CHAPTER 1

INTRODUCTION AND LITERATURE REVIEW

1.1 Thermoelasticity

Deformation of a body is associated with a change of its heat content and the time varying loading of a body causes in it not only displacement but also temperature distribution changing in time. Conversely, the heating of a body produces in it deformation and temperature change. The motion of a body is, in general, characterized by mutual interaction between deformation and temperature fields. Furthermore, while designing mechanical equipments, in most of the cases thermal effects need to be considered besides mechanical ones, especially if they involve transient effects. This is due to the fact that mechanical equipments during operation are affected by various interactions, the most significant being the mechanical and thermal effects. Mechanical and thermal loads usually occur simultaneously and as a result, the displacement and temperature fields are created in close connection with each other. Hence, these two fields are defined simultaneously by taking into account the relationship between them. Thermomechanical processes are described by the basic equations of continuum mechanics and thermodynamics. In the solution of a variety of such problems, the application of 'thermoelasticity' proves to be efficient and in engineering practice these thermomechanical problems can be described by the different theories of thermoelasticity with adequate accuracy. In this case, the internal energy of the body is a function of the deformation and temperature. As a result of the coupling between these two fields, the temperature term appears in the displacement equations of motion, and the deformation is included in the equation of heat conduction. The principal and basic steps in the evolution of the thermoelastic stress analysis technique are identified and reviewed during 1805-1990. We can note some significant contributions in this regard. It is worth mentioning that beginnings and early development of thermoelastic stress observation and analysis have been made by a school of mechanical, aerospace and civil engineering of the university of Manchester and the first thermoelastic effect is recorded by Gough's observation (1805). Some of the earlier contribution in the theory of thermoelasticity was covered by Weber (1830) who first described the effect in metals. Kelvin (Thomson) (1853; 1857) had established the classical theoretical treatment. Joule (1857; 1857; 1859) analyzed the thermoelasticity of ferruginous metals and described the thermal effects of stretching solid bodies. Further, the thermodynamic properties of thermoelastic solids are studied to find the thermal effects of the longitudinal compression of solids. The historical records on the efforts of this kind are also mentioned in the publications of Todhunter (1886) providing the history of thermoelasticity and strength of materials. However, it is worth recalling that the foundations for coupled thermoelasticity were laid by Duhamel and Neuman in the first half of the last Century. The coupling between deformation and temperature fields was first postulated by Duhamel (1837; 1838), the originator of the theory of thermal stresses. He introduced the dilatation term in the equation of thermal conductivity. Later on, Neumann (1841) also formulated independently the stress-strain-temperature relations, similar to the relations given by Duhamel. Hence, these relations are now commonly known as 'Duhamel-Neumann' relations. However, this equation was not well grounded in the thermodynamical sense. Next, an attempt with thermodynamical justification of this equation was undertaken by Voigt (1910) and Jeffreys (1930). It is known, that research in the field of thermoelasticity was preceded by broad-scale investigations within the framework of uncoupled theory with the simplifying assumption that the deformation of an elastic body does not affect the thermal conductivity of the medium. However, as recently as in 1956, Biot gave the full justification of the thermal conductivity equation on the basis of thermodynamics of irreversible processes. Biot (1956) also presented the fundamental methods for solving the coupled thermoelasticity equation and established a variational theorem. He described thermoelasticity as a broad range of phenomena- the generalization of the classical theory of elasticity and of the theory of thermal conductivity. Biot (1956), Boley and Weiner (1960), Parkus (1976), Nowinski (1978) and numerous other scientists have dealt with the solution of the problems, and as a result of their work the theory of classical linear thermoelasticity was created based on the solid foundation of reversible thermodynamics. Now, the thermoelasticity is a fully formed domain of science. The fundamental relations and differential equations have been formulated. This field has been advanced by extensive theoretical as well as experimental research work carried out during last few decades. The reason for the sudden growth of interest is that the need for designing equipment that can operate at very high temperatures arose almost simultaneously in several dynamically developing areas of industry including production of high-speed aeroplanes, design of space vehicles, rocket and jet engines, technology of large turbines and the design of nuclear reactors.

1.2 Classical Coupled Theory of Thermoelasticity

Biot (1956) worked on the field of thermoelasticity based on irreversible thermodynamics and derived the constitutive relations and basic governing equations of thermoelasticity by taking into account the coupling between thermal and strain fields on the basis of Duhamel-Neumann relations. The following fundamental equations represent the system of linear equations of the theory of coupled dynamical thermoelasticity for anisotropic materials due to Biot (1956):

Kinematic equations:

$$e_{ij} == \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{1.1}$$

where, i, j varies from 1 to 3.

Equation of motion:

$$\sigma_{ij,j} + \rho F_i = \rho \ddot{u}_i \tag{1.2}$$

$$\sigma_{ij} = \sigma_{ji} \tag{1.3}$$

Energy scale equation:

$$q_{i,i} + \rho(T_0\dot{s} - R) = 0 \tag{1.4}$$

Constitutive relations:

$$\sigma_{ij} = C_{ijkl}e_{kl} - \gamma_{ij}\theta \tag{1.5}$$

where, $i, j, k, l = 1, 2, 3$

$$q_i = -K_{ij}\theta_{,j} \tag{1.6}$$

$$\rho s = \frac{\rho c_e}{T_0}\theta - \gamma_{ij}e_{ij} \tag{1.7}$$

where R is the strength of the internal heat source, s denotes the entropy, C_{ijkl} is the elasticity tensor, γ_{ij} is the thermoelasticity tensor, K_{ij} is the thermal conductivity tensor and c_e is the specific heat per unit mass, in the isothermal state.

The Fourier law for isotropic homogeneous medium is given in the form

$$q_i = -K\theta_{,i} \tag{1.8}$$

From the equation of motion (1.2) and energy-scale equation (1.4) using the linear constitutive equations (1.5)-(1.7) we get the more general basic equations of linear thermoelasticity as

$$(C_{ijkl}e_{kl})_{,j} - (\gamma_{ij}\theta)_{,j} + \rho F_i - \rho\ddot{u}_i = 0 \tag{1.9}$$

$$(K_{ij}\theta_{,j})_{,i} - \rho c_e\dot{\theta} + \rho R + T_0\gamma_{ij}\dot{e}_{ij} = 0 \tag{1.10}$$

In the case of homogeneous, isotropic material with respect to (1.1), (1.9) and (1.10) can be transcribed as

$$\mu u_{i,jj} + (\lambda+\mu)u_{k,ki} + \gamma\theta_{,i} + \rho F_i - \rho\ddot{u}_i = 0 \tag{1.11}$$

$$K\theta_{,ii} - \rho c_e\dot{\theta} + \rho R + \gamma T_0\dot{u}_{i,i} = 0 \tag{1.12}$$

These equations are the coupled field equations referring to u_i and θ variables, for homogeneous and isotropic materials of the classical theory of linear thermoelasticity.

Biot's thermoelasticity theory as represented by above system of equations is the first coupled dynamical thermoelasticity theory that describes a broad range of phenomena. Biot's theory has been considered as an elegant model of thermoelasticity. Several eminent researchers including Boley and Weiner (1960), Chadwick (1960), Nowacki (1962, 1975), Parkus (1976), Nowinski (1978), Dhaliwal and Singh (1980), Chandrasekharaiah (1986) have contributed significantly providing the wide and detailed discussions along with interesting applications and theorems based on it. However, it has been realized through subsequent theoretical as well as experimental research work that although the theory proposed by Biot (1956) removes the drawback of uncoupled theory of thermoelasticity but it suffers from the deficiency of admitting thermal signals propagating with finite speed. This is considered as a paradox inherent in this theory. In addition to this paradox, this theory also exhibits unsatisfactory description of a solid's response to fast transient heating, like short laser pulses. Due to such shortcomings of this theory in several cases, researchers have put their efforts in recent years to modify the concept of this theory. Basically, this shortcoming arose from the inherent limitation in Fourier law of heat conduction which has been discussed in the next section.

1.3 Limitations of Fourier Law and its Generalization

From the view of Fourier law given by Eq. (1.8), it can be interpreted that heat flux is the instantaneous result of a temperature gradient established at a point of a body. The corresponding heat conduction equation in absence of any heat source for isotropic and homogeneous body is given by

$$K\nabla^2\theta = \rho C_e\dot{\theta} - \rho R \tag{1.13}$$

It is a parabolic type partial differential equation (diffusion equation). It has been realized that this law is successfully applicable to the problems that involve large spatial dimension and/or long time response. However, it is physically unrealistic for the transient behavior of heat conduction, specially at extremely short time, e.g., on the order of a fraction of second (10^{-12}s to 10^{-15}s). It has been realized that it yields unacceptable results in the cases involving high heat flux condition and short time behavior (such as laser-material interactions), high thermal gradients, etc. Moreover, heat conduction of many nano-scale devices demonstrates some distinct phenomena, that are not captured by the conventional Fourier law. In this respect, it is worth to be mentioned that in 1867, Maxwell postulated the occurrence of a wave-type heat flow and indicated that the thermal disturbance is a wave like phenomenon rather than diffusion phenomenon. Accordingly, the modification of Fourier law was suggested by him for the first time. An extensive research work has been carried out to address this apparent unrealistic prediction by Fourier law. Existence of wave type thermal signal has been discussed by some eminent researchers. The wave-type heat flow is now called as "second sound" effect (see Chandrasekharaiah (1986)). Possibility of "second sound" effect was also speculated by Nernst (1917) and later on by Landau (1941) and by Tisza (1947). Landau (1941) reported that "second sound" can be demonstrated as the propagation of phonon density disturbance for super-fluid helium and estimated its speed to be equal to $\frac{v_p}{\sqrt{3}}$ at 0K temperature, where v_p is the speed of the ordinary sound (first sound). The "second sound" was detected experimentally for the first time by Peshkov (1944) in liquid helium. He mentioned that the speed of thermal signal was found to be equal to 19 m/s at 1.4 K. Tisza's and Landau's predictions were also verified experimentally by other researchers including Maurer and Herlin (1949), Pellam and Scott (1949), and Atkins and Osborne (1950). Lifshitz (1958) observed that in fluid helium second sound occurs at low temperatures. Subsequently, "second sound" had also been detected by several workers like, Ackerman *et al.* (1966), Ackerman and Overton (1969) and Bertman and Standiford (1970), McNelly *et al.* (1970), Jackson *et al.* (1970), Jackson and Walker (1971), Rogers (1971). In this respect, we refer the detailed review article by Chandrasekharaiah (1986). Parallel to experimental research work to account for the inadequacy of Fourier law, several theoretical work have also been carried out. We give a brief discussion for some of the well established non-Fourier heat conduction models in the next section.

1.4 Non-Fourier Heat Conduction Models

Cattaneo (1958) and Vernotte (1958; 1961) recommended independently a modification to Fourier law of heat conduction for the first time. A flux rate term was introduced into Fourier law and the proposed heat conduction law was

$$\vec{q}(\vec{r},t) + \tau \frac{\partial \overrightarrow{q}(\overrightarrow{r},t)}{\partial t} = -K\overrightarrow{\nabla}\theta(\overrightarrow{r},t) \tag{1.14}$$

Here $\tau \geq 0$ is referred to as thermal relaxation time which is defined as the finite built-up time (phase-lag), for the onset of heat flow at $\overrightarrow{r}$ after a temperature gradient is imposed there. Equation (1.14) yields the following hyperbolic type heat conduction equation:

$$K\nabla^2\theta = (1 + \tau\frac{\partial}{\partial t})(\rho C_e\dot{\theta} - \rho R) \tag{1.15}$$

Clearly, the above hyperbolic type equation represents the combined diffusion and wave-like behavior of heat transport and predicts a wave-like thermal signal propagating with the finite speed, $\sqrt{\frac{K}{\rho C_e \tau}}$ when $\tau > 0$. This modified heat conduction law is called as CV law and also as Maxwell-Cattaneo law. It has been observed by several researchers (see Chandrasekharaiah (1986; 1998)) that this new heat conduction law yields more realistic results in the cases that involve a localized moving heat source with high intensity, a rapidly propagating crack tip, shock wave propagation, thermal resonance, interfacial effects between dissimilar materials, laser material processing, laser surgery which involve short time intervals and high heat fluxes. Francis (1972) provided a table of values of τ for some materials. One method for determining the value of τ for a given material was described by Mengi and Turhan (1978). They reported that the values of τ range from 10^{-10}s for gases to 10^{-14}s for metals and for liquids and insulators value falls within this range. Several authors (e.g. Boley (1964), Nowinski (1978)) have suggested that the term containing thermal relaxation time parameter may be ignored in many practical problems due to its small numerical value. But several researchers including Baumister and Hamill (1969, 1971), Chen *et al.* (1969), Maurer and Thompson (1973), Sadd and Didlake (1977), Sadd and Cha (1982) have indicated that in heat transfer problems the hyperbolic heat equation gives significantly different results than the parabolic equation. Chandrasekharaiah (1986, 1998), Hetnarski and Ignaczak (1999), Wang *et al.* (2007), Ignaczak and Ostoja-Starzewski (2010), Straughan (2011), etc. have elaborated in details about the work carried out in this aspect.

It has now been obvious that the advancement of short-pulse laser technology and their huge applications to modern micro-fabrication technology are attracting attention of the researchers towards the issues of high rate heating on thin films (Tzou (1995b)). It has been understood that laser pulses can be made shorter to the range of femtoseconds (10^{-15}s). Further, if the response time is shorter then the non-equilibrium thermodynamic transition and the microscopic effects in the energy exchange during heat transport procedure become significant. In view of recent experiments, the heat conduction theory of Cattaneo and Vernotte also fails in some cases, specially during heating of thin films (Tzou (1995a; 1995b)). In order to surmount the drawbacks of the classical heat conduction model as well as of the Cattaneo-Vernotte model, Tzou (1995a; 1995b) proposed the dual-phase-lag (DPL) theory of heat conduction. This model establishes that either the temperature gradient may dominate the heat flux or the heat flux may dominate the temperature gradient. In fact, prior to the introduction of this dual-phase-lag model of Tzou, some other models were established in order to capture the microscopic effects in heat transport mechanism. The phonon-scattering model was put forward by Joseph and Preziosi (1989) and Guyer and Krumhansl (1964). The phonon-electron interaction model was developed by Brorson *et al.* (1987), Anisimov *et al.* (1974) and Fujimoto *et al.* (1984), a microscopic two-step model was introduced by Qiu and Tien (1992, 1993). In view of the concepts of all such models, Tzou (1995b) has pronounced the effects of micro-structural interactions in the fast transient process of heat transport phenomenon by a macroscopic formulation. He developed a more generalized and accurate law of heat conduction, known as dual-phase-lag model, in the form

$$\vec{q}(\vec{r}, t+\tau_q) = -K\overrightarrow{\nabla} T(\overrightarrow{r}, t+\tau_T) \tag{1.16}$$

Here τ_q, τ_T are two delay times. τ_q represents the phase-lag of the heat flux vector and captures the thermal wave behavior, i.e,. a small-scale response in time for heat flux. τ_T is the phase-lag of the temperature gradient and captures the effect of phonon-electron interactions, a micro-scale response in space. Thus, the dual-phase-lag concept is capable of predicting the small-scale response in both space and time. The phase-lags τ_q and τ_T are assumed to be positive and they are the intrinsic properties of the medium (Tzou (1997)).

Now, applying Taylor series expansion of (1.16) by retaining terms up to the second order in τ_q but only up to the first order in τ_T (Tzou (1995a)), one can obtain

$$\vec{q} + \tau_q \frac{\partial \overrightarrow{q}}{\partial t} + \frac{\tau_q^2}{2} \frac{\partial^2 \overrightarrow{q}}{\partial t^2} = -K \left[\overrightarrow{\nabla} \theta + \tau_T \frac{\partial \overrightarrow{\nabla} \theta}{\partial t} \right] \tag{1.17}$$

Corresponding heat conduction equation is

$$(1+\tau_T \frac{\partial}{\partial t}) \nabla^2 \theta = \frac{\rho C_e}{K} (1+\tau_q \frac{\partial}{\partial t} + \frac{\tau_q^2}{2} \frac{\partial^2}{\partial t^2}) \frac{\partial \theta}{\partial t} + \frac{\rho}{K} (1+\tau_q \frac{\partial}{\partial t} + \frac{\tau_q^2}{2} \frac{\partial^2}{\partial t^2}) R \tag{1.18}$$

Equation (1.18) is clearly a hyperbolic type heat conduction equation that admits the thermal wave to propagate with a finite speed, $V_T = \frac{1}{\tau_q} \sqrt{\frac{2K\tau_T}{\rho C_e}}$ (Tzou (1995a)).

1.5 Generalized Thermoelasticity Theories

With the introduction of non-Fourier heat conduction models, parallel research activities have also been carried out in the field of thermoelasticity for providing the major growth of the area of thermoelasticity. Accordingly, several models have been proposed which are capable of removing the apparent drawbacks of classical coupled theory given by Biot (1956). Such theories are called as generalized theory of thermoelasticity or hyperbolic thermoelasticity. A brief description of such models are given below:

1.5.1 Thermoelasticity with thermal relaxation parameters

Lord and Shulman (1967) proposed one generalized thermoelastic model which includes one thermal relaxation parameter. In this model, the flux rate term was incorporated into the Fourier law of heat conduction. Basically this theory is based on Cattaneo-Vernotte law (1.14) and as a result the heat conduction equation in this theory exhibits wave-type heat phenomenon, i.e., the propagation speeds for elastic and thermal waves are finite. This thermoelasticity theory is also called as 'Extended Thermoelasticity theory (ETE)' or 'Lord-Shulman theory (LS theory)'. The LS theory is considered as the first generalization to the coupled thermoelasticity theory. This theory was extended by Dhaliwal and Sherief (1980) to general anisotropic media in the presence of heat sources.

The second generalization to the coupled thermoelasticity is known as the theory of thermoelasticity with two relaxation times or the theory of temperature-rate dependent thermoelasticity. Muller (1971) reported a review of thermodynamics of thermoelastic solids and proposed an entropy production inequality with some restrictions on a class of constitutive equations. A generalization to this inequality was then proposed by Green and Laws (1972). Subsequently, Green and Lindsay (1972) proposed a new thermoelasticity theory in which they obtained an explicit version of the constitutive equations by introducing two non negative constants that act as thermal relaxation times or temperature-rates. However, the classical Fourier law of heat conduction is not violated in this theory if the medium under consideration has a center of symmetry. This temperature rate dependent thermoelasticity theory (TRDTE) also admits "second sound" effect like the ETE theory.

1.5.2 Thermoelasticity theory of type-I, II and III (GN-I, II and III theory)

Next generalization to the coupled thermoelasticity has been made by Green and Naghdi (1991, 1992, 1993, 1995a; 1995b; 1995c) who have introduced their theory as an alternative one as compared to the conventional theory. In this theory, the propagation of heat has been modeled in a very elegant way to establish a fully consistent theory of thermoelasticity that is capable of organizing the thermal wave transmission in a reasonable manner and is based on the firm ground of thermodynamic principles. Moreover, to account for the finite speed for thermal wave, Green and Naghdi (1993) speculated a new concept in generalized thermoelasticity which is known as the thermoelasticity with no energy dissipation. The most distinct characteristic of this theory is that it is completely in contrast to the classical thermoelasticity associated with Fourier law of heat conduction. Furthermore, the potential function which is used to derive the stress tensor is used to determine the constitutive equation for the entropy flux vector. Basically, Green-Naghdi (GN) theory depends on entropy balance law rather than the usual entropy inequality. The theory proposed by Green and Naghdi (1991, 1992, 1993, 1995a; 1995b; 1995c) has been categorized into three parts which have been labeled as thermoelasticity of type-I, II and III. Linearized version of type-I theory is similar to the classical theory of thermoelasticity predicting an infinite speed of thermal wave propagation, whereas, the type-II model describes the finite speed of heat propagation wave as a special case of type III. Hence, in the heat equation of type III, the heat flux is the combination of type-I and type-II theories. For isotropic medium, the heat conduction equation for isotropic and homogeneous medium in the theory proposed by Green and Naghdi of type-III is expressed in the following way:

$$K^{*}\nabla^{2}\theta + K\nabla^{2}\dot{\theta} = \rho C_{v}\ddot{\theta} + \gamma T_{0}\ddot{u}_{i,i} \tag{1.19}$$

For the case when $K >> K^{*}$, above equation leads to the heat conduction equation of Green and Naghdi of type-I (GN-I) theory of thermoelasticity as

$$K\nabla^{2}\dot{\theta} = \rho C_{v}\ddot{\theta} + \gamma T_{0}\ddot{u}_{i,i} \tag{1.20}$$

For the case when $K^{*} >> K$, we obtain the heat conduction equation of Green and Naghdi of type-II (GN-II) theory of thermoelasticity as

$$K^{*}\nabla^{2}\theta = \rho C_{v}\ddot{\theta} + \gamma T_{0}\ddot{u}_{i,i} \tag{1.21}$$

1.5.3 Thermoelasticity with phase-lags

The review/survey articles by Chandrasekharaiah (1986, 1998) and Hetnarski and Ignaczak (1999) provided detailed discussion about ETE, TRDTE theories and Green-Naghdi theory of thermoelasticity. In the review article by Chandrasekharaiah (1998), another theory of thermoelasticity has been developed in the frame of ETE theory by introducing the dual-phase-lag heat conduction law given by Tzou (1995a, 1995b) in place of Fourier law. Two different versions of the dual-phase-lag thermoelasticity theory had been formulated in this article. Out of these two models, one model accounts for the finite speed of thermal signal.

Subsequently, Roychoudhuri (2007) proposed a model of thermoelasticity in which Fourier law of heat conduction is modified by introducing three different phase-lags for the heat flux vector, the temperature gradient and the thermal displacement gradient vectors. Hence, this model is known as the three-phase-lag thermoelasticity theory (TPLTE). The generalized heat conduction model that is proposed here is given in form

$$\vec{q}(\overrightarrow{r}, t + \tau_q) = -\left[K\overrightarrow{\nabla}T(\overrightarrow{r}, t + \tau_T) + K^{\star}\overrightarrow{\nabla}\nu(\overrightarrow{r}, t + \tau_\nu)\right] \qquad (1.22)$$

Here τ_ν denotes the phase-lag in thermal displacement gradient and $\vec{\nabla}\nu$ is the gradient of thermal displacement such that $\dot{\nu} = T$. Therefore, TPLTE is considered as the generalization of GN-III thermoelasticity theory.

1.5.4 An exact heat conduction model with a single delay term (Quintanilla (2011))

The above mentioned thermoelasticity theories have attracted the serious attention of researchers in recent years in order to find out several features of these models. Some qualitative analysis on these models are also reported. Quintanilla and Racke (2008) have discussed the stability of three-phase-lag model of heat conduction equation and the effects of considering all these three material parameters. Since, the phase-lag theories could let to parabolic-type or hyperbolic-type differential equations based on the order of Taylor's series expansion of the phase-lag parameters, a big interest has also been developed to study the different Taylor's approximations to these heat conduction equations where continuous dependence and also stability can be achieved (see Horgan and Quintanilla (2005), Mukhopadhyay and Kumar (2010a), Quintanilla (2002), Quintanilla (2003), Quintanilla and Racke (2006a,2008)). Dreher *et al.* (2009) have reported an analysis on dual-phase-lag and three-phase-lag heat conduction models. It has been analysed that if we combine the constitutive equations introduced in dual-phase-lag and three-phase-lag heat conduction theory with the energy equation, then we find a sequence of eigenvalues in a point spectrum such that its real parts tend to infinity (see Jordan *et al.* (2008), Dreher *et al.* (2009)). This implies the ill-posed behavior of the problem in Hadamard sense and we can not find the continuous dependence results of the solution with respect to initial parameters. By mentioning about these unacceptable results, Quintanilla (2011) has recently attempted to reformulate the three-phase-lag heat conduction model and suggested an alternative heat conduction theory with a single delay term. Leseduarte and Quintanilla (2013) re-examined this new model given by Quintanilla (2011) and discussed the stability and spatial behavior of the solutions under this model. They considered $\tau_\nu > \tau_q = \tau_T$ and $\tau = \tau_\nu - \tau_q$, so that the constitutive law of heat conduction was taken as

$$\vec{q}(t) = -\left[K \overrightarrow{\nabla} T(t) + K^* \overrightarrow{\nabla} \nu(t+\tau)\right] \tag{1.23}$$

The Taylor's series approximation until order l in the thermal gradient part of the above constitutive law yields

$$\vec{q}(t) = -[K\overrightarrow{\nabla}T(t)+K^*\{\overrightarrow{\nabla}\nu(t)+\tau\overrightarrow{\nabla}\dot{\nu}(t)+...+\frac{\tau^l}{l(l-1)...1}\overrightarrow{\nabla}\nu^{(l)}\}] \quad (1.24)$$

If this equation is adjoined with the energy equation, the new heat conduction equation is obtained as

$$c_e\dot{T}(t) = -[K\Delta T(t) + K^*\{\Delta\nu(t) + \tau\Delta T(t) + ... + \frac{\tau^l}{l(l-1)...1}\Delta T^{(l-1)}\}] \quad (1.25)$$

where c_e is the specific heat, $\Delta = \nabla^2$ is the Laplacian operator. Lesed-uarte and Quintanilla (2013) has shown that the solution of this equation is always stable (at least) whenever $l \leq 3$.

When we take $l = 0$, the Eq. (1.25) reduces to the form

$$c_e\dot{T}(t) = -[K\Delta T(t) + K^*\Delta\nu(t)] \quad (1.26)$$

This is the heat conduction equation under GN-III model.

When we take $l = 2$ in Eq. (1.25), we get the following equation of heat conduction:

$$c_e\dot{T}(t) = -[K\,\Delta T(t) + K^*(1+\tau\frac{\partial}{\partial t}+\frac{\tau^2}{2}\frac{\partial^2}{\partial t^2})\Delta\nu(t)] \quad (1.27)$$

If we neglect the term containing τ^2 for smallness in Eq. (1.27), then we get the following equation:

$$c_e\dot{T}(t) = -[K\Delta T(t) + K^*(1+\tau\frac{\partial}{\partial t})\Delta\nu(t)] \tag{1.28}$$

The spatial behavior of the solutions for this theory was discussed by Leseduarte and Quintanilla (2013). Furthermore, a Phragmen-Lindelof type alternative was found out. Further, it had been shown that the solutions either decay in an exponential way or blow up at infinity in an exponential way. They further extended their results to a thermoelasticity theory by considering the Taylor's series approximation of the equation of heat conduction to the delay term. Phragmen-Lindelof type alternative is also obtained for both the forward and backward in time equations. The continuous dependence results are further extended to the thermoelastic case.

1.5.5 Two-temperature thermoelasticity theory

In the mechanics of continuous media, the material is said to have hereditary characteristics or memory if the behavior of a material at time t is specified in terms of the experience of the body up to the time t. Coleman (1964) formulated a theory of materials with memory. An alternative thermoelasticity theory called as two-temperature thermoelasticity theory was proposed by Chen and Gurtin (1968), Chen and Willian (1968), and Chen *et al.* (1969). The two-temperature thermoelasticity theory proposes that the heat conduction on a deformable body depends on two different temperatures- the conductive temperature, and the thermodynamic temperature, (Gurtin and Williams (1966), Chen and Gurtin (1968), Chen and Willian (1968), and Chen *et al.* (1969)). According to this thermoelasticity theory, the entropy contribution due to heat-conduction is governed by thermodynamic temperature and that of the heat supply is governed by the conductive temperature. Further, the stress, the energy, the entropy, the heat-flux and the thermodynamic temperature at a given time depend on the histories up to the time of the deformation gradient, the conductive temperature and the gradient of this temperature. Chen *et al.* (1969) suggested that the difference between the two temperatures is proportional to heat supply. In case of the absence of heat supply, the two temperatures are equal for time-independent situation. However, for time dependent cases the two temperatures are in general different, regardless of heat supply. Uniqueness and reciprocity theorems for the two-temperature thermoelasticity theory in case of a homogeneous and isotropic solid have been provided by Iesan (1970). Subsequently, qualitative investigations on this theory were carried out by several researchers including Warren and Chen (1973), Amos (1969), Chakrabarti (1973) etc. This two-temperature thermoelasticity theory has been revisited once again in the recent years. Puri and Jordan (2006) discussed the propagation of harmonic plane waves in two-temperature theory in a detailed way and discussed several qualitative behavior of the two-temperature model. The existence, structural stability, convergence and spatial behavior of two-temperature thermoelasticity have been provided by Quintanilla (2004a). Recently, Youssef (2006b) extended this theory in the context of the generalized theory of heat conduction and formulated a generalized two-temperature theory of thermoelasticity by providing the uniqueness theorem. Magana and Quintanilla (2009) studied the uniqueness and growth of solutions of this theory. Subsequently, Youssef (2008), Youssef and Al-Lehaibi (2007) carried out some investigations on two-temperature generalized thermoelasticity and indicated some significant features of the theory.

1.6 Literature Review

The generalized thermoelasticity theories that have come into existence in recent years are applied as a wide range of extensions of classical coupled dynamical thermoelasticity. A significant attention of several active researchers has been paid on these theories to explore various problems involving thermoelastic interactions in different kinds of media and under different thermoelastic loading conditions. Major objective here is to investigate the thermoelastic behavior of field variables for homogeneous and non-homogeneous medium under various thermoelastic system. A wide literature in the reference of generalized thermoelasticity can be found in the review articles/books by Chandrasekharaiah (1986; 1998), Jou *et al.* (1996), Suhubi (1975), Iesan (1994), Ignaczak and Ostoja-Starzewski (2010), Hetnarski and Eslami (2008), etc. The Ph.D. theses of Roushan Kumar (2010), Rajesh Prasad (2012), Shweta Kothari (2013), Bharti Kumari (2017) and Shashi Kant (2018) may be referred in this respect. However, we mention below some work relevant to the present thesis.

Paria (1958) investigated a coupled thermoelastic problem of half space under Biot's theory. He obtained the short-time approximated analytical solution using Laplace transform technique. Hetnarski (1961; 1964) studied one dimensional coupled thermoelasticity problem subjected to sudden heating with constant temperature on bounding plane and further studied the short-time approximated fundamental solution of coupled thermoelastic problem. A wide variety of work on the generalized theories due to Lord and Shulman (1967), Green and Lindsay (1972), Green and Naghdi (1991; 1992; 1993), Chandrasekharaiah (1998) (dual-phase-lag theory) and Roychoudhuri (2007) (Three-phase-lag thermoelasticity theory) are reported. Sharma (1987) has solved a half space problem by first decoupling the field equations purely into a displacement equation and temperature equation and then by employing the Fourier and Laplace transforms techniques. Other related work under Lord- Shulman theory were carried out by Dhaliwal and Rokne (1988), Anwar and Sherief (1988), Rama Murthy and Sharma (1991), Bykovtsev and Shatalov (1987), Anwar (1991), Balla (1991), Sharma and Chand (1988), Roychoudhuri and Roy (1990a), Ezzat (1997), Roychoudhuri and Banerjee (1996), Chand *et al.* (1990), Saxena *et al.* (1991), Sherief and Ezzat (1996) and Misra *et al.* (1992). They reported different responses of the thermoelastic system under different types of boundary conditions. Chakravorty and Chakravorty (1998) have used the transform technique to study the thermoelastic interactions due to a line heat source moving inside a half-space. Cylindrical and spherical waves generated by various boundary loads have been discussed in the works by Sharma (1987), Misra *et al.* (1987), Furukawa *et al.* (1990), Sharma and Chand (1991; 1996), Sherief and Anwar (1988), Roychoudhuri and Bhatta (1983), Mukhopadhyay *et al.* (1991), and Banerjee and Roychoudhuri (1995). In these works, various thermo-mechanical boundary conditions were assumed and various methods are employed to study the behavior of physical variables under different cylindrical and spherical medium. Sherief and Ezzat (1994) have discussed the interactions for a problem due to an instantaneous point heat source. Das *et al.* (1997) have studied the disturbances due to plane heat source and line heat source on one-dimensional and two-dimensional problem in an unbounded body by employing the eigenvalue approach. Apart from these problems, the propagation of plane harmonic wave solution have been studied by Sharma (1986), Sharma and Sidhu (1986), and Sharma and Singh (1987; 1989; 1990).

The temperature-rate dependent thermoelasticity by Green and Lindsay (1972) has been studied by several researchers. Chandrasekharaiah (1980; 1981) have analyzed the wave propagation in a half space generated due to thermal and/or mechanical loads applied to the boundary of the medium. Jakubowaska (1985), Gladysz (1986) and Ignaczak (1978; 1989) have made general analysis on generalized theory of thermoelasticity. Chandrasekharaiah and Srikantiah (1986) have obtained small-time approximated solutions by employing the eigenvalue approach. Dhaliwal and Rokne (1989) have constructed small-time solutions under different thermoelastic systems. Sherief (1993) has studied the thermoelastic interaction in an elastic half space by using the state space approach and the Laplace transform technique. Sherief (1994) further derived small-time solution, through the Laplace transform technique, in the case of thermal shock acting for a finite period of time. Tamma and Namburu (1992) have applied a FEM-based analysis for a half-space problem. Further, Furukawa *et al.* (1989), Roychoudhuri and Roy (1989; 1990b), Chatterjee and Roychoudhuri (1990), Sherief and Saleh (1998) and Erbay *et al.* (1991) have reported some interesting results in this direction. The generated waves due to heat sources have been considered by some authors like, lgnaczak (1985), Sherief (1992), Chandrasekharaiah and Srikantiah (1987), Chandrasekharaiah and Murthy (1991), Ezzat (1995), Ignaczak and Mrowka-Matejewska (1990) and Hetnarski and lgnaczak (1993; 1994). They highlighted some detailed qualitative mathematical analysis of the results. Green and Naghdi thermoelasticity theory has been investigated by Chandrasekharaiah (1996; 1997a; 1997b; 1998), Chandrasekharaiah and Srinath (1997a; 1997b; 1998a; 1998b; 2000), and Nappa (1998), in which a detailed analysis on thermoelastic interaction under the model have been reported. Aouadi (2008) has obtained the variational principle for theory of Lord and Shulman with one relaxation time and use it to obtain a uniqueness theorem under suitable conditions. A reciprocity theorem for the formulated problem is also established. Mukhopadhyay and Kumar (2008) have considered four different theories of thermoelasticity to study the thermoelastic interactions in an unbounded elastic medium with a spherical cavity and the numerical values of the physical quantities are also computed by employing Laplace transform technique for the copper material and results are displayed in graphical forms. Further, the effects of three-phase-lags on thermoelastic interactions due to step input in temperature on the stress free boundary of a cylindrical cavity in an unbounded medium were presented by Kumar and Mukhopadhyay (2009) and significant dissimilarities between two models showing the effects of phase-lags are pointed out on the basis of analytical as well as numerical results of the problem. Ezzat and Youssef (2010) have established a three-dimensional mathematical model of the generalized thermoelasticity with one relaxation time and applied Laplace and double Fourier transform techniques for a specific problem of a half space subjected to thermal shock and traction free surface. Guo *et al.* (2014) have applied a time discontinuous Galerkin finite element method (DGFEM) for the solution of generalized thermoelastic coupled problems on the basis of well-known Lord-Shulman theory and showed that the present DGFEM shows good abilities and provides much more accurate solutions for generalized thermoelastic coupled behavior. Abbas and Zenkour (2014) have found the thermoelastic interactions in a semi-infinite medium subjected to a ramp-type heating with the aid of a finite element method under the dual-phase-lag model. Yahya and Edfawy (2014) have considered the problem of dynamic thermoelastic stresses in a spherical shell with fixed boundaries whose inner surface is subjected to a step jump in temperature under Lord and Shulman and Green and Lindsay formulations of thermoelasticity and obtained the numerical solutions for the temperature and displacement equations using the finite difference method. Abbas *et al.* (2015) have studied the deformation in a micropolar thermoelastic diffusion medium due to thermal source by the use of finite element method (FEM) in the context of Lord-Shulman (LS) theory of Thermoelasticity. El-Karamany and Ezzat (2016) have proposed three models of generalized thermoelasticity: a single-phase-lag Green-Naghdi theory of type III, a dual-phase-lag Green-Naghdi theory of type II and of type III. A uniqueness theorem is proved for the considered theories and also variational characterization of solution is established. At the same time, Alzahrani and Abbas (2016) have devoted a paper to the study of a two-dimensional thermal shock problem with weak, normal and strong conductivity using the eigenvalue approach in the context of the new consideration of heat conduction with fractional order generalized thermoelasticity of the Lord-Shulman model (LS model). Some problems on thick circular plate in the context of generalized thermoelasticity have been recently investigated by Tripathi *et al.* (2016a; 2016b; 2017).

The microscopic electron and phonon temperature distributions in the thermoelastic medium have been captured by two-temperature thermoelasticity theory. Warren and Chen (1973) have investigated the wave propagation in the two-temperature theory of thermoelasticity. Quintanilla (2004a; 2004b) has provided the existence, stability, spacial behavior, convergence and uniqueness for the theory of two-temperature thermoelasticity. Youssef and Al-Harby (2007) have employed the state-space approach to a problem of infinite body with a spherical cavity subjected to different types of thermal loading under a two-temperature generalized thermoelasticity. Youssef (2008) has studied a two-dimensional problem of a two-temperature generalized thermoelastic half-space problem subjected to ramp type heating. Abbas and Youssef (2009) and Ezzat *et.al.* (2009) analysed the thermoelastic interactions in the elastic medium for a two-temperature generalized magneto-thermoelasticity. Mukhopadhyay and Kumar (2009) have studied the thermoelastic deformation occurred in an infinite medium with a cylindrical cavity. Youssef (2010b) addressed the thermoelastic observations for an elastic half-space with constant elastic parameters with moving heat source and due to ramp type heating. Ezzat and Awad (2010) derived the equations of motion and the constitutive relations to prove the uniqueness of the solution under thermal shock problem for the theory of micropolar generalized two-temperature thermoelasticity. Kaushal *et al.* (2010) have solved a boundary-value problem by applying the Hankel transform in the contexts of two temperature based Lord-Shulman and Green-Lindsay thermoelasticity theory. Kumar *et al.* (2010) established a variational principle of convolution type and a reciprocal principle in the context of linear theory for a homogeneous and isotropic body for this model. The propagation of harmonic plane wave solution in elastic media has been discussed by Kumar and Mukhopadhyay (2010b) in the context of the two-temperature theory. Youssef (2010a) solved a problem of thermoelastic interactions in an elastic infinite medium with cylindrical cavity thermally shocked at its bounding surface and subjected to moving heat source with constant velocity. Youssef and El-Bary (2010) have studied thermoelastic problem with variable thermal conductivity and a note on the spatial decay estimates in semi-cylindrical bounded domains is provided by Awad (2011). The work by Banik and Kanoria (2011) is concerned with the determination of the thermoelastic displacement, stress, conductive temperature, and thermodynamic temperature in an infinite isotropic elastic body with a spherical cavity in the context of two-temperature Lord-Shulman model and two-temperature Green-Naghdi models of thermoelasticity. El-Karamany and Ezzat (2011a) have introduced two general models for fractional order heat conduction under non-homogeneous anisotropic elastic solid and obtained the constitutive equations for the two-temperature thermoelasticity theory by proving uniqueness and reciprocal theorems and establishing the convolutional type variational principle. El-

Karamany and Ezzat (2011b) and Youssef and Elsibai (2015) have considered the two-temperature Green–Naghdi theories and have proved that the two-temperature thermoelasticity theory admits dissipation of energy and this theory without energy dissipation is valid only when the temperature parameter becomes zero. Miglani and Kaushal (2011) have discussed about the deformation occurred in a generalized thermoelasticity with two temperatures . The theory of two-temperature thermoelasticity with two phase-lags has been studied by Mukhopadhyay *et al.* (2011) and Singh and Bijarnia (2012) have found the propagation of plane waves for the thermoelasticity without energy dissipation. Banik and Kanoria (2012) have tried to observe the effects of three-phase-lag on two-temperature thermoelasticity for infinite medium with spherical cavity. Miranville and Quintanilla (2017) considered the LS and GL theory under two-temperature thermoelasticity and have investigated the spatial behavior of the solutions.

In context of the exact heat conduction theory with a delay term proposed by Quintanilla (2011), a uniqueness theorem, variational principle as well as a reciprocity principle have been established for an anisotropic body by Kumari and Mukhopadhyay (2017b). In recent years, Magana and Quintanilla (2018), Borgmeyer *et al.* (2014), and Lesedurte and Quintanilla (2013; 2017) have made an interesting impact with the studies based on this model. Kant and Mukhopadhyay (2016) have employed the Laplace transform and Hankle transform techniques to obtain the solution for a thick plate. Here, a complete analysis on the wave propagation and discontinuities of different wave fields are found out. Further, Kumari and Mukhopadhyay (2017a) have derived the fundamental solution of equations for the case of homogeneous and isotropic bodies and have determined the effects of concentrated heat sources and body forces in an unbounded medium. They have obtained the fundamental solutions of the field equations in case of steady vibrations too. A uniqueness theorem and instability of solutions under the relaxed assumption that the elasticity tensor can be negative is established by Quintanilla (2017). Biswas *et al.* (2017a) discussed Rayleigh surface wave propagation in orthotropic thermoelastic solids under three-phase-lag model. Biswas *et al.* (2017b) have further investigated the thermal shock response in magnetothermoelastic orthotropic medium with three-phase-lag model. Eigen function expansion method has been employed by Biswas and Mukhopadhyay (2018a; 2018b) to analyze thermal shock behavior in magneto-thermoelastic orthotropic medium under three theories and to characterize Rayleigh wave propagation in orthotropic medium with phase-lags. Biswas and Shaw (2018) considered a thermodynamic framework to analyze thermal shock behavior in anisotropic hollow cylinder with energy dissipation.

In the past few years, some studies on non-homogeneous material parameters in the context of the generalized thermoelasticity theories have been carried out. Banik and Kanoria (2013) considered the three-phase-lag thermoelastic models, GN-II and GN-III models to find out thermoelastic interactions in a functionally graded isotropic unbounded medium due to the presence of periodically varying heat sources. They have applied the Laplace-Fourier double transform to solve the problem. The study of generalized solution for the vibration of functionally graded (FG) microbeam in the context of the dual-phase-lag model has been carried out by Abouelregal and Zenkour (2014). The obtained numerical results are presented for the FG beam with exponentially varying material properties through the thickness to analyse the effects of ramp type heating. Abbas (2015) obtained the solution of a problem on thermoelastic interactions in a functional graded material due to thermal shock in the context of the fractional order three-phase-lag model. The analytical solution in the transform domain is obtained by using the eigenvalue approach for functionally graded materials. Sherief and Abd El-Latief (2016) considered a problem in the context of the generalized theory of thermoelasticity for a half space. Here, the material of the half space is functionally graded in which Lame's modulii are functions of the vertical distance from the surface of the medium. The surface is traction free and subjected to a time dependent thermal shock. The problem has been solved by using the Laplace transform method together with the perturbation technique. Further, the generalized coupled thermoelasticity based on the Lord-Shulman (LS) theory is employed by Heydarpour and Aghdam (2016) to study the transient thermoelastic behavior of rotating functionally graded (FG) truncated conical shells subjected to thermal shock with different boundary conditions. Sharma and Mishra (2017) have studied a problem on functionally graded sphere and analyzed the results in the context of linear theory of generalized thermoelasticity with one relaxation time. Here, Laplace transform has been used to solve the problem which yields natural frequencies of free vibrations without performing inversion of the transform. Thermal behavior of functionally graded solid sphere with non-uniform heat generation has been discussed by Pawar *et al.* (2017). Sur and Kanoria (2017) have attempted to discuss about the behaviour of displacements, temperature and stress distributions for the three-phase-lag and Green-Naghdi heat equations in a functionally graded transversely isotropic plate subjected to a spatially varying heat source by employing the Laplace-Fourier double transform. Abbas and Mohamed (2017) have considered the problem of magneto-thermoelastic interactions in a functionally graded material (FGM) under dual-phase-lag model in the presence of thermal shock. The propagation of thermal and thermoelastic waves have been studied by Wang *et al.*(2018) for a hollow cylinder whose material properties are spatially graded and temperature dependent. They employ the thermoelasticity theory with one relaxation parameter (LS Theory) and solved the problem by using Laplace transform technique. Very recently, Nikolarakis and Theotokoglou (2018) have analyzed the effects of functionally graded ceramic/metal layer under Lord-Shulman theory with the aid of finite element method.

1.7 Objective of the Thesis

Main Objective of the present thesis is concerned with the mathematical modeling on various problems involving thermoelastic interactions. A mathematical model is an abstraction or simplification that allows one to describe or summarize a system. Since, the modeling of any device and phenomena is very essential part to both the engineering and science, therefore, engineers and scientists have the perfect practical reasons for doing mathematical modeling. The thesis aims to study the physical behavior of field variables of various thermoelastic systems under different thermoelastic models and thereby to understand the basic differences among these models with respect to the responses of the field variables due to thermoelastic interactions. The thesis is basically divided into two different parts concerning with mathematical modeling on various types of coupled thermo-mechanical problems in the contexts of some recent models of thermoelasticity. It is aimed at analyzing various aspects of these recently proposed thermoelasticity theories by investigating some problems involving thermoelastic interactions inside a medium due to various types of thermo-mechanical loads. In the first part of the thesis, we concentrate on the two-temperature thermoelastic model with two relaxation parameters and the second part considers the thermoelastic model with a single delay term. Different types of coupled problems are solved in both the parts. Problems on homogeneous as well as non-homogeneous medium are considered and different methodologies are applied to solve the problems. In order to consider the non-homogeneity of the medium, we considered two problems with temperature-dependent thermal conductivity and one problem with functionally graded material (FGM) properties. FGMs are composite media that have continuously changing material properties and they constitute a new branch of materials developed with the purpose of designing structures to withstand suddenly applied loads as compared to traditional laminated composites. It is very important to highlight the thermo-mechanical responses of structural elements with FGM properties. Work carried out in this direction is rare as the mathematical formulation on coupling effects of FGMs with thermal field is a challenging task. It is worth pursuing research in this direction. Hence, an attempt has been made to investigate a problem of FGMs and focus on mathematical formulation of governing equations by employing the recent theory of heat conduction. The formulation of such problems further follows solution of a non-linear system of coupled equations by applying the numerical scheme: hybrid finite element method. It is aimed that the thesis will bring some light into the effects of applying the recently developed non-Fourier heat conduction models to understand thermo-mechanical responses under some situations.