CHAPTER 5

AN INVESTIGATION ON THERMOELASTIC INTERACTIONS UNDER AN EXACT HEAT CONDUCTION MODEL WITH A DELAY TERM[1]

5.1 Introduction

The thermoelasticity theories with phase-lags have been introduced to overcome the inadequacy of the classical theory. These theories have drawn the serious attention of researchers in recent years who have investigated several features of various models. Qualitative analyses on dual-phase-lag thermoelasticity have been reported by Quintanilla (2003) and Quintanilla and Racke (2006a; 2006b). The stability of the three-phase-lag heat conduction equation was discussed by Quintanilla and Racke (2008). Quintanilla (2009) also studied the spatial behavior of solutions of the three-phase-lag heat equation. Subsequently, some critical analysis on these models are also reported. Dreher *et al.* (2009) have reported a critical analysis on dual-phase-lag and three-phase-lag heat conduction models and have shown that when we adjoin these constitutive equation with energy equation, there always exists a sequence of eigenvalues in the point spectrum such that its real parts tend to infinity. This implies the ill-posedness of the problem in

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the Hadamard sense. That is, we can not obtain continuous dependence of the solution with respect to initial parameters (see Quintanilla (2011), Leseduarte and Quintanilla (2013)). Subsequently, an attention has been paid on different Taylor series approximations to these heat conduction equations (see Quintanilla and Racke (2008) and the references therein) to investigate the continuous dependence results and stability analysis. Quintanilla (2011) has reformulated the three-phase-lag model in an alternative way by defining $\tau_q = \tau_T$ and $\tau = \tau_q - \tau_\nu > 0$. He obtained the heat conduction equation with a single delay term in the form

$$\ddot{\upsilon} - \frac{K^* \tau^2}{2} \nabla \ddot{\upsilon} = (K + K^* \tau) \nabla T + K^* \nabla \upsilon \tag{i}$$

This is termed as exact heat conduction with a delay. Later on, Leseduarte and Quintanilla (2013) investigated the stability and spatial behavior of the solutions of this newly proposed model with single delay term. A Phragmen-Lindelof type alternative is obtained and it has been shown that the solutions either decay in an exponential way or blow-up at infinity in an exponential way. The obtained results are extended to a thermoelasticity theory by considering the Taylor series approximation of the equation of heat conduction to the delay term and Phragmen-Lindelof type alternative is obtained for the forward and backward in time equations. Kumari and Mukhopadhyay (2017b) made an attempt to establish some important theorems in this context. A uniqueness theorem has been established for an anisotropic body and a variational principle as well as a reciprocity principle is established too. For the half space problem, a detailed analysis of analytical and numerical results under the current theory is provided by Kant and Mukhopadhyay (2016).

In the present chapter, we consider this newly proposed model with a single delay term (Quintanilla (2011), Leseduarte and Quintanilla (2013)). This model considers all the microstructural effects in the heat transport phenomenon like dual-phase-lag and three-phase-lag model. We make an attempt to investigate a problem of thermoelastic interactions in the context of this model. The state-space approach is employed to formulate the problem and the formulation is then is applied to solve a boundary value problem of an isotropic elastic half space with its plane boundary subjected to sudden increase in temperature and zero stress. The Laplace transform is applied to obtain the solution of the problem. The short-time approximated solution for the field variables is obtained analytically. A detailed analysis of analytical results is provided. An attempt has also been made to illustrate the problem and numerical values of field variables are obtained for a particular material. Results are analyzed with different graphs and a comparison of the results with others existing models of thermoelasticity is made. To the best of the author's knowledge, this thermoelastic model has not yet received much attention of researchers. Hence, this problem is considered to understand the basic features of this new model with respect to other well established models of thermoelasticity.

5.2 Basic Governing Equations

We employ the thermoelasticity theory based on the exact heat conduction model with a delay term as proposed by Leseduarte and Quintanilla (2013) to consider the thermoelastic interactions in a homogeneous and isotropic solid in the absence of body forces and heat sources. The equation of motion, stress-strain-temperature relations and the unified heat conduction equation in the contexts of the theory of extended thermoelasticity due to Lord and Shulman (LS model), thermoelasticity without energy dissipation due to GN-II model, thermoelasticity of type III (GN-III model) and the new model with a delay parameter as proposed by Leseduarte and Quintanilla (2013) can be considered in a unified way as follows:

Equation of motion:

$$\mu\nabla^2\mathbf{u} + (\lambda+\mu)\,\vec{\vec{\nabla}}\,div\mathbf{u} - \gamma\nabla T = \rho\ddot{\mathbf{u}} \tag{5.1}$$

Stress-strain-temperature relation:

$$\vec{\sigma} = \lambda(div\mathbf{u})\vec{I} + \mu\left(\vec{\nabla}\mathbf{u} + \vec{\nabla}\mathbf{u}^{\mathbf{T}}\right) - \gamma T\vec{I} \tag{5.2}$$

Unified heat conduction equation:

$$\left(\delta_{1j} + \xi_0\delta_{2j}\frac{\partial}{\partial t}\right)K\nabla^2 T + \delta_{2j}K^*\left[\nabla^2 T + \xi_1\left(\tau_1\nabla^2\dot{T} + \frac{\tau_1^2}{2}\nabla^2\ddot{T}\right)\right]$$
$$= \rho c_E\left[\delta_{1j}\dot{T} + (\delta_{1j}\tau_0 + \delta_{2j})\,\ddot{T}\right] + \gamma T_0\left[\delta_{1j}div\mathbf{u} + (\delta_{1j}\tau_0 + \delta_{2j})\,div\ddot{\mathbf{u}}\right] \tag{5.3}$$

where τ_0 is the thermal relaxation parameter due to Lord -Shulman model, and τ_1 is the delay time due to new model (Leseduarte and Quintanilla (2013)).

Eqs. (5.1)-(5.3) reduce to the equations of different theories of thermoelasticity as follows:

1. **LS model** : j =1 (i.e. $\delta_{2j} = 0$), $\tau_0 > 0$

2. **GN – II model** : j = 2,(i.e. $\delta_{2j} = 1$), $\xi_0 = 0 = \xi_1$

3. **GN – III model** : j = 2, $\xi_0 = 1$, $\xi_1 = 0$

4. **New model** : j = 2, $\xi_0 = 1 = \xi_1$, $\tau_1 > 0$

5.3 Problem Formulation

We consider a thermoelastic medium in unbounded space, $x \geq 0$, whose state variables depend only on space variable x and time t. For simplicity of equations (5.1)-(5.3), we use the following non-dimensional variables and notations:

$x' = c_1\eta x,\ u' = c_1\eta u,\ t' = c_1^2\eta t,\ \tau_0' = c_1^2\eta\tau_0,\ \tau_1' = c_1^2\eta\tau_1,\ \theta = \frac{T}{T_0}$, $\sigma'_{xx} = \frac{\sigma_{xx}}{\lambda+2\mu}$, $c_1^2 = \frac{\lambda+2\mu}{\rho}$, $K^* = \frac{\rho c_E(\lambda+2\mu)}{4}$ and $\eta = \frac{\rho c_E}{K}$.

Therefore, in terms of the above non-dimensional variables, after dropping the primes, equations (5.1)-(5.3) can be transformed to the non-dimensional forms as follows:

$$\frac{\partial^2 u}{\partial x^2} - a_1\frac{\partial\theta}{\partial x} = \frac{\partial^2 u}{\partial t^2} \tag{5.4}$$

$$\sigma_{xx} = \frac{\partial u}{\partial x} - a_1\theta \tag{5.5}$$

$$\begin{aligned}&\left[\delta_{1j} + \delta_{2j}\left\{\xi_0\frac{\partial}{\partial t} + \frac{1}{4} + \frac{\xi_1}{4}\left(\tau_1\frac{\partial}{\partial t} + \frac{\tau_1^2}{2}\frac{\partial^2}{\partial t^2}\right)\right\}\right]\frac{\partial^2\theta}{\partial x^2}\\ &= \left[\delta_{1j}\frac{\partial\theta}{\partial t} + (\delta_{1j}\tau_0 + \delta_{2j})\frac{\partial^2\theta}{\partial t^2}\right] + a_2\left[\delta_{1j}\frac{\partial}{\partial t} + (\delta_{1j}\tau_0 + \delta_{2j})\frac{\partial^2}{\partial t^2}\right]\frac{\partial u}{\partial x}\end{aligned} \tag{5.6}$$

where, we define the non-dimensional quantities $a_1 = \frac{\gamma T_0}{\lambda+2\mu}$, and $a_2 = \frac{\gamma}{K\eta}$.

5.4 Solution of the Problem

Applying Laplace transform on time under homogeneous initial conditions to the equations (5.4)-(5.6), we get

$$\frac{\partial^2 \bar{u}}{\partial x^2} - a_1 \frac{\partial \bar{\theta}}{\partial x} = s^2 \bar{u} \tag{5.7}$$

$$\bar{\sigma}_{xx} = \frac{\partial \bar{u}}{\partial x} - a_1 \bar{\theta} \tag{5.8}$$

$$\begin{aligned} &\left[\delta_{1j} + \delta_{2j}\left\{\xi_0 s + \frac{1}{4} + \frac{\xi_1}{4}\left(\tau_1 s + \frac{\tau_1^2 s^2}{2}\right)\right\}\right]\frac{\partial^2 \overline{\theta}}{\partial x^2} \\ &= \left[\delta_{1j} s + \left(\delta_{1j}\tau_0 + \delta_{2j}\right) s^2\right]\overline{\theta} + a_2\left[\delta_{1j} s + \left(\delta_{1j}\tau_0 + \delta_{2j}\right) s^2\right]\frac{\partial \overline{u}}{\partial x} \end{aligned} \tag{5.9}$$

where, s is the Laplace transform parameter

By using the equations (5.7)-(5.9), we find two coupled equations in state variables, $\bar{\theta}$ and $\overline{\sigma}_{xx}$ as

$$\begin{aligned} \frac{\partial^2 \overline{\theta}}{\partial x^2} \quad = \quad & \frac{8\left[\delta_{1j}\left(1+\epsilon\right)s + \left(\delta_{1j}\tau_0 + \delta_{2j} + \delta_{1j}\tau_0\epsilon + \delta_{2j}\epsilon\right)s^2\right]}{\xi_1\tau_1^2 s^2 + 2\delta_{2j}\left(\xi_1\tau_1 + 4\xi_0\right)s + 2\left(4\delta_{1j} + \delta_{2j}\right)}\overline{\theta} \\ & + \frac{8a_2\left[\delta_{1j}s + \left(\delta_{1j}\tau_0 + \delta_{2j}\right)s^2\right]}{\xi_1\tau_1^2 s^2 + 2\delta_{2j}\left(\xi_1\tau_1 + 4\xi_0\right)s + 2\left(4\delta_{1j} + \delta_{2j}\right)}\overline{\sigma}_{xx} \end{aligned} \tag{5.10}$$

$$\frac{\partial^2\bar{\sigma}_{xx}}{\partial x^2} = a_1 s^2\bar{\theta} + s^2\bar{\sigma}_{xx} \tag{5.11}$$

where $\epsilon = a_1 a_2$.

Now, we write equations (5.10) and (5.11) in the matrix forms as

$$\frac{d^2\bar{V}(x,s)}{dx^2} = P(s)\bar{V}(x,s) \tag{5.12}$$

where,

$$\bar{V}(x,s) = \begin{bmatrix} \bar{\theta}(x,s) \\ \bar{\sigma}_{xx}(x,s) \end{bmatrix} \tag{5.13}$$

$$P(s) = \begin{bmatrix} L_1 & L_2 \\ M_1 & M_2 \end{bmatrix} \tag{5.14}$$

$$L_1 = \frac{8\left[\delta_{1j}\left(1+\epsilon\right)s + \left(\delta_{1j}\tau_0 + \delta_{2j} + \delta_{1j}\tau_0\epsilon + \delta_{2j}\epsilon\right)s^2\right]}{\xi_1\tau_1^2 s^2 + 2\delta_{2j}\left(\xi_1\tau_1 + 4\xi_0\right)s + 2\left(4\delta_{1j} + \delta_{2j}\right)}$$

$$L_2 = \frac{8a_2\left[\delta_{1j}s + \left(\delta_{1j}\tau_0 + \delta_{2j}\right)s^2\right]}{\xi_1\tau_1^2 s^2 + 2\delta_{2j}\left(\xi_1\tau_1 + 4\xi_0\right)s + 2\left(4\delta_{1j} + \delta_{2j}\right)}$$

$$M_1 = a_1 s^2, \quad M_2 = s^2$$

Solution of the equation (5.12) is given by

$$\bar{V}(x,s) = \bar{V}(0,s)e^{-\left[\sqrt{P(s)}\right]x} \tag{5.15}$$

where,

$$\bar{V}(0,s) = \begin{bmatrix} \bar{\theta}\,(0,s) \\ \bar{\sigma}_{xx}(0,s) \end{bmatrix} = \begin{bmatrix} \bar{\theta}_0 \\ \bar{\sigma}_0 \end{bmatrix}, \text{ say.}$$

In the above solutions, we have omitted the part of the exponential that has a positive power to obtain bounded solutions for large x.

Now, using Cayley Hamilton's theorem, $e^{-\left[\sqrt{P(s)}\right]x}$ can be expressed in the form

$$e^{-\left[\sqrt{P(s)}\right]x} = L(x,s) = b_0 I + b_1 Q(s) \tag{5.16}$$

where, b_0 and b_1 are constants depending on s and x to be determined and I is 2×2 identity matrix. $Q(s)$ is a 2×2 matrix to be determined.

The characteristic equation of matrix $P(s)$ is obtained as

$$k^2 - (L_1 + M_2)\,k + (L_1 M_2 - L_2 M_1) = 0 \tag{5.17}$$

where, two roots k_1 and k_2 of the equation (5.17) will satisfy

$$k_1 + k_2 = L_1 + M_2 \tag{5.18}$$

$$k_1 k_2 = L_1 M_2 - L_2 M_1 \tag{5.19}$$

Further, the spectral decomposition of $P(s)$ can be written as

$$P(s) = k_1 E_1 + k_2 E_2 \tag{5.20}$$

where, E_1 and E_2 are projections of $P(s)$ which follow the conditions

$$E_1+E_2 = I,\ E_1E_2 = E_2E_1 = \text{zero matrix},\ E_1^2 = E_1 \text{ and } E_2^2 = E_2 \quad (5.21)$$

By using equations (5.20) and (5.21), we have

$$\sqrt{P(s)} = \sqrt{k_1}E_1 + \sqrt{k_2}E_2 \quad (5.22)$$

where,

$$E_1 = \frac{1}{k_1 - k_2}\begin{bmatrix} L_1 - k_1 & L_2 \\ M_1 & M_2 - k_1 \end{bmatrix} \quad (5.23)$$

and

$$E_2 = \frac{1}{k_1 - k_2}\begin{bmatrix} k_1 - L_1 & -L_2 \\ M_1 & k_1 - M_2 \end{bmatrix} \quad (5.24)$$

Now the characteristic roots r_1 and r_2 of the matrix $\sqrt{P(s)}$ are given by

$$r_1 = \sqrt{k_1} \;\; and \;\; r_2 = \sqrt{k_2} \quad (5.25)$$

Since, equation (5.16) is Cayley-Hamilton form of the matrix $\sqrt{P(s)}$, hence r_1 and r_2 will also satisfy equation (5.16). Therefore, we get the following system of linear equations:

$$e^{[-r_1x]} = b_0 + b_1r_1 \quad (5.26)$$

$$e^{[-r_2x]} = b_0 + b_1r_2 \quad (5.27)$$

Solving the system of linear equations (5.26) and (5.27), we find

$$b_0 = \frac{1}{r_1 - r_2}\left[r_1 e^{-r_2 x} - r_2 e^{-r_1 x}\right] \tag{5.28}$$

$$b_1 = \frac{1}{r_1 - r_2}\left[e^{-r_1 x} - e^{-r_2 x}\right] \tag{5.29}$$

Substituting b_0 and b_1 in equation (5.16), we find the elements L_{ij} $(i,j=1,2)$ of matrix $L(x,s)$ in the following forms:

$$\begin{aligned}
L_{11} &= \tfrac{1}{k_1-k_2}\left[\left(k_1 - L_1\right)e^{-\sqrt{k_2}x} - \left(k_2 - L_1\right)e^{-\sqrt{k_1}x}\right] \\
L_{12} &= \tfrac{L_2}{k_1-k_2}\left[e^{-\sqrt{k_1}x} - e^{-\sqrt{k_2}x}\right] \\
L_{21} &= \tfrac{M_1}{k_1-k_2}\left[e^{-\sqrt{k_1}x} - e^{-\sqrt{k_2}x}\right] \\
L_{22} &= \tfrac{1}{k_1-k_2}\left[\left(k_1 - M_2\right)e^{-\sqrt{k_2}x} - \left(k_2 - M_2\right)e^{-\sqrt{k_1}x}\right]
\end{aligned}$$

Therefore, the solution of equation (5.12) can be written as

$$\bar{V}(x,s) = L(x,s)\bar{V}(0,s) \tag{5.30}$$

Hence, component-wise solutions for $\bar{\theta}$ and $\bar{\sigma}_{xx}$ can be written as

$$\bar{\theta} = \frac{1}{k_1 - k_2}\left[\left(k_1\bar{\theta}_0 - L_1\bar{\theta}_0 - L_2\bar{\sigma}_0\right)e^{-\sqrt{k_2}x} - \left(k_2\bar{\theta}_0 - L_1\bar{\theta}_0 - L_2\bar{\sigma}_0\right)e^{-\sqrt{k_1}x}\right] \tag{5.31}$$

$$\bar{\sigma}_{xx} = \frac{1}{k_1 - k_2}\left[\left(k_1\bar{\sigma}_0 - M_2\bar{\sigma}_0 - M_1\bar{\theta}_0\right)e^{-\sqrt{k_2}x} - \left(k_2\bar{\sigma}_0 - M_2\bar{\sigma}_0 - M_1\bar{\theta}_0\right)e^{-\sqrt{k_1}x}\right] \tag{5.32}$$

From equations (5.7) and (5.8), we get

$$\bar{u} = \frac{1}{s^2}\frac{\partial\sigma}{\partial x}$$

$$i.e.\quad \bar{u} = \frac{1}{s^2\left(k_1-k_2\right)}\Big[\left(M_1\bar{\theta}_0 + M_2\bar{\sigma}_0 - k_1\bar{\sigma}_0\right)\sqrt{k_2}e^{-\sqrt{k_2}x}$$
$$-\left(M_1\bar{\theta}_0 + M_2\bar{\sigma}_0 - k_2\bar{\sigma}_0\right)\sqrt{k_1}e^{-\sqrt{k_1}x}\Big] \tag{5.33}$$

5.4.1 Boundary Conditions

We consider a half space $x \geq 0$ occupied by a homogeneous and isotropic thermoelastic solid with homogeneous initial conditions. Suppose that the boundary $x = 0$ of the half space is stress free and is subjected to a unit-step increase in temperature. Therefore, boundary conditions can be taken as follows:

$$\sigma_{xx}(0,t) = \sigma_0 = 0 \tag{5.34}$$

$$\theta(0,t) = \theta_0 = \theta^* H(t) \tag{5.35}$$

where, θ^* is a constant temperature and $H(t)$ is Heaviside unit-step function.

Now, after taking the Laplace transform of equation (5.34) and (5.35), we get

$$\bar{\sigma}_{xx}(0,s) = \bar{\sigma}_0 = 0 \tag{5.36}$$

$$\bar{\theta}(0,s) = \bar{\theta}_0 = \frac{\theta^*}{s} \tag{5.37}$$

Using the boundary conditions (5.36) and (5.37) into the equations (5.31)-(5.33), we obtain the solutions for the field variables in the Laplace transform domain as

$$\bar{\theta} = \frac{\theta^*}{s\,(k_1 - k_2)} \left[(k_1 - L_1)\, e^{-\sqrt{k_2}x} - (k_2 - L_1)\, e^{-\sqrt{k_1}x} \right] \tag{5.38}$$

$$\bar{\sigma}_{xx} = \frac{\theta^* M_1}{s\,(k_1 - k_2)} \left[e^{-\sqrt{k_1}x} - e^{-\sqrt{k_2}x} \right] \tag{5.39}$$

$$\bar{u} = \frac{\theta^* M_1}{s^3\,(k_1 - k_2)} \left[\sqrt{k_2} e^{-\sqrt{k_2}x} - \sqrt{k_1} e^{-\sqrt{k_1}x} \right] \tag{5.40}$$

5.5 Short-time Approximated Solutions

In the previous section, we found the solutions of the field variables in the Laplace transform domain. The distributions of stress, temperature and displacement fields in the physical domain (x,t) will be determined by taking Laplace inversions of $\bar{\sigma}_{xx}$, $\bar{\theta}$ and $\bar{u}$ given by (5.38)-(5.40). However, to the solution of (5.17) to find k_1 and k_2 exactly for all value of s is unmanageable and hence closed form solutions (transform-inversion) for all field variables for all values of s is also practically impossible to construct. Therefore, we will make attempt to find small-time approximated solutions of the the field variables for which we assume that Laplace transform parameter s is very large (see, Dhaliwal and Rokne(1988; 1989), Chandrasekharaiah and Srinath(1996), Mukhopadhyay and Kumar(2010b)). For this, we concentrate our attention for the new model only by substituting j =2, $\xi_0 = 1 = \xi_1$, $\tau_1 > 0$ in equation (5.10) and all the subsequent equations.

In view of this, the square root of roots k_1 and k_2 of equation (5.17) can be found by using Maclaurin's series expansion and neglecting the higher powers of small terms as follows:

$$\sqrt{k_1} = s + \frac{m_1}{s} - \frac{m_2}{s^2} \tag{5.41}$$

$$\sqrt{k_2} = n_1 - \frac{n_2}{s} + \frac{n_3}{s^2} \tag{5.42}$$

where, $m_1 = \frac{4\epsilon}{\tau_1^2}$, $m_2 = \frac{8\epsilon(\tau_1+4)}{\tau_1^4}$, $n_1 = \frac{2\sqrt{2}}{\tau_1}$, $n_2 = \frac{2\sqrt{2}(\tau_1+4)}{\tau_1^3}$, $n_3 = \frac{\sqrt{2}\left(48+24\tau_1+\tau_1^2-8\epsilon\tau_1^2\right)}{\tau_1^5}$.

Substituting the values of k_1 and k_2 in equations (5.38)-(5.40) and after simplification, we get the short-time approximated solutions for the variables in Laplace transform domain as

$$\overline{\theta}\,(t,x) \simeq \sum_{i=1}^{2} \left[A_i \frac{e^{-\left(s+\frac{m_1}{s}\right)x}}{s^{2i}} - B_i \frac{e^{-\left(n_1-\frac{n_2}{s}\right)x}}{s^{i+3}} \right] \tag{5.43}$$

$$\overline{\sigma}_{xx}\,(t,x) \simeq a_1 \sum_{i=1}^{2} C_i \left[\frac{e^{-\left(s+\frac{m_1}{s}\right)x}}{s^{2i-1}} - \frac{e^{-\left(n_1-\frac{n_2}{s}\right)x}}{s^{2i-1}} \right] \tag{5.44}$$

$$\overline{u}\,(t,x) \simeq a_1 \sum_{i=1}^{2} \left[-D_i \frac{e^{-\left(s+\frac{m_1}{s}\right)x}}{s^{2i}} + n_i \frac{e^{-\left(n_1-\frac{n_2}{s}\right)x}}{s^{i+2}} \right] \tag{5.45}$$

where, we used the following notations:

$A_1 = C_1 = D_1 = 1$, $A_2 = -\frac{8\epsilon}{\tau_1^2}$, $B_1 = \frac{8(1-a_2)}{\tau_1^2}$, $B_2 = \frac{-16(\tau_1+4)(1-a_2)}{\tau_1^4}$, $C_2 = \frac{8(1-\epsilon)}{\tau_1^2}$, $D_2 = m_1 + C_2$.

Now, taking inverse Laplace transforms of equations (5.43)-(5.45), we get the approximated analytical solution for the distributions of displacement, temperature and stress in space -time (x,t) domain as

$$\begin{aligned}\theta\left(x,t\right) &= \sum_{i=1}^{2} A_i \left(\frac{t-x}{m_1 x}\right)^{2i-1/2} J_{2i-1}\left(2\sqrt{m_1 x\left(t-x\right)}\right) H(t-x) \\ &- \sum_{i=1}^{2} B_i e^{-n_1 x} \left(\frac{t}{n_2 x}\right)^{i+2/2} I_{i+2}\left(2\sqrt{n_2 x t}\right) \end{aligned} \tag{5.46}$$

$$\begin{aligned}\sigma_{xx}\left(t,x\right) &= a_1 \sum_{i=1}^{2} C_i \left(\frac{t-x}{m_1 x}\right)^{i-1} J_{2i-2}\left(2\sqrt{m_1 x\left(t-x\right)}\right) H(t-x) \\ &- a_1 \sum_{i=1}^{2} C_i e^{-n_1 x} \left(\frac{t}{n_2 x}\right)^{i-1} I_{2i-2}\left(2\sqrt{n_2 x t}\right) \end{aligned} \tag{5.47}$$

$$\begin{aligned}u\left(t,x\right) &= -a_1 \sum_{i=1}^{2} D_i \left(\frac{t-x}{m_1 x}\right)^{2i-1/2} J_{2i-1}\left(2\sqrt{m_1 x\left(t-x\right)}\right) H(t-x) \\ &+ a_1 \sum_{i=1}^{2} n_i e^{-n_1 x} \left(\frac{t}{n_2 x}\right)^{i+1/2} I_{i+1}\left(2\sqrt{n_2 x t}\right) \end{aligned} \tag{5.48}$$

5.6 Analysis of Analytical Results

From the solutions obtained in equations (5.46)-(5.48), we observe that each of the distributions of temperature, displacement and stresses consists of two main parts. In the first part, the term involving $H(t-x)$ corresponds to a wave propagating with finite speed (unity) which can be identified as predominantly elastic wave. The expressions given by Eqns. (5.46)-(5.48) reveal that under this new theory of thermoelasticity, the modified elastic wave propagates without any attenuation. This is a distinct feature predicted by the theory and it has a similarity with the thermoelasticity without energy dissipation (GN-II) model. The second part of the solutions of the field variables does not indicate to be a contribution of a wave, instead it is diffusive in nature indicating an exponential decay with distance with an attenuating coefficient n_1. This represents that under this present model, the thermal wave do not propagate with finite wave speed like, other generalized thermoelasticity theories: LS model, GL model, GN-II model, dual-phase-lag model or three-phase-lag model. However, a similar nature is observed under GN-III model (see Mukhopadhyay and Kumar(2010b)). Furthermore, we note that the solutions for temperature and displacement (see equations (5.46) and (5.48)) are continuous in nature. However the analytical results given by equation (5.47) shows that stress distribution has discontinuity with finite jumps at the elastic wave front. The finite jump is obtained as follows:

$$\begin{aligned} \sigma_{xx}|_{x=t} &= \sigma_{xx}|_{x=t^+} - \sigma_{xx}|_{x=t^-} \\ &= a_1 C_1 \end{aligned}$$

It must be mentioned here that this behavior of solutions in the context of the present new model has similarity with the corresponding results under GN-III model (see Mukhopadhyay and Kumar (2010b)).

5.7 Numerical Results and Discussion

In this section, we illustrate the problem with numerical values of the field variables like displacement, temperature and stress for a material in space-time domain. For the inversion of Laplace transforms, we employ a numerical method as described by Bellman *et al.* (1966). With the help of MATLAB software, we compute values of the variables by using solutions obtained in (5.43)-(5.45) and employing the numerical method. We choose the copper material for this purpose and physical data for it are given as follows (2005):

$\lambda = 7.76 \times 10^{10}\ Nm^{-2}$, $\mu = 3.86 \times 10^{10}\ Nm^{-2}$, $\alpha_t = 1.78 \times 10^{-5}\ K^{-1}$, $K = 8886.73\ sm^{-2}$, $c_E = 383.1\ JKg^{-1}K^{-1}$, $\rho = 8954\ Kg\ m^{-3}$.

We assume

$$\theta^* = 1,\ \tau_1 = 0.1,\ \tau_0 = 0.1.$$

For the purpose of comparing our results in the present context, we also carry out calculations for field variables under new mode and other three models like, GN-II, GN-III and LS models as described in the section of basic equations. Variations of field variables for different values of non-dimensional space co-ordinate, x ($x \geq 0$) and for non-dimensional time t are computed and values are plotted in different figures.

Figs. 5.1(a,b,c) show the variation of displacement with respect to distance, x under four models (new model, GN-II, GN-III and LS model) at three different times $t = 0.1, 0.7, 1.2$ and indicate that when the boundary of the half space is subjected to a thermal shock and zero stress, the displacement shows an oscillatory nature near the boundary. It attains a stationary maximum value (+ve) after some distance from the boundary and finally it decreases to zero value. At larger time, it covers little more

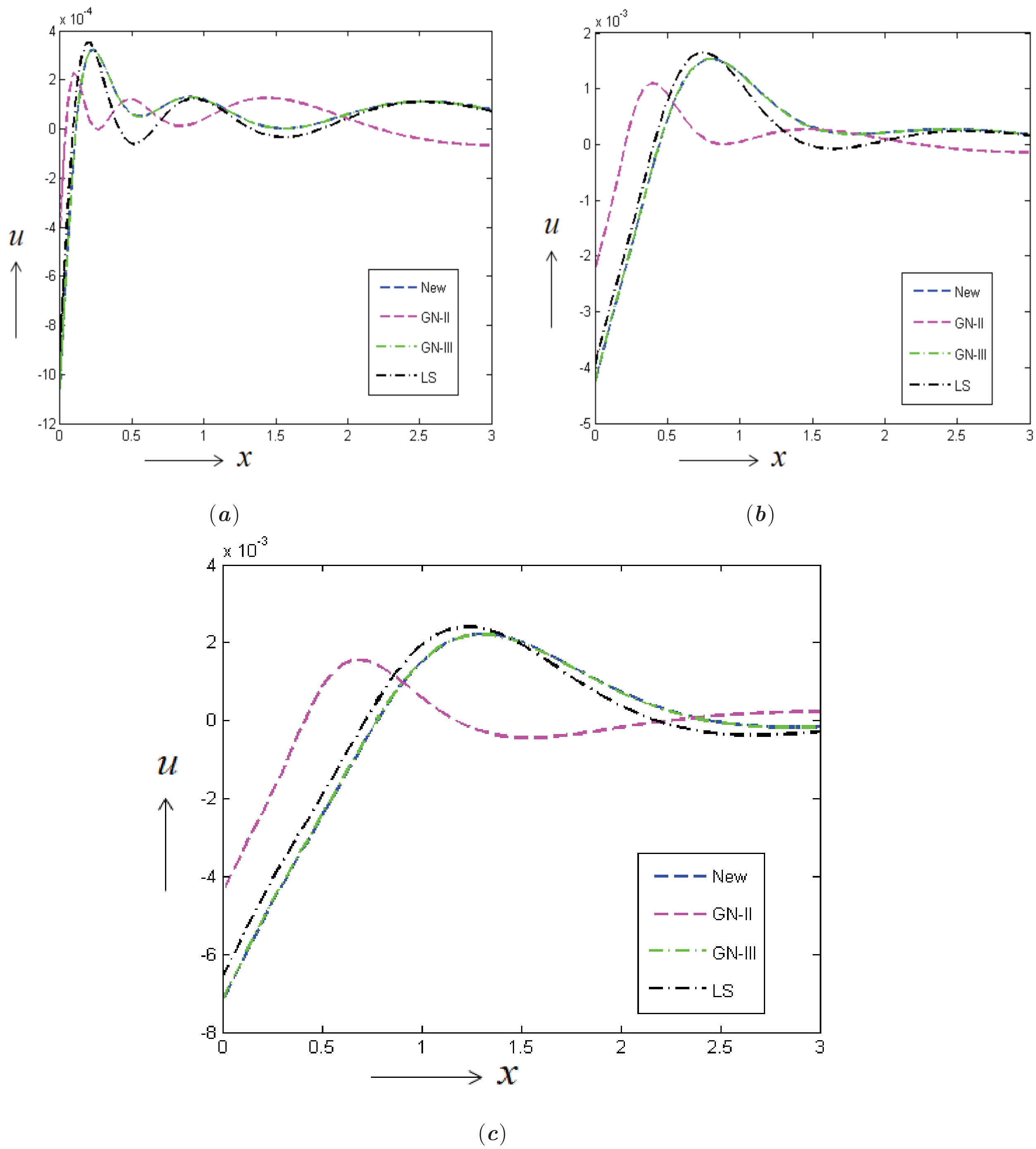


Figure 5.1: $(\boldsymbol{a})$.Variation of displacement u vs. x at $t = 0.1$, $(\boldsymbol{b})$. Variation of displacement u vs. x at $t = 0.7$ and $(\boldsymbol{c})$. Variation of displacement u vs. x at $t = 1.2$

distance to get zero value, i.e. the effective region of displacement increases with time. However, we notice that the stationary point at which displacement attains a maximum value shifts to the right in the x-axis as time is increased. A significant difference is noted for the displacement field predicted by different models. However, it is observed that the new model predicts very much similar result like GN-III model. After traveling a distance, the displacement distribution under GN-III and new model almost merged to each other. At all time, near the boundary of the half space and till the middle of the region of effect, this filed shows significantly different values in the contexts of GN-II and LS models as compared to GN-III model and present model although after some distance the difference decreases between LS model and GN-III model or new model. The difference between GN-II model and other three models is very much prominent. The region of influence is much larger in cases of GN-III model and new model as compared to other models.

Variation of temperature is depicted in Figs. 5.2(a,b,c) which indicate that temperature field has maximum value unity at the boundary of the half space (which agrees with the boundary condition of our problem), then it decreases to zero value after some distance from the boundary. A prominent difference is observed for the values of temperature under different models although the profiles under new model shows very close nature with GN-III model and other two models (LS and GN-II) predict significantly different values as compared to these two models. The difference is more prominent near the boundary and at higher time. A very significant nature of temperature distribution is observed under different models:

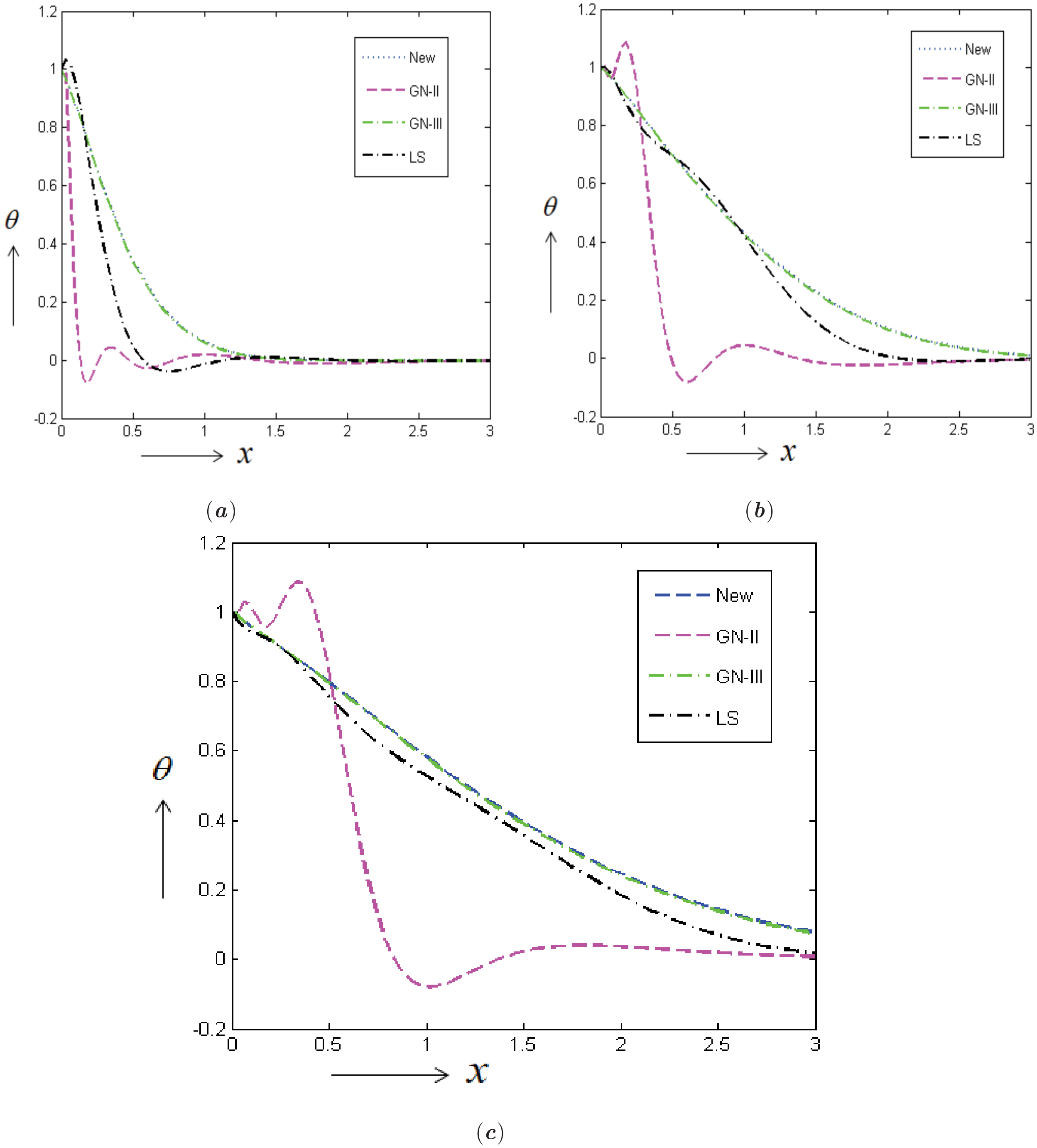
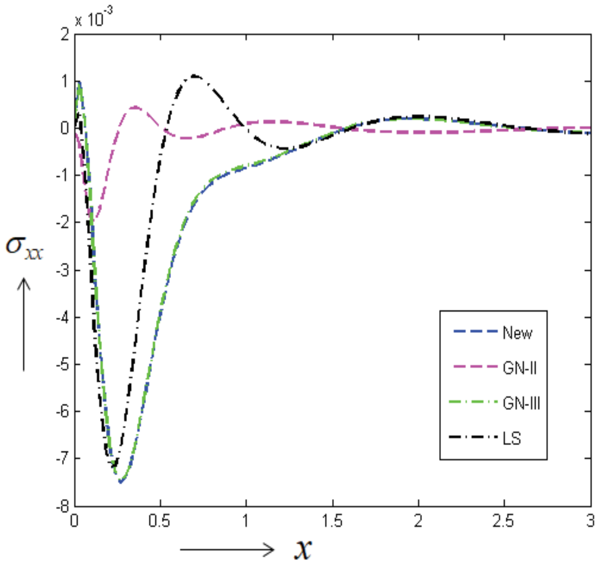


Figure 5.2: **(a).**Variation of temperature θ vs. x at $t = 0.1$, **(b).** Variation of temperature θ vs. x at $t = 0.7$ and **(c).** Variation of temperature θ vs. x at $t = 1.2$

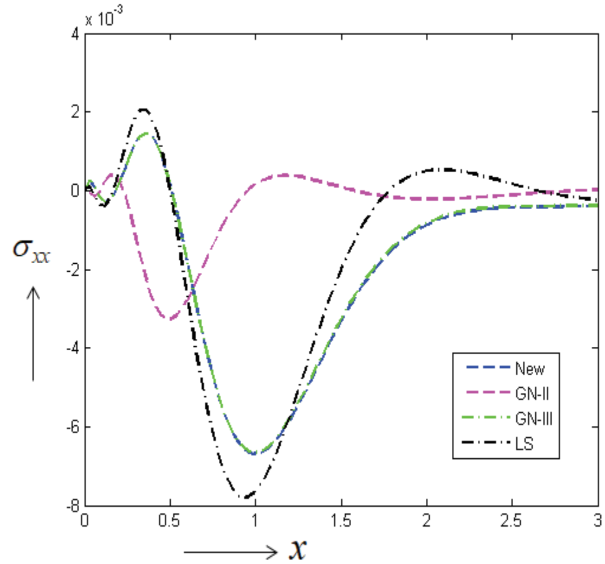
temperature field shows negative value (i.e., temperature goes less than the reference temperature) for a region after some distance from the boundary in the context of GN-II model. However, this behavior is not observed under LS model, GN-III model or new model. At higher time, the effective region of temperature field is much more larger in the contexts of GN-III model and new model as compared to LS and GN-II models.

Figs. 5.3(a,b,c) show the variations of stress field under four different thermoelastic models with respect to distance x and for three different times t. It shows that the stress is zero at $x = 0$ which satisfies the boundary condition of the problem. It is evident from Figs. 5.3(a)-(c) that the stress field shows oscillatory behavior near the boundary of the half space. The stress field becomes compressive for a part of the effective region just after some distance from the boundary and it attains an absolute maximum value in this region under all models. This absolute maximum value increases with time. The maximum value is attained at the earliest under GN-II model. At higher time, this value is maximum in the context of LS model. With the increase of time, the location of the stationary points at which the absolute maximum values is attained shifts towards the right in x-axis (see Figs. 5.3(a,b,c)). Finally the stress field ends to zero value, though the region of influence is much larger in cases of GN-III model and new model as compared to other two models.

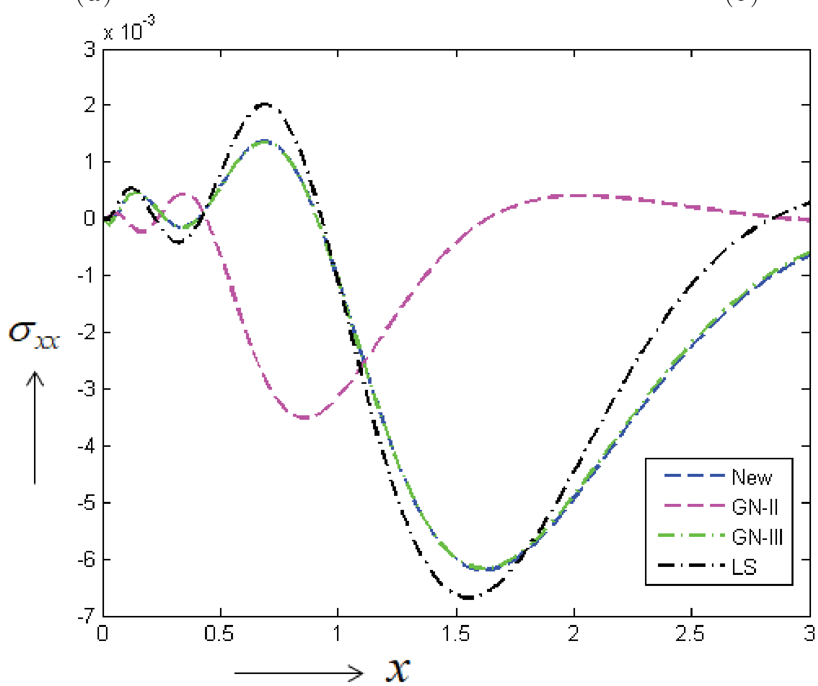
All the Figures indicate that the new thermoelastic model based on the exact heat conduction model with a delay as proposed by Leseduarte and Quintanilla(2013) predicts almost similar results like GN-III model as compared to GN-II and LS models.



($\boldsymbol{a}$)



($\boldsymbol{b}$)



($\boldsymbol{c}$)

Figure 5.3: **($\boldsymbol{a}$).**Variation of stress σ_{xx} vs. x at $t = 0.1$, **($\boldsymbol{b}$).** Variation of stress σ_{xx} vs. x at $t = 0.7$ and **($\boldsymbol{c}$).** Variation of stress σ_{xx} vs. x at $t = 1.2$