they discussed problem and gave corresponding formulation and in the second part they discussed numerical evaluation of the problem. Well- posedness of the Green-Lindsay variational problem of dynamic thermoelasticity is discussed by Chyr and Shynkarenko (2017).

The theory of thermoelasticity by Green and Naghdi (1991, 1992, 1993, 1995) have also been studied by several researchers. The uniqueness of the solution of governing equations for the GN theory formulated in term of stress and energy-flux is established in Chandrasekharaiah (1996a). Chandrasekharaiah (1996b) studied the one-dimensional thermal wave propagation in a half-space based on the GN model due to a sudden exposure of temperature to the boundary, using the Laplace transform method. Chandrasekharaiah (1997) also presented the complete solutions of the governing field equations for the GN theory. Chandrasekharaiah and Srinath (1996c) then extended this problem for rotating body. A problem of cylindrical and spherical cavity in an unbounded medium whose boundary is subjected to some loads and due to heat source in an unbounded medium are studied by Chandrasekharaiah and Srinath (1997a,1997b, 1998a, 1998b) under GN-II theory. Sharma and Chauhan (2001) have investigated the disturbances produced in a half-space by the application of a mechanical point load and thermal source acting on the boundary of the half-space. Taheri *et al.* (2004) studied the problem of coupled thermoelasticity of a layer based on the GN theory. The problem was transformed into the Laplace domain, where the analytical solution was obtained. An inverse numerical method was then employed to obtain the solution in real time domain. For GN-II model, Roychoudhuri and Dutta (2005) have studied thermoelastic interactions in an isotropic homogeneous thermoelastic solid containing periodically varying time-dependent distributed heat sources. Some qualitative analyses on thermoelasticity theories of Green and Naghdi have been reported by Quintanilla (2001a, 2001b, 2003a), Quintanilla and Straughan (2004) and Quintanilla (2007). A problem of thermoelastic interactions in an unbounded medium with a spherical cavity due to a thermal shock at the boundary under GN-II theory has been studied by Mukhopadhyay (2002). Issan and Quintanilla (2016) discussed strain gradient theory of chiral Cosserat thermoelasticity without energy dissipation. Energy decay rate of transmission problem between thermoelasticity of type I and type II is studied by Wang *et al.* (2017). Various problems under GN-III model have also been reported by Taheri *et al.* (2005), Mallik and Kanoria (2006), Kar and Kanoria (2006, 2007a), Roychoudhuri and Bandyopadhyay (2007), Banik *et al.* (2007), Kar and Kanoria (2007b) and Mukhopadhyay and Kumar (2008a, 2008b). Mallik and Kanoria (2008) have investigated a two dimensional problem of transversely isotropic problem based on GN-II and GN-III theories. Harmonic plane wave propagation in thermoelastic medium under GN-III model is reported in a detailed study by Puri and Jordan (2004) and later on, by Kovalev and Radayev (2010) and Kothari and Mukhopadhyay (2012). Chirita and Ciarletta (2010) and Mukhopadhyay and Prasad (2011) established the convolution type variational and reciprocity theorems in the context of linear theory of GN-II and GN-III respectively. Other studies related to the Green- Naghdi theory are carried out recently by Bakhshi *et al.* (2006), Mukhopadhyay and Kumar (2008a, 2008b 2010a, 2010b), Abbas (2012), Hosseini and Abolbashari (2012), and El-karamany and Ezzat

(2016). Recently, a stability result for the vibrations given by the standard linear model with thermoelasticity of type III is reported by Apalara and Messaoudi (2017).

Considerable problems concerned with thermoelasticity with dual phase-lags are studied by following researchers. Roychoudhuri (2007) investigated a problem of one-dimensional waves in an elastic half space with its plane subjected to some boundary conditions to analyze the effect of phase-lags. Some qualitative analyses on dual phase-lag thermoelasticity have been reported by Quintanilla (2003a, 2003b, 2002) and Quintanilla and Racke (2006a, 2006b). Prasad *et al.* (2010) have investigated the propagation of harmonic plane waves by obtaining the dispersion relation for an isotropic and homogeneous medium in the context of this theory. Abouelregal (2011) has discussed the effects of two phase-lags of the model on Rayleigh waves in a homogeneous and isotropic elastic half space. The uniqueness and reciprocal theorem for dual phase lag thermoelasticity with two differnt approach is discussed by El-karamany and Ezzat (2014) and Kothari and Mukhopadhyay (2013) respectively. They also established variational principle for linear anisotropic homogeneous and non homogeneous thermoelasticity theory respectively. Abdallah (2009) studied a problem based on the dual Phase Lag (DPL) heat conduction equation in which he investigated the thermoelastic properties of a semi-infinite medium induced by a homogeneously illuminating ultrashort pulsed laser heating. Abouelregal (2011a, 2011b) also solved some problems in the context of the dual-phase-lag thermoelastic model. Ezzat *et.al.* (2017) discussed a problem on dual-phase-lag thermoelasticity theory with memory-dependent derivatives. Zenkour and Abouelregal (2017a, 2017b) also studied a problem thermoelastic interaction in an orthotropic continuum of a variable thermal conductivity with a cylindrical hole. Qualitative properties of solutions in the time differential dual-phase-lag model of heat conduction analyzed very recently by Chirita *et al.* (2017).

The three-phase-lag heat conduction model is an extension of the dual-phase-lag. Using Taylor approximations, it is proved that this theory also covers the Green-Naghdi theories. Many studies that highlight the beneficial effect in terms of application of this theory have been published. The stability of the three phase-lag heat conduction equation has been analyzed in details by Quintanilla and Racke (2008). Quintanilla (2009) also studied the spatial behavior of solutions of the three phase-lags heat equation. The effects of three-phase-lags on an infinite medium with cylindrical cavity are studied by Kumar and Mukhopadhyay (2009), and the generalized thermoelastic functionally graded orthotropic hollow sphere under thermal shock with three-phase-lag effect is studied by Kar and Kanoria (2009). Kothari et.al. (2010) discussed the fundamental solution of this theory. Some author also studied dual phase-lag and three phase- lag thermoelasticity theories in combine way. The effects of phase-lags on wave propagation in a thick plate under axis-symmetric temperature distribution based on dual and three phase-lags thermoelasticity theories have been disscused by Mukhopadhyay and Kumar (2010b) and the effects of phase-lags on wave propagation in an infinite solid due to a continuous line heat source is examined by Prasad *et al.* (2011). Recently, Mukhopadhyay *et al.* (2014) have reported a qualitative study to discuss the similarity and dissimilarity of various models as mentioned above and proposed rational material laws to consider more general medium.

There are some very interesting discussions in the field of thermoelasticity have been reported in recent time. Some of which are as follows: One dimensional model of nonlinear thermoelasticity at low temperature and small strain is studied by Ignaczak and Domanski (2016). Gadain (2016) explored coupled singular and non singular thermoelastic system by double Laplace decomposition method. Liu *et al.* (2016) investigated the time decay of solutions for non simple elasticity with voids. Knops and Quintanilla (2015) analyzed spatial behaviour in thermoelastostatic cylinders of indefinitely increasing cross-section. A problem of transient coupled thermoelasticity of an annular fin is studied by Abd-Alla *et al.* (2012). Large time behaviour for non-simple thermoelasticity with second sound is examined by Rivera and Vega (2017). Mustafa (2017) explored uniform stability of second sound thermoelasticity with distributed delay and Freed (2017) investigated explicit and implicit theories of thermoelasticity for anisotropic materials. The thermoelastic model by Quintanilla (2011) with a single delay term is introduced very recently and is yet to be studied. Recently, a uniqueness theorem and instability of solutions under the relaxed assumption that the elasticity tensor can be negative is established by Quintanilla (2016). Kumar and Mukhopadhyay (2016) investigated a problem of thermoelastic interactions on this theory in which state-space approach is used to formulate the problem and the formulation is then applied to a problem of an isotropic elastic half-space with its plane boundary subjected to sudden increase in temperature and zero stress. Some interesting studies based on this model are carried out very recently by Magana and Quintanilla (2017), Borgmeyer *et al.* (2014), and Lesedurte and Quintanilla (2015, 2017).

1.3 Objective of the Present Thesis

The term thermoelasticity involves a large category of phenomena. It comprises the general theory of heat conduction, thermal stresses, and strains set up by thermal flow in elastic bodies and the reverse result of temperature distribution caused by the elastic deformation itself. It is well known that the latter is an important cause of internal damping in elastic bodies. The absence of any elasticity term in the heat conduction equation for uncoupled thermoelasticity appears to be unrealistic, since the produced strain causes variation in the temperature field due to the mechanical loading of an elastic body. The classical theories of thermoelasticity also have infinite speed of propagation of thermal signals, that contradict physical facts. During the last three decades, non-classical theories involving finite speed of heat transportation in elastic solids have been developed. The various non-Fourier heat conduction models have been introduced with the purpose to eliminate the inherent drawback in classical heat conduction theory and/or classical coupled thermoelasticity theory. Hence, in-depth research is necessary to carry out on generalized thermoelasticity theories in solving thermoelastic problems in place of the classical uncoupled/coupled theory of thermoelasticity in order to understand the specific features of the concerned theories as compared to the classical theory. Furthermore, due to the advancement of pulsed lasers, fast burst nuclear reactors, and particle accelerators, which can supply heat pulses with a very fast time-rise, generalized thermoelasticity theory is receiving serious attention of researchers.

The main objective of the present thesis is to discuss some important aspects of some recent non-Fourier heat conduction models by applying them for coupled thermo-mechanical problems. We concentrate on various problems of real continuous medium in which deformation occurs within the range of elasticity. Since the deformation as well as temperature increase is assumed to be very small, the linearized theories connecting temperature and strain have been considered. We pay attention to Green-Naghdi theory of type- II and the most recent thermoelasticity theory and aim to establish some important theorems on these models. Solutions of some physical problems associated with thermoelastic interactions based on these theories are derived. We make attempt to investigate the effects of employing these models in various cases of mutual interactions. Attempts are also made to understand the detailed aspects on the propagation of harmonic plane wave inside the medium and to compare the results predicted by concerned models with the corresponding results of other models of thermoelasticity. We aim to highlight some significant points corresponding to characteristic features of the models.