CHAPTER 2

AN INVESTIGATION ON THERMOELASTIC INTERACTIONS UNDER TWO-TEMPERATURE THERMOELASTICITY WITH TWO RELAXATION PARAMETERS[1]

2.1 Introduction

A thermoelasticity theory, called as two-temperature thermoelasticity theory is proposed by Chen and Gurtin (1968), Chen and William (1968), and Chen *et al.* (1969). This two-temperature thermoelasticity theory proposes that the heat conduction on a deformable body depends on two different temperatures- the conductive temperature, and the thermodynamic temperature (see Gurtin and Williams (1966), Chen and Gurtin (1968), Chen and William (1968), and Chen *et al.* (1969)). Here, the entropy contribution due to heat conduction is governed by thermodynamic temperature and that of the heat supply by the conductive temperature. The stress, energy, entropy, heat-flux and the thermodynamic temperature at a given time depend on the histories up to that time of the deformation gradient, the conductive temperature and the gradient of this temperature. Further, the difference between these two temperatures is proportional to the heat supply. In the case of the absence of a

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heat supply the two temperatures are equal for the time-independent situation. However, for time-dependent cases, the two temperatures are, in general, different. Boley and Tolins (1962) discussed one transient coupled thermo-plastic boundary value problem in half space by employing this theory. Uniqueness and reciprocity theorems for the two-temperature thermoelasticity theory in case of a homogeneous and isotropic solid have been reported by Iesan (1970). Later on, wave propagation in the two-temperature theory was investigated by Warren and Chen (1973). An analytical study of a one-dimensional conductive temperature equation in the uncoupled context in a half space with the Heaviside boundary condition was studied by Amos (1969). Recently, this two-temperature model of thermoelasticity has drawn the serious attention of researchers. Puri and Jordan (2006) reported a detailed investigation on a plane harmonic wave under this theory. Youssef (2006b) extended this theory in the frame of the generalized theory of heat conduction and formulated two versions of two-temperature theory with relaxation parameters by providing the uniqueness theorem. Subsequently, some investigations (Youssef (2006a), Youssef and Al-Lehaibi (2007) and Youssef and Al-Harby (2007)) are carried out on the basis of this two-temperature thermoelastic model with one relaxation time. Recently, the same model has been investigated in a detailed manner by considering two aspects by Mukhopadhyay and Kumar (2009) and Kumar and Mukhopadhyay (2010a).

The present chapter aims at investigating the thermoelastic interactions in an isotropic homogeneous elastic medium with a cylindrical cavity in the context of the two-temperature theory of thermoelasticity with two relaxation time parameters. Chandrasekharaiah and Keshavan (1992) have studied axisymmetric thermoelastic interactions in an unbounded body with a cylindrical cavity by using the classical coupled thermoelastic model, the Lord–Shulman model and Green–Lindsay (GL) model in a unified way. The results of the present work are compared with the corresponding results reported in Chandrasekharaiah and Keshavan (1992) and Mukhopadhyay and Kumar (2009). Some distinct predictions of the two-temperature model as compared to the conventional one-temperature model are investigated through this study, which indicate some significant features of the two-temperature thermoelastic models.

2.2 Problem Formulation: The basic governing equations

In the absence of body forces and heat sources, the basic governing equations for thermoelastic interactions in a homogeneous isotropic solid in the contexts of the GL theory and the two-temperature Green–Lindsay (TGL) theory can be written in a unified way as follows (Youssef (2006b)):

stress–strain temperature relation

$$\sigma_{ij} = \lambda e \delta_{ij} + 2\mu e_{ij} - \gamma \left(\theta + \tau_1 \frac{\partial \theta}{\partial t} \right) \delta_{ij} \tag{2.1}$$

strain-displacement relation

$$e_{ij} = \frac{1}{2} \left(u_{i,j} + u_{j,i} \right) \tag{2.2}$$

and heat conduction equation

$$K\phi_{,ii} = \rho c_E \left(\frac{\partial \theta}{\partial t} + \tau_0 \frac{\partial^2 \theta}{\partial t^2} \right) + \gamma T_0 \frac{\partial e}{\partial t} \tag{2.3}$$

where the dilatation e is given by

$$e = e_{kk} \tag{2.4}$$

The conductive temperature is related to thermodynamic temperature as

$$\phi - \theta = \alpha \phi_{,ii} \tag{2.5}$$

where $\alpha > 0$, the two-temperature parameter, is a scalar parameter and called the temperature discrepancy (Chen *et al.* (1969)).

The corresponding equations in the cases of the GL model and the TGL model correspond to the cases of $\alpha = 0$ and $\alpha > 0$ in equation (2.5) respectively.

We consider an infinite isotropic elastic medium with a cylindrical cavity of radius a. The center of the cavity is taken to be the origin of the cylindrical polar coordinate system (r, φ, z). In the absence of body forces and heat sources and considering the axisymmetric plane strain problem, the displacement and temperature are taken to be functions of r and t only. Therefore, the equation of motion is given by

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\phi\phi}}{r} = \rho \frac{\partial^2 u}{\partial t^2} \tag{2.6}$$

The non-zero strain components are obtained from equation (2.2) as

$$e_{rr} = \frac{\partial u}{\partial r}, \quad e_{\varphi\varphi} = \frac{u}{r} \tag{2.7}$$

and from equation (2.4), we get

$$e = \frac{\partial u}{\partial r} + \frac{u}{r} \tag{2.8}$$

Equations (2.1) and (2.7) then yield the non-zero stress components as

$$\sigma_{rr} = (\lambda + 2\mu)\frac{\partial u}{\partial r} + \lambda\frac{u}{r} - \gamma\left(1 + \tau_1\frac{\partial}{\partial t}\right)\theta \tag{2.9}$$

$$\sigma_{\varphi\varphi} = (\lambda + 2\mu)\frac{u}{r} + \lambda\frac{\partial u}{\partial r} - \gamma\left(1 + \tau_1\frac{\partial}{\partial t}\right)\theta \tag{2.10}$$

From equations (2.3) and (2.5), we get

$$\left[K + \alpha\rho c_E\left(\frac{\partial}{\partial t} + \tau_0\frac{\partial^2}{\partial t^2}\right)\right]\nabla^2\phi = \rho c_E\left(\frac{\partial \phi}{\partial t} + \tau_0\frac{\partial^2\phi}{\partial t^2}\right) + \gamma T_0\frac{\partial e}{\partial t} \tag{2.11}$$

where $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}$

From equation (2.5),(2.6),(2.9) and (2.10) , we get

$$(\lambda + 2\mu)\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right) - \gamma\left(1 + \tau_1\frac{\partial}{\partial t}\right)\left[\frac{\partial \phi}{\partial r} - \alpha\frac{\partial}{\partial r}\left(\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r}\right)\right] = \rho\frac{\partial^2 u}{\partial t^2} \tag{2.12}$$

Now, in what follows we will use the following dimensionless terms and variables:

$$(r'.u', \xi_0) = \frac{c_0}{\kappa}(r, u, a), \; (t'.\tau_0', \tau_1') = \frac{c_0^2}{\kappa}(t, \tau_0, \tau_1), \; (\theta', \phi') = \frac{1}{T_0}(\theta, \phi), \omega = \alpha c_0^2\eta^2,$$

$$\sigma'_{rr} = \frac{\sigma_{rr}}{(\lambda + 2\mu)},\ \sigma'_{\varphi\varphi} = \frac{\sigma_{\varphi\varphi}}{(\lambda + 2\mu)},\ e' = e,\ c_0^2 = \frac{(\lambda + 2\mu)}{\rho},\ \kappa = \frac{K}{\rho c_E},\ \lambda_1 = \frac{\lambda}{(\lambda + 2\mu)}$$

Equations (2.9)–(2.12) then transform to the dimensionless forms (after dropping the primes for convenience) as

$$\left[1 + \omega\left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2}\right)\right]\nabla^2\phi = \left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2}\right)\phi + a_2 \frac{\partial e}{\partial t} \tag{2.13}$$

$$\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right) - a_1\left(1 + \tau_1 \frac{\partial}{\partial t}\right)\left[\frac{\partial \phi}{\partial r} - \omega \frac{\partial}{\partial r}\left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r}\right)\right] = \frac{\partial^2 u}{\partial t^2} \tag{2.14}$$

$$\sigma_{rr} = \frac{\partial u}{\partial r} + \lambda_1 \frac{u}{r} - a_1\left(1 + \tau_1 \frac{\partial}{\partial t}\right)\theta \tag{2.15}$$

$$\sigma_{\varphi\varphi} = \frac{u}{r} + \lambda_1 \frac{\partial u}{\partial r} - a_1\left(1 + \tau_1 \frac{\partial}{\partial t}\right)\theta \tag{2.16}$$

where we have used the notations

$$a_1 = \frac{\gamma\theta_0}{(\lambda + 2\mu)}, \quad a_2 = \frac{\gamma\kappa}{K}, \quad \lambda_1 = \frac{\lambda}{(\lambda + 2\mu)}$$

2.2.1 Boundary Conditions

To consider the thermoelastic interactions in the medium, we assume that the surface of the cavity ($r = \xi$) is stress free and is subjected to a thermal shock. The boundary conditions of the problem are therefore taken as

$$\sigma_{rr} \mid_{r=\xi} = 0 \tag{2.17}$$

$$\phi \mid_{r=\xi} = \phi_0 H(t) \tag{2.18}$$

where ϕ_0 is a constant temperature and $H(t)$ is the Heaviside unit function.

2.3 Solution of the Problem

For the solution of the problem, we apply Laplace transform defined by

$$\bar{f}(p) = \int_0^{\infty} f(t) \mathrm{e}^{-pt} dt \tag{2.19}$$

Therefore, by considering homogeneous initial conditions and applying Laplace transform on equations (2.13)–(2.18), we obtain

$$\left[1 + \omega\left(p + \tau_0 p^2\right)\right] \left[\frac{\partial^2 \bar{\phi}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{\phi}}{\partial r}\right] = \left(p + \tau_0 p^2\right) \bar{\phi} + a_2 p \bar{e} \tag{2.20}$$

$$\left(\frac{\partial^2 \bar{u}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{u}}{\partial r} - \frac{\bar{u}}{r^2}\right) - a_1 (1 + \tau_1 p) \left[\frac{\partial \bar{\phi}}{\partial r} - \omega \frac{\partial}{\partial r}\left(\frac{\partial^2 \bar{\phi}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{\phi}}{\partial r}\right)\right] = p^2 \bar{u} \tag{2.21}$$

$$\bar{\sigma}_{rr} = \frac{\partial \bar{u}}{\partial r} + \lambda_1 \frac{\bar{u}}{r} - a_1 (1 + \tau_1 p) \bar{\theta} \tag{2.22}$$

$$\bar{\sigma}_{\varphi\varphi} = \frac{\bar{u}}{r} + \lambda_1 \frac{\partial \bar{u}}{\partial r} - a_1 (1 + \tau_1 p) \bar{\theta} \tag{2.23}$$

$$\bar{\phi} - \bar{\theta} = \omega \nabla^2 \bar{\phi} \tag{2.24}$$

$$\bar{\sigma}_{rr} \mid_{r=\xi} = 0 \tag{2.25}$$

$$\bar{\phi} \mid_{r=\xi} = \frac{\phi_0}{p} \tag{2.26}$$

$$\bar{e} = \frac{\partial \bar{u}}{\partial r} + \frac{\bar{u}}{r}$$

Then, by decoupling equations (2.20) and (2.21), we obtain

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - m_1^2 \right) \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - m_2^2 \right) \bar{\phi} = 0 \tag{2.27}$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} - m_1^2 \right) \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} - m_2^2 \right) \bar{u} = 0 \tag{2.28}$$

where m_i, $(i = 1, 2)$ are the roots of the equation

$$\left[1 + \omega\epsilon p + \omega\beta p^2\right] m^4 - \left[p\epsilon + (\beta + 1)\, p^2 + \omega p^3 + \omega\tau_0 p^4\right] m^2 + (p^3 + \tau_0 p^4) = 0 \tag{2.29}$$

In equation (2.29), $\varepsilon = (1 + \varepsilon_0)$, $\beta = \tau_0 + \varepsilon_0\tau_1$ where $\varepsilon_0 = a_1 a_2 = \frac{\gamma^2 T_0}{\rho^2 c_E c_0^2}$ is the thermoelastic coupling constant.

Now, the solutions of equations (2.27) and (2.28) bounded at infinity are given by

$$\bar{\phi} = \sum_{i=1}^{2} A_i K_0(m_i r) \tag{2.30}$$

$$\bar{u} = \sum_{i=1}^{2} B_i K_1(m_i r) \tag{2.31}$$

where $K_0(m_i r)$ and $K_1(m_i r)$ are the modified Bessel functions of the second kind of order zero and one, respectively and A_i and B_i are the arbitrary constants.

The solution for $\bar{\theta}$ is now obtained from equations (2.24) and (2.30) as

$$\bar{\theta} = \sum_{i=1}^{2} A_i (1 - \omega m_i^2) K_0(m_i r) \tag{2.32}$$

Using equations (2.20), (2.30) and (2.31), we obtain the relations between the constants A_i and B_i as follows:

$$B_i = F_i A_i, \quad where \quad F_i = \frac{\left(p + \tau_0 p^2\right) - m_i^2 \left[1 + \omega \left(p + \tau_0 p^2\right)\right]}{a_2 p} \tag{2.33}$$

Now, using equations (2.22) and (2.23) and the recurrence relations

$$x K_n^{'}(x) = K_n(x) - x K_{n+1}(x)$$

$$x K_n^{'}(x) = -n K_n(x) - x K_{n-1}(x)$$

we obtain the solutions for the radial and circumferential stresses as

$$\bar{\sigma}_{rr} = \sum_{i=1}^{2} A_i S_i^r \tag{2.34}$$

$$\bar{\sigma}_{\varphi\varphi} = \sum_{i=1}^{2} B_i K_1 S_i^{\phi} \tag{2.35}$$

where

$$S_i^r = F_i \left\{ -m_i K_0(m_i r) + \frac{(\lambda_1 - 1)}{r} K_1(m_i r) \right\} - a_1 (1 + \tau_1 p)(1 - \omega m_i^2) K_0(m_i r), \ \ i = 1, 2$$

$$S_i^{\phi} = F_i \left\{ -m_i K_0(m_i r) + \frac{(1 - \lambda_1)}{r} K_1(m_i r) \right\} - a_1 (1 + \tau_1 p)(1 - \omega m_i^2) K_0(m_i r), \ \ i = 1, 2$$

The boundary conditions in equations (2.25) and (2.26) yield the constants A_1 and A_2 as

$$A_1 = -\frac{S_2^{\xi} \phi_0}{p \left[S_1^{\xi} K_0(m_2 \xi) - S_2^{\xi} K_0(m_1 \xi) \right]}$$

$$A_1 = \frac{S_1^{\xi} \phi_0}{p \left[S_1^{\xi} K_0(m_2 \xi) - S_2^{\xi} K_0(m_1 \xi) \right]}$$

where

$$S_i^{\xi} = F_i \left\{ -m_i K_0(m_i \xi) + \frac{(\lambda_1 - 1)}{\xi} K_1(m_i \xi) \right\} - a_1 (1 + \tau_1 p)(1 - \omega m_i^2) K_0(m_i \xi), \quad i = 1, 2$$

Equations (2.30)–(2.32), (2.34) and (2.35) constitute the solution of the problem in the transformed domain (r, p).

2.4 Small-time Approximated Solutions

The solutions in the physical domain are determined by inverting the Laplace transforms involved in the expressions of equation (2.30)–(2.35). But because of the dependency of m_1 and m_2 on the Laplace transform parameter p, it is impossible to carry out this operation exactly for all values of p. However, the present study is concerned with the thermoelasticity theory with relaxation parameters which is more relevant for the problems involving a short duration of time. Therefore, in this section, we will now confine our attention on obtaining the short-time approximated solutions of the field variables for which we assume p is very large.

Therefore, with the help of Maclaurin's series expansions and neglecting the higher powers of the small terms, we get the roots of equation (2.29) for large values of p as follows:

$$m_i \approx b_{i0}p + b_{i1} + b_{i2}\frac{1}{p}, \quad i = 1, 2 \qquad \text{(for the GL model)} \tag{2.36}$$

$$\left.\begin{aligned} m_1 &\approx \sqrt{\tfrac{\tau_0}{\beta}}p + \tfrac{1}{2}\tfrac{\beta-\varepsilon\tau_0}{\beta\sqrt{\tau_0\beta}} - C\tfrac{1}{p} \\ m_2 &\approx \tfrac{1}{\sqrt{\omega}} - \tfrac{1}{2\tau_0\omega\sqrt{\omega}}\tfrac{1}{p^2} \end{aligned}\right\} \qquad \text{(for the TGL model)} \tag{2.37}$$

where

$$a_{i0} = \frac{\beta + (-1)^{i+1}\sqrt{\beta^2 - 4\tau_0}}{2}, \quad a_{i1} = \frac{1}{2}\left(\varepsilon + (-1)^{i+1}\frac{\varepsilon\beta - 2}{\sqrt{\beta^2 - 4\tau_0}}\right)$$

$$a_{i2} = (-1)^{i+1} \frac{1}{4\sqrt{\beta^2 - 4\tau_0}} \left(\varepsilon^2 - \frac{(\beta - \varepsilon\tau_0)^2}{\beta^2 - 4\tau_0} \right)$$

$$C = \frac{2\beta\tau_0 (2\tau_0 + \varepsilon\omega) - \beta (4\tau_0 - \omega) - 3\varepsilon^2\tau_0^2\omega}{8\beta^2\tau_0\omega\sqrt{\beta\tau_0}}$$

with $b_{i0} = \sqrt{a_{i0}}$, $b_{i1} = \frac{1}{2}\frac{a_{i1}}{\sqrt{a_{i0}}}$, $b_{i2} = \frac{4a_{i2}a_{10} - (a_{i1})^2}{8(a_{i0})^{3/2}}$

Now, we substitute m_1 and m_2 from equations (2.36) and (2.37) into equations (2.30)–(2.35) and use the following approximation formula for the modified Bessel function of order ν

$$K_\nu(z) \approx \sqrt{\frac{\pi}{2z}} \left[1 + \frac{4\nu^2 - 1}{8z} + \frac{(4\nu^2 - 1)(4\nu^2 - 9)}{2!\,(8z)^2} + ... \right]$$

Therefore, after detailed calculations, we obtain the expressions for the short-time approximated solutions of displacement, thermodynamic temperature, conductive temperature and radial stress in the Laplace transform domain (r, p) for two different models as follows:

For the TGL model:

$$\bar{u} = \sqrt{\frac{\xi}{r}}\phi_0 \left(\sum_{j=1}^{2} \mathrm{e}^{-m_1(r-\xi)} \left[\frac{u_{1j}}{p^{j+2}} \right] + \sum_{j=1}^{2} \mathrm{e}^{-m_2(r-\xi)} \left[\frac{u_{2j}}{p^{j+3}} \right] \right) \tag{2.38}$$

$$\bar{\phi} = \sqrt{\frac{\xi}{r}}\phi_0 \left(\sum_{j=1}^{2} -\mathrm{e}^{-m_1(r-\xi)} \left[\frac{\phi_{1j}}{p^{j+5}} \right] + \sum_{j=1}^{2} \mathrm{e}^{-m_2(r-\xi)} \left[\frac{\phi_{2j}}{p^{j}} \right] \right) \tag{2.39}$$

$$\bar{\theta} = \sqrt{\frac{\xi}{r}}\phi_0 \left(\sum_{j=1}^{2} e^{-m_1(r-\xi)} \left[\frac{\theta_{1j}}{p^{j+3}} \right] + \sum_{j=1}^{2} e^{-m_2(r-\xi)} \left[\frac{\theta_{2j}}{p^{j+2}} \right] \right) \tag{2.40}$$

$$\bar{\sigma}_{rr} = \frac{\phi_0}{a_2 D}\sqrt{\frac{\xi}{r}} \sum_{i=1}^{2}\sum_{j=1}^{2} e^{-m_i(r-\xi)}(-1)^i \left[\frac{\sigma^r_{ij}}{p^{j+1}} \right] \tag{2.41}$$

where we have used the following notations

$$u_{11} = \frac{-a_1\tau_1\sqrt{\beta}}{\omega\tau_0\sqrt{\tau_0}},\ \phi_{11} = \frac{a_1a_2\tau_1\beta\sqrt{\beta}}{\omega^2\tau_0^3\sqrt{\tau_0}},\ \theta_{11} = -\frac{a_1a_2\tau_1\sqrt{\beta}}{\omega\tau_0^2\sqrt{\tau_0}},\ \sigma^r_{21} = -\frac{a_1\tau_1\xi^2\left(128r^2-16r\sqrt{\omega}+9\omega\right)}{r^2\tau_0\left(16\xi-9\sqrt{\omega}\right)\omega\sqrt{\omega}},$$

$$u_{12} = \frac{a_1\left[8a_1a_2r\tau_1^2\beta\sqrt{\beta}\xi-3\tau_1\beta\xi\tau_0\sqrt{\beta}+r\left\{-8\beta\xi\tau_0\sqrt{\beta}+\tau_1\left(12\beta\xi\sqrt{\tau_0}+\beta\sqrt{\beta}(7-8\lambda_1)\tau_0-4\varepsilon\xi\tau_0\sqrt{\tau_0}\right)\right\}\right]}{8r\omega\xi\tau_0^3\sqrt{\beta}}.$$

$$u_{21} = \frac{\xi^2\left(128r^2+48r\sqrt{\omega}-15\omega\right)(-3\tau_0+4\omega)}{4a_2r^2\tau_0^2\left(16\xi-9\sqrt{\omega}\right)\omega^2\sqrt{\omega}},$$

$$u_{22} = \frac{\xi^2\left(128r^2+48r\sqrt{\omega}-15\omega\right)\left[-27\tau_0\omega+\xi\left\{-48\tau_0\left(\sqrt{\beta\tau_0}-\sqrt{\omega}\right)+64\omega\sqrt{\beta\tau_0}\right\}\right]}{4a_2r^2\tau_0^3\left(16\xi-9\sqrt{\omega}\right)\omega^3},$$

$$\phi_{12} = \frac{a_1a_2\sqrt{\beta}\left[8a_1a_2r\tau_1^2\beta\sqrt{\beta}\xi+\tau_1\beta\xi\tau_0\sqrt{\beta}+r\left\{-8\beta\xi\tau_0\sqrt{\beta}+\tau_1\left(28\beta\xi\sqrt{\tau_0}+\beta\sqrt{\beta}(7-8\lambda_1)\tau_0-12\varepsilon\xi\tau_0\sqrt{\tau_0}\right)\right\}\right]}{8r\omega^2\xi\tau_0^5},$$

$$\phi_{21} = -\frac{\xi^2\left(128r^2-16r\sqrt{\omega}+9\omega\right)}{r^2\left(16\xi\sqrt{\omega}-9\omega\right)},\ \phi_{22} = \frac{16\beta\varepsilon^2\left(128r^2-16r\sqrt{\omega}+9\omega\right)}{r^2\sqrt{\beta\tau_0}\left(16\xi-9\sqrt{\omega}\right)\omega},$$

$$\theta_{12} = \frac{a_1a_2\left[8a_1a_2r\tau_1^2\beta\sqrt{\beta}\xi+\tau_1\beta\xi\tau_0\sqrt{\beta}+r\left\{-8\beta\xi\tau_0\sqrt{\tau_0}+\tau_1\left(20\beta\xi\sqrt{\tau_0}+\beta\sqrt{\beta}(7-8\lambda_1)\tau_0-4\varepsilon\xi\tau_0\sqrt{\tau_0}\right)\right\}\right]}{8r\omega\xi\tau_0^4\sqrt{\beta}},$$

$$\theta_{21} = -\frac{\xi^2\left(128r^2-16r\sqrt{\omega}+9\omega\right)}{r^2\tau_0\left(16\xi-9\sqrt{\omega}\right)\omega\sqrt{\omega}},\ \theta_{22} = \frac{\varepsilon^2\left[-16\beta\left(\sqrt{\beta\tau_0}-\sqrt{\omega}\right)-9\omega\right]\left(128r^2-16r\sqrt{\omega}+9\omega\right)}{r^2\tau_0^2\left(16\xi-9\sqrt{\omega}\right)^2\omega^2},$$

$$\sigma^r_{11} = \frac{a_1\tau_1}{\omega\tau_0},\ \sigma^r_{12} = \frac{a_1\left[(7-8\lambda_1)\xi\sqrt{\beta\tau_0}+r\left\{8\xi\tau_0+\tau_1\left(-8\xi+(7-8\lambda_1)\sqrt{\beta\tau_0}\right)\right\}\right]}{8r\omega\xi\tau_0^2},$$

$$\sigma^r_{22} = \frac{a_1\xi^2\left[16\xi\tau_0\sqrt{\omega}-9\omega\tau_0+\tau_1\left\{\left(\sqrt{\beta\tau_0}-\sqrt{\omega}\right)+9\omega\right\}\right]\left(128r^2-16r\sqrt{\omega}+9\omega\right)}{r^2\tau_0^2\left(16\xi-9\sqrt{\omega}\right)^2\omega^2}$$

For the GL model:

$$\bar{u} = \frac{\phi_0}{a_2 D_1}\sqrt{\frac{\xi}{r}} \sum_{i=1}^{2}\sum_{j=1}^{2} (-1)^i e^{-m_i(r-\xi)} \left[\frac{u_{ij}}{p^j} \right] \tag{2.42}$$

$$\bar{\theta} = \frac{\phi_0}{D_1}\sqrt{\frac{\xi}{r}} \sum_{i=1}^{2}\sum_{j=1}^{2} (-1)^i e^{-m_i(r-\xi)} \left[\frac{\theta_{ij}}{p^j} \right] \tag{2.43}$$

$$\bar{\sigma}_{rr} = \frac{\phi_0}{a_2 D_1}\sqrt{\frac{\xi}{r}}\sum_{i=1}^{2}\sum_{j=1}^{2}(-1)^i e^{-m_i(r-\xi)}\left[\frac{\sigma_{ij}^r}{p^{j-1}}\right] \tag{2.44}$$

where the corresponding notations of the TGL model reduce to the following expressions for this case

$D_1 = S_{11}^{\xi} - S_{21}^{\xi}$, $D_2 = S_{12}^{\xi} - S_{22}^{\xi} + \left(\frac{S_{21}^{\xi}}{b_{10}} - \frac{S_{11}^{\xi}}{b_{20}}\right)\frac{1}{8\xi}$,

$S_{i1}^r = -b_{i0}F_{i1} - a_1a_2\tau_1$, $S_{i2}^r = b_{i0}F_{i2} - b_{i1}F_{i1} - a_1a_2 + \frac{a_1a_2\tau_1}{8rb_{i0}} + \frac{(8\lambda_1-7)F_{i1}}{8r}$, $(i=1,2)$,

$S_{ij}^{\xi} = S_{ij}^r\Big|_{r=\xi}$ $(i,j=1,2)$,

$F_{i1} = \left(\frac{\tau_0}{b_{i0}} - \frac{\tau_0 b_{i1}}{(b_{i0})^2} - b_{i0}\right)$, $F_{i2} = \left(\frac{1}{b_{i0}} - b_{i1}\right)$,

$u_{11} = F_{11}S_{21}^{\xi}$, $u_{12} = F_{12}S_{21}^{\xi} + F_{11}S_{22}^{\xi} - \frac{D_2}{D_1}F_{11}S_{21}^{\xi} + \frac{3F_{11}S_{21}^{\xi}}{8rb_{10}}$,

$u_{21} = F_{21}S_{11}^{\xi}$, $u_{22} = F_{22}S_{11}^{\xi} + F_{21}S_{12}^{\xi} - \frac{D_2}{D_1}F_{12}S_{11}^{\xi} + \frac{3F_{21}S_{11}^{\xi}}{8rb_{20}}$,

$\theta_{11} = S_{21}^{\xi}$, $\theta_{12} = S_{22}^{\xi} - \frac{D_2}{D_1}S_{21}^{\xi} - \frac{S_{21}^{\xi}}{8rb_{10}}$,

$\theta_{21} = S_{11}^{\xi}$, $\theta_{12} = S_{12}^{\xi} - \frac{D_2}{D_1}S_{11}^{\xi} - \frac{S_{11}^{\xi}}{8rb_{20}}$,

$\sigma_{11}^r = S_{21}^{\xi}S_{11}^r$, $\sigma_{12}^r = S_{22}^{\xi}S_{11}^r + S_{21}^{\xi}S_{12}^r - \frac{D_2}{D_1}S_{21}^{\xi}S_{11}^r$,

$\sigma_{21}^r = S_{21}^r S_{11}^{\xi}$, $\sigma_{22}^r = S_{12}^{\xi}S_{21}^r + S_{11}^{\xi}S_{22}^r - \frac{D_2}{D_1}S_{11}^{\xi}S_{21}^r$

2.5 Analytical Results

To invert the Laplace transforms involved in equations (2.38)–(2.44), we use the convolution theorem of Laplace transform and the following formulas (Oberhettinger and Badii,1973)

$$L^{-1}\left[\frac{e^{-\frac{a}{p}}}{p^{\nu+1}}\right] = \left(\frac{t}{a}\right)^{\frac{\nu}{2}} J_\nu\left(2\sqrt{at}\right), \quad Re(\nu) > -1,\ a > 0$$

$$L^{-1}\left[\frac{e^{\frac{a}{p}}}{p^{\nu+1}}\right] = \left(\frac{t}{a}\right)^{\frac{\nu}{2}} I_\nu\left(2\sqrt{at}\right), \quad Re(\nu) > -1,\ a > 0$$

$$L^{-1}\left[e^{\frac{a}{p}}\right]=\delta(t)+\sqrt{\frac{t}{a}}I_1\left(2\sqrt{at}\right),\qquad a>0$$

$$L^{-1}\left[e^{-\frac{a}{p}}\right]=\delta(t)-\sqrt{\frac{t}{a}}J_1\left(2\sqrt{at}\right),\qquad a>0$$

where J_ν and I_ν are the Bessel function and the modified Bessel function respectively. Since it can be shown that $b_{12}<0$ and $b_{21}>0$ for the GL model and $C>0$ for the TGL model, the solutions of all field variables in the physical domain (r,t) are obtained as follows:

For the TGL model:

$$u(r,t)=\phi_0\sqrt{\frac{\xi}{r}}\left[e^{-\frac{\beta-\varepsilon\tau_0}{2\beta\sqrt{\tau_0\beta}}(r-\xi)}\sum_{j=1}^{2}\left\{u_{1j}\left(\frac{\eta_1}{C(r-\xi)}\right)^{\frac{j+1}{2}}J_{j+1}(\eta_1^{'})H(\eta_1)\right\}+\mathrm{e}^{-\frac{1}{\sqrt{\omega}}(r-\xi)}\sum_{j=1}^{2}\left\{u_{2j}\frac{t^{j+2}}{(j+2)!}\right\}\right]$$

$$\phi(r,t)=\phi_0\sqrt{\frac{\xi}{r}}\left[e^{-\frac{\beta-\varepsilon\tau_0}{2\beta\sqrt{\tau_0\beta}}(r-\xi)}\sum_{j=1}^{2}\left\{\phi_{1j}\left(\frac{\eta_1}{C(r-\xi)}\right)^{\frac{j+4}{2}}J_{j+4}(\eta_1^{'})H(\eta_1)\right\}+\mathrm{e}^{-\frac{1}{\sqrt{\omega}}(r-\xi)}\sum_{j=1}^{2}\left\{\phi_{2j}\frac{t^{j-1}}{(j-1)!}\right\}\right]$$

$$\theta(r,t)=\phi_0\sqrt{\frac{\xi}{r}}\left[e^{-\frac{\beta-\varepsilon\tau_0}{2\beta\sqrt{\tau_0\beta}}(r-\xi)}\sum_{j=1}^{2}\left\{\theta_{1j}\left(\frac{\eta_1}{C(r-\xi)}\right)^{\frac{j+2}{2}}J_{j+2}(\eta_1^{'})H(\eta_1)\right\}+\mathrm{e}^{-\frac{1}{\sqrt{\omega}}(r-\xi)}\sum_{j=1}^{2}\left\{\theta_{2j}\frac{t^{j+1}}{(j+1)!}\right\}\right]$$

$$\sigma_{rr}(r,t)=\phi_0\sqrt{\frac{\xi}{r}}\left[e^{-\frac{\beta-\varepsilon\tau_0}{2\beta\sqrt{\tau_0\beta}}(r-\xi)}\sum_{j=1}^{2}\left\{\sigma^r_{1j}\left(\frac{\eta_1}{C(r-\xi)}\right)^{\frac{j}{2}}J_{j}(\eta_1^{'})H(\eta_1)\right\}+\mathrm{e}^{-\frac{1}{\sqrt{\omega}}(r-\xi)}\sum_{j=1}^{2}\left\{\sigma^r_{2j}\frac{t^{j}}{j!}\right\}\right]$$

where

$$\eta_1=t-\sqrt{\frac{\tau_0}{\beta}}(r-\xi),\quad \eta_1^{'}=2\sqrt{C(r-\xi)\left(t-\sqrt{\frac{\tau_0}{\beta}}(r-\xi)\right)}$$

and clearly the definitions of β and ε imply

$$\frac{\beta-\varepsilon\tau_0}{\beta\sqrt{\tau_0\beta}}=\frac{a_1a_2\left(\tau_1-\tau_0\right)}{\beta\sqrt{\tau_0\beta}}>0$$

For the GL model:

$$u(r,t) = \frac{\phi_0}{D_1 a_2}\sqrt{\frac{\xi}{r}}\left[\begin{array}{c} \sum_{j=1}^{2} -e^{-b_{11}(r-\xi)}\left\{u_{1j}\left(\frac{\eta_1}{b_{12}(r-\xi)}\right)^{\frac{j-1}{2}} J_{j-1}\left(2\sqrt{b_{12}(r-\xi)\eta_1}\right)H(\eta_1)\right\} \\ +\sum_{j=1}^{2} -e^{-b_{21}(r-\xi)}\left\{u_{2j}\left(\frac{\eta_2}{-b_{22}(r-\xi)}\right)^{\frac{j-1}{2}} I_{j-1}\left(2\sqrt{-b_{22}(r-\xi)\eta_2}\right)H(\eta_2)\right\} \end{array}\right]$$

$$\theta(r,t) = \frac{\phi_0}{D_1}\sqrt{\frac{\xi}{r}}\left[\begin{array}{c} \sum_{j=1}^{2} -e^{-b_{11}(r-\xi)}\left\{\theta_{1j}\left(\frac{\eta_1}{b_{12}(r-\xi)}\right)^{\frac{j-1}{2}} J_{j-1}\left(2\sqrt{b_{12}(r-\xi)\eta_1}\right)H(\eta_1)\right\} \\ +\sum_{j=1}^{2} e^{-b_{21}(r-\xi)}\left\{\theta_{2j}\left(\frac{\eta_2}{-b_{22}(r-\xi)}\right)^{\frac{j-1}{2}} I_{j-1}\left(2\sqrt{-b_{22}(r-\xi)\eta_2}\right)H(\eta_2)\right\} \end{array}\right]$$

$$\sigma_{rr}(r,t) = \frac{\phi_0}{D_1 a_2}\sqrt{\frac{\xi}{r}}\left[\begin{array}{c} -e^{-b_{11}(r-\xi)}\left[\begin{array}{c} \sigma^r_{11}\left\{\delta(\eta_1) - \left(\frac{b_{12}(r-\xi)}{\eta_1}\right)^{\frac{1}{2}} J_1\left(2\sqrt{b_{12}(r-\xi)\eta_1}\right)H(\eta_1)\right\} \\ +\sigma^r_{12}J_0\left(2\sqrt{b_{12}(r-\xi)\eta_1}\right)H(\eta_1) \end{array}\right] \\ +e^{-b_{21}(r-\xi)}\left[\begin{array}{c} \sigma^r_{21}\left\{\delta(\eta_2) + \left(\frac{b_{22}(r-\xi)}{\eta_2}\right)^{\frac{1}{2}} J_1\left(2\sqrt{-b_{22}(r-\xi)\eta_2}\right)H(\eta_2)\right\} \\ +\sigma^r_{22}I_0\left(2\sqrt{-b_{22}(r-\xi)\eta_2}\right)H(\eta_2) \end{array}\right] \end{array}\right]$$

where $\eta_1 = t - b_{10}(r-\xi)$, $\eta_2 = t - b_{20}(r-\xi)$.

2.6 Discussions

The short-time approximated solutions obtained above reveal that in the case of the TGL model, the solution for each field consists of two different parts. The first part, containing the term $H(\eta_1)$, represents the contribution of a wave (predominantly an elastic wave) near the wave front $r = t\sqrt{\frac{\beta}{\tau_0}} + \xi$ which propagates with finite velocity $\sqrt{\frac{\beta}{\tau_0}}$ and with an exponential decay. It is observed that neither the speed nor the decay coefficient depends on the two-temperature parameter. The other part of the solution does not represent the contribution of any wave but is distributed throughout the medium decaying exponentially with radial distance r. The decaying exponent is clearly observed to depend on the two-temperature parameter, α. The two-temperature model therefore does not indicate a finite speed of the thermal signal or a predominantly thermal wave as a contribution to the solution of the field variables.

On the contrary, in the case of the GL model, the solutions of all fields consist of two coupled waves. The term containing $H(\eta_1)$ represents the predominantly elastic wave near the wave front $r = \frac{t}{b_{10}} + \xi$ with velocity $\frac{1}{b_{10}}$ (called V_1). The second part represents the predominantly thermal wave propagating with finite speed $\frac{1}{b_{20}}$ (called V_2). Both the waves are seen to decay exponentially with radial distance.

The short-time approximated solutions further indicate a significant dissimilarity between the GL and TGL models regarding the nature of different filed variables at the wave fronts. In the case of the TGL model, all the field variables are continuous in nature, where in the case of the GL model the temperature and displacement fields have discontinuities with finite jumps at both the wave fronts $r = tV_1 + \xi$ and $r = tV_2 + \xi$. Furthermore, both the stress fields are observed to experience a δ-function singularity and infinite singularity near $r = tV_1 + \xi$ and $r = tV_2 + \xi$. A discontinuity in the displacement field is obviously an unrealistic feature of the GL model for this particular problem. The magnitudes of finite discontinuity in the case of the GL model are obtained as

$$u^+ - u^-\Big|_{r=tV_1+\xi} = \frac{\phi_0}{a_2 D_1}\sqrt{\frac{\xi}{r}}\left[-e^{-b_{11}\frac{t}{b_{10}}}u_{11}\right], \quad u^+ - u^-\Big|_{r=tV_2+\xi} = \frac{\phi_0}{a_2 D_1}\sqrt{\frac{\xi}{r}}\left[e^{-b_{21}\frac{t}{b_{20}}}u_{21}\right]$$

$$\sigma_{rr}^+ - \sigma_{rr}^-\Big|_{r=tV_1+\xi} = \frac{\phi_0}{a_2 D_1}\sqrt{\frac{\xi}{r}}\left[-e^{-b_{11}\frac{t}{b_{10}}}\sigma_{12}^r\right], \quad \sigma_{rr}^+ - \sigma_{rr}^-\Big|_{r=tV_2+\xi} = \frac{\phi_0}{a_2 D_1}\sqrt{\frac{\xi}{r}}\left[e^{-b_{21}\frac{t}{b_{20}}}\sigma_{22}^r\right]$$

$$\theta^+ - \theta^- \Big|_{r=tV_1+\xi} = \frac{\phi_0}{D_1}\sqrt{\frac{\xi}{r}}\left[-e^{-b_{11}\frac{t}{b_{10}}}\theta_{11}\right], \quad \theta^+ - \theta^- \Big|_{r=tV_2+\xi} = \frac{\phi_0}{D_1}\sqrt{\frac{\xi}{r}}\left[e^{-b_{21}\frac{t}{b_{20}}}\theta_{21}\right]$$

It must be mentioned here that the results for the GL model in the present study agree with the corresponding results reported by Chandrasekharaiah and Keshavan (1992).

By comparing the results of the two-temperature Lord–Shulman (TLS) model with one relaxation as reported by Mukhopadhyay and Kumar (2009) we also observe a very significant dissimilarity between the TLS and TGL models. In the case of the TLS model, unlike the TGL model, the predominantly elastic wave propagates without any attenuation.

Therefore, it can be emphasized that the discontinuities of finite jump for the temperature and displacement fields and the delta function singularities experienced by the stress fields for the case of the GL model are not observed in the case of the TGL model, which is a significant feature of the two-temperature thermoelastic model with two relaxation parameters. However, the two-temperature model does not predict a finite speed of the thermal signal and a predominantly thermal wave is not found as a contribution to the solution of the physical fields, although these theories include the thermal relaxation parameters. This is obviously a significant feature of the two-temperature model and these models therefore may not be referred to as generalized thermoelastic models which have a significant feature of admitting a finite speed of the thermal wave.