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Ekeland's variational principle for interval-valued functions

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Abstract

In this paper, we attempt to propose Ekeland's variational principle for interval-valued functions (IVFs). To develop the variational principle, we study a concept of sequence of intervals. In the sequel, the idea of gH-semicontinuity for IVFs is explored. A necessary and sufficient condition for an IVF to be gH-continuous in terms of gH-lower and upper semicontinuity is given. Moreover, we prove a characterization for gH-lower semicontinuity by the level sets of the IVF. With the help of this characterization result, we ensure the existence of a minimum for an extended gH-lower semicontinuous, level-bounded and proper IVF. To find an approximate minima of a gH-lower semicontinuous and gH-Gâteaux differentiable IVF, the proposed Ekeland's variational principle is used.

Keywords Interval-valued functions · gH-semicontinuity · gH-Gâteaux differentiability · Ekeland's variational principle

Mathematics Subject Classification 26A24 · 90C30 · 65K05

1 Introduction

In real analysis, we deal with real-valued functions and their calculus. Similarly, interval analysis deals with interval-valued functions (IVFs), where uncertain variables are represented by intervals. The analysis of IVFs enables one to effectively deal with the errors/uncertainties that appear while modeling the real-life problems.

To identify characteristic of IVFs, calculus plays a significant role. Wu (2007) proposed the concepts of limit, continuity, and H-differentiability for IVFs. The concept of H-differentiability uses H-difference to find the difference between elements of $I(\mathbb{R})$,

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and hence it is restrictive (Stefanini and Bede 2009). To overcome the shortcomings of H-differentiability, Stefanini and Bede (2009) introduced gH-differentiability for IVFs. Thereafter, using gH-differentiability, Chalco-Cano et al. (2013) developed the calculus for IVFs. In the same article (Chalco-Cano et al. 2013), the fundamental theorem of calculus for IVFs has been presented. With the help of the parametric representation of an IVF, the notions of gH-gradient and gH-partial derivative of an IVF has been discussed in Ghosh (2017). Recently, Ghosh et al. (2020) introduced the concepts of gH-Gâteaux and Fréchet derivatives for IVFs with the help of linear IVFs. Further, researchers have also discussed concepts of differential equations with IVFs (Wu 2007; Ahmad et al. 2019; Chen et al. 2004; Van Hoa 2015). In order to study the interval fractional differential equations, Lupulescu (2015) developed the theory of fractional calculus for IVFs.

In developing mathematical theory for optimization with IVFs, apart from calculus of IVFs, an appropriate choice for ordering of intervals is necessary since the set of intervals is not linearly ordered (Ghosh et al. 2020) like the set of real numbers. Hence, the very definition of optimality gets differed than that of conventional one. However, one can use some partial ordering structures on the set of intervals. Some partial orderings of intervals are discussed by Ishibuchi and Tanaka in their 1990 paper (Ishibuchi and Tanaka 1990). By making use of these partial orderings and H-differentiability, Wu (2007) proposed KKT optimality conditions for an IOP. In a set of two papers, Wu (2008a, b) solved four types of IOPs and presented weak and strong duality theorems for IOPs using H-differentiability. Chalco-Cano et al. (2013) used a more general concept of differentiability (gH-differentiability) and provided KKT type optimality conditions for IOPs. Singh et al. (2016) investigated a class of interval-valued multiobjective programming problems and proposed the concept of Pareto optimal solutions for this class of optimization problems. Unlike the earlier approaches, in 2017, Osuna-Gómez et al. (2017) provided efficiency conditions for an IOP without converting it into a real-valued optimization problem. In 2018, Zhang et al. (2018) and Gong et al. (2016) proposed genetic algorithms to solve IOPs. Ghosh et al. (2019) reported generalized KKT conditions to obtain the solution of constrained IOPs. Recently, Van Su and Dinh (2020) presented duality results for interval-valued pseudoconvex optimization problems with equilibrium constraints. Many other authors have also proposed optimality conditions and solution concepts for IOP, for instances, see Ghosh (2017), Ahmad et al. (2019), Ghosh et al. (2018), Wolfe (2000) and the references therein.

1.1 Motivation and work done

So far, all the solution concepts in interval analysis to find minima of an IVF are for those IVFs that are gH-continuous and gH-differentiable. However, while modeling the real-world problems, we may get an objective function that is neither gH-differentiable nor gH-continuous.[1] We thus, in this study, introduce the notions of gH-semicontinuity and give results which guarantees the existence of a minima and an approximate minima for an IVF which need not be gH-differentiable or gH-continuous.

For a nonsmooth optimization problem, it is not always easy to find an exact optima (Facchinei and Pang 2007). In such situations, one attempts to find approximate optima. It is a well-known fact that Ekeland's variational principle (Ekeland 1974) is helpful to give approximate solutions (Facchinei and Pang 2007). Also, it is widely known that in the conventional and vector optimization problems, the concept of weak sharp minima (Burke and Deng 2002)

[1] Analytical models of some interesting real-world problems with neither differentiable nor continuous objective functions can be found in Clarke's book Clarke (1990)

plays an important role. It is closely related to sensitive analysis and convergence analysis of optimization problems (Burke and Ferris 1993; Henrion and Outrata 2001). Ekeland's variational principle is a useful tool to show the existence of weak sharp minima for a constrained optimization problem with nonsmooth objective function (Facchinei and Pang 2007). Moreover, Ekeland's variational principle (Ekeland 1974) is one of the most powerful tools for nonlinear analysis. It has applications in different areas including optimization theory, fixed point theory, and global analysis, for instances, see Borwein et al. (1999), Ekeland (1979), Fabian et al. (1996), Fabian and Mordukhovich (1998), Georgiev (1988), Kruger (2003), Penot (1986). Due to all these wide applications of Ekeland's variational principle in different areas, especially in nonsmooth optimization and control theory, we attempt to study this principle for gH-lower semicontinuous IVFs in this article. Further, we also give Ekeland's variational principle for gH-Gâteaux differentiable IVFs.

1.2 Delineation

The proposed study is presented in the following manner. In Sect. 2, basic terminologies and definitions on intervals and IVFs are provided. In Sect. 3, we define gH-semicontinuity for IVFs and give a characterization for gH-continuity of an IVF in terms of gH-lower and upper semicontinuity. Also, we give a characterization of gH-lower semicontinuity, and using this we prove that an extended gH-lower semicontinuous, level-bounded and proper IVF attains its minimum. Further, a characterization of the set argument minimum of an IVF is given. After that, we present Ekeland's variational principle for IVFs and its application in Sect. 4. Lastly, the conclusion and future scopes are given in Sect. 5.

2 Preliminaries and terminologies

In this article, the following notations are used.

- $\mathbb{R}$ denotes the set of real numbers
- $\mathbb{R}^+$ denotes the set of nonnegative real numbers
- $I(\mathbb{R})$ represents the set of all closed and bounded intervals
- Bold capital letters are used to represent the elements of $I(\mathbb{R})$
- $\overline{I(\mathbb{R})} = I(\mathbb{R}) \cup \{-\infty, +\infty\}$
- $\mathbf{0}$ represents the interval $[0, 0]$
- $\mathcal{X}$ denotes a finite dimensional Banach space
- $B_\delta(\bar{x})$ is an open ball of radius δ centered at $\bar{x}$.

Consider two intervals $\mathbf{A} = [\underline{a}, \overline{a}]$ and $\mathbf{B} = \left[\underline{b}, \overline{b}\right]$. The *addition* of $\mathbf{A}$ and $\mathbf{B}$, denoted $\mathbf{A} \oplus \mathbf{B}$, is defined by

$$\mathbf{A} \oplus \mathbf{B} = \left[\, \underline{a} + \underline{b},\ \overline{a} + \overline{b} \,\right].$$

The *addition* of $\mathbf{A}$ and a real number a, denoted $\mathbf{A} \oplus a$, is defined by

$$\mathbf{A} \oplus a = \mathbf{A} \oplus [a, a] = \left[\, \underline{a} + a,\ \overline{a} + a \,\right].$$

The *subtraction* of $\mathbf{B}$ from $\mathbf{A}$, denoted $\mathbf{A} \ominus \mathbf{B}$, is defined by

$$\mathbf{A} \ominus \mathbf{B} = \left[\, \underline{a} - \overline{b},\ \overline{a} - \underline{b} \,\right].$$

The *multiplication* by a real number μ to $\mathbf{A}$, denoted $\mu \odot \mathbf{A}$ or $\mathbf{A} \odot \mu$, is defined by

$$\mu \odot \mathbf{A} = \mathbf{A} \odot \mu = \begin{cases} [\mu \underline{a},\ \mu \overline{a}], & \text{if } \mu \geq 0 \\ [\mu \overline{a},\ \mu \underline{a}], & \text{if } \mu < 0. \end{cases}$$

Definition 2.1 (*gH-difference of intervals* Stefanini and Bede 2009). Let $\mathbf{A}$ and $\mathbf{B}$ be two elements of $I(\mathbb{R})$. The gH-difference between $\mathbf{A}$ and $\mathbf{B}$ is defined as the interval $\mathbf{C}$ such that

$$\mathbf{C} = \mathbf{A} \ominus_{gH} \mathbf{B} \Longleftrightarrow \begin{cases} \mathbf{A} = \mathbf{B} \oplus \mathbf{C} \\ \text{or} \\ \mathbf{B} = \mathbf{A} \ominus \mathbf{C}. \end{cases}$$

For $\mathbf{A} = [\underline{a}, \overline{a}]$ and $\mathbf{B} = [\underline{b}, \overline{b}]$, $\mathbf{A} \ominus_{gH} \mathbf{B}$ is given by (see Stefanini and Bede 2009)

$$\mathbf{A} \ominus_{gH} \mathbf{B} = [\min\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\}, \max\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\}].$$

Also, if $\mathbf{A} = [\underline{a}, \overline{a}]$ and a be any real number, then we have

$$\mathbf{A} \ominus_{gH} a = \mathbf{A} \ominus_{gH} [a, a] = [\min\{\underline{a} - a, \overline{a} - a\}, \max\{\underline{a} - a, \overline{a} - a\}] = [\underline{a} - a, \overline{a} - a].$$

Definition 2.2 (*Dominance of intervals* Wu 2008b). Let $\mathbf{A} = [\underline{a}, \overline{a}]$ and $\mathbf{B} = [\underline{b}, \overline{b}]$ be two elements of $I(\mathbb{R})$. Then,

(i) $\mathbf{B}$ is said to be dominated by $\mathbf{A}$ if $\underline{a} \leq \underline{b}$ and $\overline{a} \leq \overline{b}$, and then we write $\mathbf{A} \preceq \mathbf{B}$;
(ii) $\mathbf{B}$ is said to be strictly dominated by $\mathbf{A}$ if $\mathbf{A} \preceq \mathbf{B}$ and $\mathbf{A} \neq \mathbf{B}$, and then we write $\mathbf{A} \prec \mathbf{B}$. Equivalently, $\mathbf{A} \prec \mathbf{B}$ if and only if any of the following cases hold:
   - **Case** 1. $\underline{a} < \underline{b}$ and $\overline{a} \leq \overline{b}$,
   - **Case** 2. $\underline{a} \leq \underline{b}$ and $\overline{a} < \overline{b}$,
   - **Case** 3. $\underline{a} < \underline{b}$ and $\overline{a} < \overline{b}$;

(iii) if neither $\mathbf{A} \preceq \mathbf{B}$ nor $\mathbf{B} \preceq \mathbf{A}$, we say that none of $\mathbf{A}$ and $\mathbf{B}$ dominates the other, or $\mathbf{A}$ and $\mathbf{B}$ are not comparable. Equivalently, $\mathbf{A}$ and $\mathbf{B}$ are not comparable if either '$\underline{a} < \underline{b}$ and $\overline{a} > \overline{b}$' or '$\underline{a} > \underline{b}$ and $\overline{a} < \overline{b}$';
(iv) $\mathbf{B}$ is said to be not dominated by $\mathbf{A}$ if either $\mathbf{B} \preceq \mathbf{A}$ or $\mathbf{A}$ and $\mathbf{B}$ are not comparable, and then we write $\mathbf{A} \nprec \mathbf{B}$. Similarly, a real number a is said to be not dominated by $\mathbf{A}$ if either $[a, a] \preceq \mathbf{A}$ or $\mathbf{A}$ and $[a, a]$ are not comparable, and then we write $\mathbf{A} \nprec a$.

Remark 2.1 By Definition 2.2, it is easy to see that for any $\mathbf{A},\ \mathbf{B} \in I(\mathbb{R})$ either $\mathbf{A} \prec \mathbf{B}$ or $\mathbf{A} \nprec \mathbf{B}$.

Note 2.1 It is to be mentioned that dominance relation (Definition 2.2) and the interval arithmetic used in this article are also proposed independently by Kulisch and Miranker (1981).

In the following two lemmas, we give a few inequalities about intervals and their norms. The norm of an interval $\mathbf{A} = \left[\underline{a}, \bar{a}\right]$ is defined by (see Moore 1966)

$$\|\mathbf{A}\|_{I(\mathbb{R})} = \max\{|\underline{a}|, |\bar{a}|\}.$$

It is noteworthy that the set $I(\mathbb{R})$ equipped with the norm $\|.\|_{I(\mathbb{R})}$ is a normed quasilinear space with respect to the operations $\oplus$, $\ominus_{gH}$ and $\odot$ (see Lupulescu 2015).

Next, we recall the definition of quasilinear normed space.

Definition 2.3 (See Markov 2000). $I(R)$ equipped with the norm $\|.\|_{I(\mathbb{R})}$ with operations of addition ($\oplus$) and multiplication by real scalars ($\odot$), is called a quasilinear normed space if the following laws hold:

(i) $(\mathbf{A} \oplus \mathbf{B}) \oplus \mathbf{C} = \mathbf{A} \oplus (\mathbf{B} \oplus \mathbf{C})$ for all $\mathbf{A}, \mathbf{B}, \mathbf{C} \in I(\mathbb{R})$,
(ii) There exists an element $\mathbf{E} \in I(\mathbb{R})$ such that $\mathbf{A} \oplus \mathbf{E} = \mathbf{E} \oplus \mathbf{A} = \mathbf{A}$ for all $\mathbf{A} \in I(\mathbb{R})$,
(iii) $\mathbf{A} \oplus \mathbf{B} = \mathbf{B} \oplus \mathbf{A}$ for all $\mathbf{A}, \mathbf{B} \in I(\mathbb{R})$,
(iv) If $\mathbf{A}, \mathbf{B}, \mathbf{C} \in I(\mathbb{R})$ and $\mathbf{A} \oplus \mathbf{C} = \mathbf{B} \oplus \mathbf{C}$ implies $\mathbf{A} = \mathbf{B}$
(v) $\lambda \odot (\mu \odot \mathbf{A}) = (\lambda\mu) \odot \mathbf{A}$ for all $\lambda, \mu \in \mathbb{R}$ and $\mathbf{A} \in I(\mathbb{R})$,
(vi) $1 \odot \mathbf{A} = \mathbf{A}$ for all $\mathbf{A} \in I(\mathbb{R})$,
(vii) $\lambda \odot (\mathbf{A} \oplus \mathbf{B}) = \lambda \odot \mathbf{A} \oplus \lambda \odot \mathbf{B}$ for all $\lambda \in \mathbb{R}$ and $\mathbf{A}, \mathbf{B} \in I(\mathbb{R})$, and
(viii) $(\lambda + \mu) \odot \mathbf{A} = \lambda \odot \mathbf{A} \oplus \mu \odot \mathbf{A}$ for all $\mathbf{A} \in I(\mathbb{R})$ and $\lambda, \mu \in \mathbb{R}$ with $\lambda\mu \geq 0$.

Lemma 2.1 *Let $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ and $\boldsymbol{D}$ be elements of $I(\mathbb{R})$. Then,*

(i) $\|\boldsymbol{A} \oplus \boldsymbol{B}\|_{I(\mathbb{R})} \leq \|\boldsymbol{A}\|_{I(\mathbb{R})} + \|\boldsymbol{B}\|_{I(\mathbb{R})}$ (triangle inequality for the elements of $I(\mathbb{R})$),
(ii) if $\boldsymbol{A} \preceq \boldsymbol{C}$ and $\boldsymbol{B} \preceq \boldsymbol{D}$, then $\boldsymbol{A} \oplus \boldsymbol{B} \preceq \boldsymbol{C} \oplus \boldsymbol{D}$,

Proof See A. □

Lemma 2.2 *(*Properties of the elements of $I(\mathbb{R})$ under gH-difference*). For all elements $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D} \in I(\mathbb{R})$ and $\epsilon > 0$, we have*

(i) $\|\boldsymbol{A} \ominus_{gH} \boldsymbol{B}\|_{I(\mathbb{R})} < \epsilon \iff \boldsymbol{B} \ominus_{gH} [\epsilon, \epsilon] \prec \boldsymbol{A} \prec \boldsymbol{B} \oplus [\epsilon, \epsilon]$,
(ii) $\boldsymbol{A} \ominus_{gH} [\epsilon, \epsilon] \not\prec \boldsymbol{B} \implies \boldsymbol{A} \not\preceq \boldsymbol{B}$.

Proof See B. □

Definition 2.4 (*Infimum of a subset of* $\overline{I(\mathbb{R})}$). Let $\mathbf{S} \subseteq \overline{I(\mathbb{R})}$. An interval $\bar{\mathbf{A}} \in I(\mathbb{R})$ is said to be a lower bound of $\mathbf{S}$ if $\bar{\mathbf{A}} \preceq \mathbf{B}$ for all $\mathbf{B}$ in $\mathbf{S}$. A lower bound $\bar{\mathbf{A}}$ of $\mathbf{S}$ is called an infimum of $\mathbf{S}$ if for all lower bounds $\mathbf{C}$ of $\mathbf{S}$ in $I(\mathbb{R})$, $\mathbf{C} \preceq \bar{\mathbf{A}}$. We denote infimum of $\mathbf{S}$ by $\inf \mathbf{S}$.

Example 2.1 Let $\mathbf{S} = \left\{\left[\frac{1}{n}, 1\right] : n \in \mathbb{N}\right\}$. The set of lower bounds of $\mathbf{S}$ is

$$\{[\alpha, \beta] : -\infty < \alpha \leq 0 \text{ and } -\infty < \beta \leq 1\}.$$

Therefore, the infimum of $\mathbf{S}$ is $[0, 1]$ because $[\alpha, \beta] \preceq [0, 1]$ for all $-\infty < \alpha \leq 0$ and $-\infty < \beta \leq 1$.

Definition 2.5 (*Supremum of a subset of* $\overline{I(\mathbb{R})}$). Let $\mathbf{S} \subseteq \overline{I(\mathbb{R})}$. An interval $\bar{\mathbf{A}} \in I(\mathbb{R})$ is said to be an upper bound of $\mathbf{S}$ if $\mathbf{B} \preceq \bar{\mathbf{A}}$ for all $\mathbf{B}$ in $\mathbf{S}$. An upper bound $\bar{\mathbf{A}}$ of $\mathbf{S}$ is called a supremum of $\mathbf{S}$ if for all upper bounds $\mathbf{C}$ of $\mathbf{S}$ in $I(\mathbb{R})$, $\bar{\mathbf{A}} \preceq \mathbf{C}$. We denote supremum of $\mathbf{S}$ by $\sup \mathbf{S}$.

Example 2.2 Let $\mathbf{S} = \left\{\left[\frac{1}{n^2} + 1, 3\right] : n \in \mathbb{N}\right\}$. The set of upper bounds of $\mathbf{S}$ is

$$\{[\alpha, \beta] : 2 \leq \alpha < +\infty \text{ and } 3 \leq \beta < +\infty\}.$$

Therefore, the supremum of $\mathbf{S}$ is $[2, 3]$ because $[2, 3] \preceq [\alpha, \beta]$ for all $2 \leq \alpha < +\infty$ and $3 \leq \beta < +\infty$.

Remark 2.2 Let $\mathbf{S} = \left\{[a_\alpha, b_\alpha] \in \overline{I(\mathbb{R})} : \alpha \in \Lambda \text{ and } \Lambda \text{ being an index set}\right\}$. Then, by Definition 2.4 and 2.5, it follows that $\inf \mathbf{S} = \left[\inf_{\alpha \in \Lambda} a_\alpha, \inf_{\alpha \in \Lambda} b_\alpha\right]$ and $\sup \mathbf{S} = \left[\sup_{\alpha \in \Lambda} a_\alpha, \sup_{\alpha \in \Lambda} b_\alpha\right]$. It is evident that if $\inf \mathbf{S}$ and $\sup \mathbf{S}$ exist for an $\mathbf{S}$, then they are unique.

Note 2.2 Infimum and supremum of a subset of $I(\mathbb{R})$ may not exist. For instance, consider $\mathbf{S} = \{[-2,-1],[-3,-1],[-4,-1],\ldots\}$. Here, $\mathbf{S}$ has no lower bound in $I(\mathbb{R})$ as $\{-2,-3,-4,\ldots\}$ has no lower bound in $\mathbb{R}$. Therefore, infimum of $\mathbf{S}$ does not exist in $I(\mathbb{R})$.

Remark 2.3 (i) It is noteworthy that infimum and supremum of a subset of $\overline{I(\mathbb{R})}$ always exist in $\overline{I(\mathbb{R})}$. For instance, consider $\mathbf{S}$ as in Note 2.2. Here, infimum of $\mathbf{S}$ does not exist in $I(\mathbb{R})$ but exists in $\overline{I(\mathbb{R})}$. Note that infimum of $\mathbf{S}$ is $-\infty$.

(ii) Infimum and supremum of a finite subset S of real numbers always belong to the set S but this is not true for a finite subset of $I(\mathbb{R})$. For instance, consider $\mathbf{S} = \{[-2,4],[-1,3]\}$. Then, $\inf \mathbf{S} = [-2,3]$ and $\sup \mathbf{S} = [-1,4]$.

Definition 2.6 (*Infimum of an IVF*). Let $\mathcal{S}$ be a nonempty subset of $\mathcal{X}$ and $\mathbf{F} : \mathcal{S} \to \overline{I(\mathbb{R})}$ be an extended IVF. Then infimum of $\mathbf{F}$, denoted as $\inf\limits_{x\in\mathcal{S}} \mathbf{F}(x)$ or $\inf\limits_{\mathcal{S}} \mathbf{F}$, is equal to the infimum of the range set of $\mathbf{F}$, i.e.,

$$\inf_{\mathcal{S}} \mathbf{F} = \inf\{\mathbf{F}(x) : x \in \mathcal{S}\}.$$

Similarly, the supremum of an IVF is defined by

$$\sup_{\mathcal{S}} \mathbf{F} = \sup\{\mathbf{F}(x) : x \in \mathcal{S}\}.$$

Definition 2.7 (*Sequence in* $I(\mathbb{R})$). An IVF $\mathbf{F} : \mathbb{N} \to I(\mathbb{R})$ is called a sequence in $I(\mathbb{R})$.

The image of nth element, $\mathbf{F}(n)$, is said to be the nth element of the sequence $\mathbf{F}$. We denote a sequence $\mathbf{F}$ by $\{\mathbf{F}(n)\}$.

Example 2.3 (i) $\mathbf{F} : \mathbb{N} \to I(\mathbb{R})$ that is defined by $\mathbf{F}(n) = [n, n+1]$ is a sequence.

(ii) $\mathbf{F} : \mathbb{N} \to I(\mathbb{R})$ that is defined by $\mathbf{F}(n) = \left[\frac{n}{4}, \frac{n}{2}\right]$ is also a sequence.

Definition 2.8 (*Convergence of a sequence in* $I(\mathbb{R})$).

1. A sequence $\{\mathbf{F}(n)\}$ is said to converge to $\mathbf{L} \in I(\mathbb{R})$ if for each $\epsilon > 0$, there exists an integer $m > 0$ such that

$$\|\mathbf{F}(n) \ominus_{gH} \mathbf{L}\|_{I(\mathbb{R})} < \epsilon \text{ for all } n \geq m.$$

   The interval $\mathbf{L}$ is called limit of the sequence $\{\mathbf{F}(n)\}$ and is presented by $\lim\limits_{n\to+\infty} \mathbf{F}(n) = \mathbf{L}$ or $\mathbf{F}(n) \to \mathbf{L}$.
2. We say the limit of a sequence $\{\mathbf{F}(n)\}$ is $+\infty$ if for every real number $a > 0$, there exists an integer $m > 0$ such that

$$[a, a] \prec \mathbf{F}(n) \text{ for all } n \geq m.$$

3. We say the limit of a sequence $\{\mathbf{F}(n)\}$ is $-\infty$ if for every real number $a > 0$, there exists an integer $m > 0$ such that

$$\mathbf{F}(n) \prec [-a, -a] \text{ for all } n \geq m.$$

Example 2.4 Consider the sequence $\mathbf{F}(n) = \left[\frac{1}{n}, 1\right],\ n \in \mathbb{N}$, in $I(\mathbb{R})$.
Let $\epsilon > 0$ be given. Note that

$$\|\mathbf{F}(n) \ominus_{gH} [0,1]\|_{I(\mathbb{R})} = \left\|\left[\frac{1}{n}, 1\right] \ominus_{gH} [0,1]\right\|_{I(\mathbb{R})} = \left\|\left[0, \frac{1}{n}\right]\right\|_{I(\mathbb{R})} = \frac{1}{n} < \epsilon \text{ whenever } n > \frac{1}{\epsilon}.$$

So, by taking $m = \lfloor \frac{1}{\epsilon} \rfloor + 1$, where $\lfloor \cdot \rfloor$ is the floor function, we have

$$\|\mathbf{F}(n) \ominus_{gH} [0,1]\|_{I(\mathbb{R})} < \epsilon \text{ for all } n \geq m.$$

Thus, $\lim\limits_{n\to+\infty} \mathbf{F}(n) = \mathbf{L} = [0,1]$.

Note 2.3 Let $\{\mathbf{F}(n)\}$ be a sequence in $I(\mathbb{R})$ with $\mathbf{F}(n) = \left[\underline{f}(n), \overline{f}(n)\right]$, where $\left\{\underline{f}(n)\right\}$ and $\left\{\overline{f}(n)\right\}$ be two convergent sequences in $\mathbb{R}$. Then, $\{\mathbf{F}(n)\}$ is convergent and

$$\lim_{n\to+\infty} \mathbf{F}(n) = \left[\lim_{n\to+\infty} \underline{f}(n), \lim_{n\to+\infty} \overline{f}(n)\right].$$

The reason is as follows.
Suppose $\underline{f}(n)$ and $\overline{f}(n)$ are convergent sequences with limits l_1 and l_2, respectively. Then, for each $\epsilon > 0$, there exist positive integers m_1 and m_2 such that

$$\left|\underline{f}(n) - l_1\right| < \epsilon \text{ for all } n \geq m_1, \text{ and } \left|\overline{f}(n) - l_2\right| < \epsilon \text{ for all } n \geq m_2$$

$$\iff \max\left\{\left|\underline{f}(n) - l_1\right|, \left|\overline{f}(n) - l_2\right|\right\} < \epsilon \text{ for all } n \geq m,$$

where $m = \max\{m_1, m_2\}$

$$\iff \left\|\left[\underline{f}(n), \overline{f}(n)\right] \ominus_{gH} [l_1, l_2]\right\|_{I(\mathbb{R})} < \epsilon \text{ for all } n \geq m$$

$$\text{i.e., } \left\|\mathbf{F}(n) \ominus_{gH} [l_1, l_2]\right\|_{I(\mathbb{R})} < \epsilon \text{ for all } n \geq m.$$

Thus,

$$\lim_{n\to+\infty} \mathbf{F}(n) = [l_1, l_2] = \left[\lim_{n\to+\infty} \underline{f}(n), \lim_{n\to+\infty} \overline{f}(n)\right].$$

Definition 2.9 (*Bounded sequence in* $I(\mathbb{R})$). A sequence $\{\mathbf{F}(n)\}$ is said to be bounded above if there exists an interval $\mathbf{K}_1 \in I(\mathbb{R})$ such that

$$\mathbf{F}(n) \preceq \mathbf{K}_1 \text{ for all } n \in \mathbb{N}.$$

A sequence $\{\mathbf{F}(n)\}$ is said to be bounded below if there exists an interval $\mathbf{K}_2 \in I(\mathbb{R})$ such that

$$\mathbf{K}_2 \preceq \mathbf{F}(n) \text{ for all } n \in \mathbb{N}.$$

A sequence $\{\mathbf{F}(n)\}$ is said to be bounded if it is both bounded above and below.

Definition 2.10 A sequence $\{\mathbf{F}(n)\}$ is said to be monotonic increasing sequence if $\mathbf{F}(n) \preceq \mathbf{F}(n+1)$ for all $n \in \mathbb{N}$.

Lemma 2.3 *A bounded above monotonic increasing sequence of intervals is convergent and converges to its supremum.*

Proof Let $\{\mathbf{F}(n)\}$ be a bounded above monotonic increasing sequence and $\mathbf{M}$ be its supremum. Then, by Definition 2.5, we have

(i) $\mathbf{F}(n) \preceq \mathbf{M}$ for all $n \in \mathbb{N}$ and
(ii) for a given $\epsilon > 0$, there exists an integer $m > 0$ such that $\mathbf{M} \ominus_{gH} [\epsilon, \epsilon] \prec \mathbf{F}(m)$.

Since $\{\mathbf{F}(n)\}$ is a monotonic increasing sequence,

$$\mathbf{M} \ominus_{gH} [\epsilon, \epsilon] \prec \mathbf{F}(m) \preceq \mathbf{F}(m+1) \preceq \mathbf{F}(m+2) \preceq \cdots \preceq \mathbf{M}.$$

That is, $\mathbf{M} \ominus_{gH} [\epsilon, \epsilon] \prec \mathbf{F}(m) \prec \mathbf{M} \oplus [\epsilon, \epsilon]$ for all $n \geq m$. Thus, the sequence $\{\mathbf{F}(n)\}$ is convergent and $\lim_{n\to+\infty} \mathbf{F}(n) = \mathbf{M}$. ◻

Definition 2.11 (*Lower limit and upper limit of a sequence in $I(\mathbb{R})$*). Let $\{\mathbf{F}(n)\}$ be a sequence. The lower limit of $\{\mathbf{F}(n)\}$, denoted $\liminf \mathbf{F}(n)$, is defined by

$$\liminf \mathbf{F}(n) = \lim_{n\to+\infty} \inf\{\mathbf{F}(n), \mathbf{F}(n+1), \mathbf{F}(n+2), \ldots\}.$$

Similarly, upper limit of $\{\mathbf{F}(n)\}$ is defined by

$$\limsup \mathbf{F}(n) = \lim_{n\to+\infty} \sup\{\mathbf{F}(n), \mathbf{F}(n+1), \mathbf{F}(n+2), \ldots\}.$$

Example 2.5 Consider the following sequence in $I(\mathbb{R})$:

$$\mathbf{F}(n) = \begin{cases} \left[\frac{1}{n^2}, \frac{1}{n^2}+1\right] & \text{if } n \text{ is odd} \\ [n, n^2+1] & \text{if } n \text{ is even.} \end{cases}$$

It is easy to see that $\inf\limits_{n\in\mathbb{N}} \left[\frac{1}{n^2}, \frac{1}{n^2}+1\right] = [0, 1]$ and $\inf\limits_{n\in\mathbb{N}} \left[n, n^2+1\right] = [1, 2]$. Therefore,

$$\lim_{n\to+\infty} \inf\{\mathbf{F}(n), \mathbf{F}(n+1), \mathbf{F}(n+2), \ldots\} = [0, 1] \text{ and hence, } \liminf \mathbf{F}(n) = [0, 1].$$

Note that $\sup\limits_{n\in\mathbb{N}} \left[\frac{1}{n^2}, \frac{1}{n^2}+1\right] = [1, 2]$ and $\sup\limits_{n\in\mathbb{N}} \left[n, n^2+1\right] = +\infty$. Thus,

$$\lim_{n\to+\infty} \sup\{\mathbf{F}(n), \mathbf{F}(n+1), \mathbf{F}(n+2), \ldots\} = +\infty \text{ and hence, } \limsup \mathbf{F}(n) = +\infty.$$

Next, we state the conventional Ekeland's variational principle.

Theorem 2.1 *(Ekeland's variational principle for real-valued functions Ekeland (1974)).* *Let $f : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ be a lsc extended real-valued function and $\epsilon > 0$. Assume that*

$$\inf_{\mathcal{X}} f \text{ is finite and } f(\bar{x}) < \inf_{\mathcal{X}} \boldsymbol{f} + \epsilon.$$

Then, for any $\delta > 0$, there exists an $x_0 \in \mathcal{X}$ such that

(i) $\|x_0 - \bar{x}\|_{\mathcal{X}} < \frac{\epsilon}{\delta}$,
(ii) $f(x_0) \leq f(\bar{x})$, *and*
(iii) $\underset{x\in\mathcal{X}}{\operatorname{argmin}}\{f(x) + \delta\|x - x_0\|_{\mathcal{X}}\} = \{x_0\}$.

3 *gH*-continuity and *gH*-semicontinuity of interval-valued functions

In this section, we define gH-lower and gH-upper semicontinuity for extended IVFs and show that gH-continuity of an IVF implies gH-lower and upper semicontinuity and vice-versa. Further, we give a characterization of gH-lower semicontinuity in terms of the level sets of the IVF (Theorem 3.6) and use this to prove that an extended gH-lower semicontinuous, level-bounded and proper IVF attains its minimum (Theorem 3.7). We also give a characterization of the set argument minimum of an IVF (Theorem 3.8).

Throughout this section, an extended IVF is an IVF with domain $\mathcal{X}$ and codomain $\overline{I(\mathbb{R})}$.

Definition 3.1 (*gH-limit of an IVF*). Let $\mathbf{F} : \mathcal{S} \to I(\mathbb{R})$ be an IVF on a nonempty subset $\mathcal{S}$ of $\mathcal{X}$. The function $\mathbf{F}$ is called tending to a limit $\mathbf{L} \in I(\mathbb{R})$ as x tends to $\bar{x}$, denoted by $\lim\limits_{x\to\bar{x}} \mathbf{F}(x)$, if for each $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\|\mathbf{F}(x) \ominus_{gH} \mathbf{L}\|_{I(\mathbb{R})} < \epsilon \text{ whenever } 0 < \|x - \bar{x}\|_{\mathcal{X}} < \delta.$$

Definition 3.2 (gH*-continuity*). Let $\mathbf{F} : \mathcal{S} \to I(\mathbb{R})$ be an IVF on a nonempty subset $\mathcal{S}$ of $\mathcal{X}$. The function $\mathbf{F}$ is said to be gH-continuous at $\bar{x} \in \mathcal{S}$ if for each $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\|\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x})\|_{I(\mathbb{R})} < \epsilon \text{ whenever } \|x - \bar{x}\|_{\mathcal{X}} < \delta.$$

Definition 3.3 (*Lower limit and gH-lower semicontinuity of an extended IVF*). The lower limit of an extended IVF $\mathbf{F}$ at $\bar{x} \in \mathcal{X}$, denoted $\liminf_{x \to \bar{x}} \mathbf{F}(x)$, is defined by

$$\begin{aligned} \liminf_{x \to \bar{x}} \mathbf{F}(x) &= \lim_{\delta \downarrow 0} (\inf\{\mathbf{F}(x) : x \in B_\delta(\bar{x})\}) \\ &= \sup_{\delta > 0} (\inf\{\mathbf{F}(x) : x \in B_\delta(\bar{x})\}) . \end{aligned}$$

$\mathbf{F}$ is called gH-lower semicontinuous (gH-lsc) at a point $\bar{x}$ if

$$\mathbf{F}(\bar{x}) \preceq \liminf_{x \to \bar{x}} \mathbf{F}(x). \tag{1}$$

Further, $\mathbf{F}$ is called gH-lsc on $\mathcal{X}$ if (1) holds for every $\bar{x} \in \mathcal{X}$.

Example 3.1 Consider the following IVF $\mathbf{F} : \mathbb{R}^2 \to I(\mathbb{R})$:

$$\mathbf{F}(x_1, x_2) = \begin{cases} [1, 2] \odot \sin\left(\frac{1}{x_1}\right) \oplus \cos^2 x_2 & \text{if } x_1 x_2 \neq 0 \\ [-2, -1] & \text{if } x_1 x_2 = 0. \end{cases}$$

The lower limit of $\mathbf{F}$ at $(0, 0)$ is given by

$$\liminf_{(x_1, x_2) \to (0,0)} \mathbf{F}(x_1, x_2) = \lim_{\delta \downarrow 0} (\inf\{\mathbf{F}(x_1, x_2) : (x_1, x_2) \in B_\delta(0, 0)\}) .$$

Note that as $x_1 \to 0$, $\sin\left(\frac{1}{x_1}\right)$ oscillates between -1 and 1. Therefore, for any $\delta > 0$,

$$\inf_{(x_1, x_2) \in B_\delta(0,0)} \mathbf{F}(x_1, x_2) = [1, 2] \odot (-1) = [-2, -1].$$

Also, note that when $(x_1, x_2) = (0, 0)$, $\mathbf{F}(x_1, x_2) = [-2, -1]$. Thus,

$$\liminf_{(x_1, x_2) \to (0,0)} \mathbf{F}(x_1, x_2) = [-2, -1].$$

Since $\mathbf{F}(0, 0) = [-2, -1] \preceq [-2, -1] = \liminf_{(x_1, x_2) \to (0,0)} \mathbf{F}(x_1, x_2)$, the function $\mathbf{F}$ is gH-lsc at $(0, 0)$.

Note 3.1 Let $\mathbf{F}$ be an extended IVF with $\mathbf{F}(x) = \left[\underline{f}(x), \overline{f}(x)\right]$, where $\underline{f}, \overline{f} : \mathcal{X} \to \mathbb{R} \cup \{-\infty, +\infty\}$ be two extended real-valued functions. Then, $\mathbf{F}$ is gH-lsc at $\bar{x} \in \mathcal{X}$ if and only if $\underline{f}$ and $\overline{f}$ both are lsc at $\bar{x}$. The reason is as follows.

$$\begin{aligned} \underline{f} \text{ and } \overline{f} \text{ are lsc at } \bar{x} &\iff \underline{f}(\bar{x}) \leq \liminf_{x \to \bar{x}} \underline{f}(x) \text{ and } \overline{f}(\bar{x}) \leq \liminf_{x \to \bar{x}} \overline{f}(x) \\ &\iff \left[\underline{f}(\bar{x}), \overline{f}(\bar{x})\right] \preceq \left[\liminf_{x \to \bar{x}} \underline{f}(x), \liminf_{x \to \bar{x}} \overline{f}(x)\right] \\ &\iff \left[\underline{f}(\bar{x}), \overline{f}(\bar{x})\right] \preceq \liminf_{x \to \bar{x}} \left[\underline{f}(x), \overline{f}(x)\right], \text{ by Remark 2.2} \\ &\text{i.e., } \quad \mathbf{F}(\bar{x}) \preceq \liminf_{x \to \bar{x}} \mathbf{F}(x). \end{aligned}$$

Note 3.1 reduces our efforts to check gH-lower semicontinuity of extended IVFs that are given in the form $\mathbf{F}(x) = \left[\underline{f}(x), \overline{f}(x)\right]$. For example, consider $\mathbf{F} : \mathbb{R}^2 \to I(\mathbb{R})$ as

$$\mathbf{F}(x_1, x_2) = \begin{cases} \left[\frac{|x_1 x_2|}{2{x_1}^2 + {x_2}^2}, \frac{e^{|6x_1 x_2|}}{{x_1}^2 + {x_2}^2}\right] & \text{if } x_1 x_2 \neq 0 \\ \mathbf{0} & \text{if } x_1 x_2 = 0 \end{cases}$$

and take $\bar{x} = (0, 0)$. It is easy to see that both

$$\underline{f}(x_1, x_2) = \begin{cases} \frac{|x_1 x_2|}{2{x_1}^2 + {x_2}^2} & \text{if } x_1 x_2 \neq 0 \\ 0 & \text{if } x_1 x_2 = 0 \end{cases}$$

and

$$\overline{f}(x_1, x_2) = \begin{cases} \frac{e^{|6x_1 x_2|}}{{x_1}^2 + {x_2}^2} & \text{if } x_1 x_2 \neq 0 \\ 0 & \text{if } x_1 x_2 = 0 \end{cases}$$

are lsc at $\bar{x}$. Thus, by Note 3.1, $\mathbf{F}$ is gH-lsc at $\bar{x}$.

Theorem 3.1 *Let $\boldsymbol{F}$ be an extended IVF. Then, $\boldsymbol{F}$ is gH-lsc at $\bar{x} \in \mathcal{X}$ if and only if for each $\epsilon > 0$, there exists a $\delta > 0$ such that $\boldsymbol{F}(\bar{x}) \ominus_{gH} [\epsilon, \epsilon] \prec \boldsymbol{F}(x)$ for all $x \in B_\delta(\bar{x})$.*

Proof Let $\mathbf{F}$ be gH-lsc at $\bar{x}$.
To the contrary, suppose there exists an $\epsilon_0 > 0$ such that for all $\delta > 0$, $\mathbf{F}(\bar{x}) \ominus_{gH} [\epsilon_0, \epsilon_0] \not\prec \mathbf{F}(x)$ for at least one x in $B_\delta(\bar{x})$.
Then,

$$\begin{aligned} &\mathbf{F}(\bar{x}) \ominus_{gH} [\epsilon_0, \epsilon_0] \not\prec \inf\{\mathbf{F}(x) : x \in B_\delta(\bar{x})\} \text{ for all } \delta > 0 \\ &\implies \mathbf{F}(\bar{x}) \ominus_{gH} [\epsilon_0, \epsilon_0] \not\prec \lim_{\delta \downarrow 0}(\inf\{\mathbf{F}(x) : x \in B_\delta(\bar{x})\}) \\ &\implies \mathbf{F}(\bar{x}) \ominus_{gH} [\epsilon_0, \epsilon_0] \not\prec \liminf_{x \to \bar{x}} \mathbf{F}(x) \\ &\implies \mathbf{F}(\bar{x}) \not\preceq \liminf_{x \to \bar{x}} \mathbf{F}(x), \quad \text{by } (ii) \text{ of Lemma 2.2,} \end{aligned}$$

which contradicts that $\mathbf{F}$ is gH-lsc at $\bar{x}$. Thus, for each $\epsilon > 0$, there exists a $\delta > 0$ such that $\mathbf{F}(\bar{x}) \ominus_{gH} [\epsilon, \epsilon] \prec \mathbf{F}(x)$ for all $x \in B_\delta(\bar{x})$.

Conversely, suppose for a given $\epsilon > 0$, there exists a $\delta > 0$ such that $\mathbf{F}(\bar{x}) \ominus_{gH} [\epsilon, \epsilon] \prec \mathbf{F}(x)$ for all $x \in B_\delta(\bar{x})$. Then,

$$\begin{aligned} &\mathbf{F}(\bar{x}) \ominus_{gH} [\epsilon, \epsilon] \preceq \inf\{\mathbf{F}(x) : x \in B_\delta(\bar{x})\} \\ &\implies \mathbf{F}(\bar{x}) \ominus_{gH} [\epsilon, \epsilon] \preceq \lim_{\delta \downarrow 0}(\inf\{\mathbf{F}(x) : x \in B_\delta(\bar{x})\}) \\ &\implies \mathbf{F}(\bar{x}) \ominus_{gH} [\epsilon, \epsilon] \preceq \liminf_{x \to \bar{x}} \mathbf{F}(x). \end{aligned}$$

As $\mathbf{F}(\bar{x}) \ominus_{gH} [\epsilon, \epsilon] \preceq \liminf_{x \to \bar{x}} \mathbf{F}(x)$ for every $\epsilon > 0$, we have $\mathbf{F}(\bar{x}) \preceq \liminf_{x \to \bar{x}} \mathbf{F}(x)$. Thus, $\mathbf{F}$ is gH-lsc at $\bar{x}$. □

Definition 3.4 (*Upper limit and gH-upper semicontinuity of an extended IVF*). The upper limit of an extended IVF $\mathbf{F}$ at $\bar{x} \in \mathcal{X}$, denoted $\limsup_{x \to \bar{x}} \mathbf{F}(x)$, is defined as

$$\begin{aligned} \limsup_{x \to \bar{x}} \mathbf{F}(x) &= \lim_{\delta \downarrow 0} \left(\sup\{\mathbf{F}(x) : x \in B_\delta(\bar{x})\}\right) \\ &= \inf_{\delta > 0} \left(\sup\{\mathbf{F}(x) : x \in B_\delta(\bar{x})\}\right). \end{aligned}$$

$\mathbf{F}$ is called gH-upper semicontinuous (gH-usc) at $\bar{x}$ if

$$\limsup_{x \to \bar{x}} \mathbf{F}(x) \preceq \mathbf{F}(\bar{x}). \tag{2}$$

Further, $\mathbf{F}$ is called gH-usc on $\mathcal{X}$ if (2) holds for every $\bar{x} \in \mathcal{X}$.

Note 3.2 Let $\mathbf{F}$ be an extended IVF with $\mathbf{F}(x) = \left[\underline{f}(x), \overline{f}(x)\right]$, where $\underline{f}, \overline{f} : \mathcal{X} \to \mathbb{R} \cup \{-\infty, +\infty\}$ be two extended real-valued functions. Then, because of a similar reason as in Note 3.1, $\mathbf{F}$ is gH-usc at $\bar{x} \in \mathcal{X}$ if and only if $\underline{f}$ and $\overline{f}$ are usc at $\bar{x}$.

Theorem 3.2 *Let $\mathbf{F}$ be an extended IVF. Then, $\mathbf{F}$ is gH-usc at $\bar{x} \in \mathcal{X}$ if and only if for each $\epsilon > 0$, there exists a $\delta > 0$ such that $\mathbf{F}(x) \prec \mathbf{F}(\bar{x}) \oplus [\epsilon, \epsilon]$ for all $x \in B_\delta(\bar{x})$.*

Proof Similar to the proof of Theorem 3.1. □

Theorem 3.3 *An extended IVF $\mathbf{F}$ is gH-continuous if and only if $\mathbf{F}$ is both gH-lower and upper semicontinuous.*

Proof Let $\mathbf{F}$ be gH-continuous at $\bar{x} \in \mathcal{X}$. Then, for each $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\begin{aligned}
&\|\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x})\|_{I(\mathbb{R})} < \epsilon \text{ for all } x \in B_\delta(\bar{x}) \\
&\quad \Longleftrightarrow \mathbf{F}(\bar{x}) \ominus_{gH} [\epsilon, \epsilon] \prec \mathbf{F}(x) \prec \mathbf{F}(\bar{x}) \oplus [\epsilon, \epsilon] \text{ for all } x \in B_\delta(\bar{x}), \quad \text{by } (ii) \text{ of Lemma 2.2} \\
&\quad \Longleftrightarrow \mathbf{F}(\bar{x}) \text{ is } gH\text{-lsc and } gH\text{-usc at } \bar{x}, \quad \text{by Theorems 3.1 } \textit{and } 3.2.
\end{aligned}$$

□

Definition 3.5 (*Proper IVF*). An extended IVF $\mathbf{F}$ is called a proper function if there exists an $\bar{x} \in \mathcal{X}$ such that $\mathbf{F}(\bar{x}) \prec [+\infty, +\infty]$ and $[-\infty, -\infty] \prec \mathbf{F}(x)$ for all $x \in \mathcal{X}$.

Example 3.2 Consider the IVF $\mathbf{F} : \mathbb{R}^2 \to \overline{I(\mathbb{R})}$ that is given by $\mathbf{F}(x_1, x_2) = \left[x_1, e^{x_1} + {x_2}^2\right]$. Note that $\mathbf{F}(0, 0) = [0, 1] \prec [+\infty, +\infty]$. Also, $[-\infty, -\infty] \prec \mathbf{F}(x_1, x_2)$ for all $(x_1, x_2) \in \mathbb{R}^2$. Therefore, $\mathbf{F}$ is a proper function.

Lemma 3.1 *Let $\mathbf{F}_1$ and $\mathbf{F}_2$ be two proper extended IVFs, and $\mathcal{S}$ be a nonempty subset of $\mathcal{X}$. Then,*

(i) $\inf_{x \in \mathcal{S}} \mathbf{F}_1(x) \oplus \inf_{x \in \mathcal{S}} \mathbf{F}_2(x) \preceq \inf_{x \in \mathcal{S}} \{\mathbf{F}_1(x) \oplus \mathbf{F}_2(x)\}$ *and*
(ii) $\sup_{x \in \mathcal{S}} \{\mathbf{F}_1(x) \oplus \mathbf{F}_2(x)\} \preceq \sup_{x \in \mathcal{S}} \mathbf{F}_1(x) \oplus \sup_{x \in \mathcal{S}} \mathbf{F}_2(x)$.

Proof Let $\boldsymbol{\alpha}_1 = \inf_{x \in \mathcal{S}} \mathbf{F}_1(x)$ and $\boldsymbol{\alpha}_2 = \inf_{x \in \mathcal{S}} \mathbf{F}_2(x)$. Then,

$$\begin{aligned}
&\boldsymbol{\alpha}_1 \preceq \mathbf{F}_1(x) \text{ for all } x \in \mathcal{S} \text{ and } \boldsymbol{\alpha}_2 \preceq \mathbf{F}_2(x) \text{ for all } x \in \mathcal{S} \\
&\quad \Longrightarrow \boldsymbol{\alpha}_1 \oplus \boldsymbol{\alpha}_2 \preceq \mathbf{F}_1(x) \oplus \mathbf{F}_2(x) \text{ for all } x \in \mathcal{S}, \quad \text{by } (ii) \text{ of Lemma 2.1} \\
&\quad \Longrightarrow \boldsymbol{\alpha}_1 \oplus \boldsymbol{\alpha}_2 \preceq \inf_{x \in \mathcal{S}} (\mathbf{F}_1(x) \oplus \mathbf{F}_2(x)) \\
&\quad \text{i.e., } \inf_{x \in \mathcal{S}} \mathbf{F}_1(x) \oplus \inf_{x \in \mathcal{S}} \mathbf{F}_2(x) \preceq \inf_{x \in \mathcal{S}} \{\mathbf{F}_1(x) \oplus \mathbf{F}_2(x)\}.
\end{aligned}$$

Part (ii) can be similarly proved. □

Theorem 3.4 *Let $\mathbf{F}_1$ and $\mathbf{F}_2$ be two proper extended IVFs, and $\mathcal{S}$ be a nonempty subset of $\mathcal{X}$. Then,* *(i)* $\liminf_{x\to\bar{x}} \mathbf{F}_1(x) \oplus \liminf_{x\to\bar{x}} \mathbf{F}_2(x) \preceq \liminf_{x\to\bar{x}} (\mathbf{F}_1 \oplus \mathbf{F}_2)(x)$ *and*

(ii) $\limsup_{x\to\bar{x}} (\mathbf{F}_1 \oplus \mathbf{F}_2)(x) \preceq \limsup_{x\to\bar{x}} \mathbf{F}_1(x) \oplus \limsup_{x\to\bar{x}} \mathbf{F}_2(x)$.

Proof

$$\begin{aligned}
\liminf_{x\to\bar{x}} \mathbf{F}_1(x) \oplus \liminf_{x\to\bar{x}} \mathbf{F}_2(x) &= \lim_{\delta\downarrow 0} \inf_{x\in B_\delta(\bar{x})} \mathbf{F}_1(x) \oplus \lim_{\delta\downarrow 0} \inf_{x\in B_\delta(\bar{x})} \mathbf{F}_2(x), \quad \text{by Definition 3.3} \\
&\preceq \lim_{\delta\downarrow 0} \left(\inf_{x\in B_\delta(\bar{x})} \mathbf{F}_1(x) \oplus \inf_{x\in B_\delta(\bar{x})} \mathbf{F}_2(x) \right) \\
&\preceq \lim_{\delta\downarrow 0} \inf_{x\in B_\delta(\bar{x})} (\mathbf{F}_1 \oplus \mathbf{F}_2)(x), \quad \text{by } (i) \text{ of Lemma 3.1} \\
&= \liminf_{x\to\bar{x}} (\mathbf{F}_1 \oplus \mathbf{F}_2)(x).
\end{aligned}$$

This completes the proof of (i). Part (ii) can be similarly proved. □

Theorem 3.5 *Let* $\mathbf{F}_1$ *and* $\mathbf{F}_2$ *be two proper and* gH*-lsc extended IVFs. Then,* $\mathbf{F}_1 \oplus \mathbf{F}_2$ *is* gH*-lsc.*

Proof Take $\bar{x} \in \mathcal{X}$. Since $\mathbf{F}_1$ and $\mathbf{F}_2$ are gH-lsc at $\bar{x}$, we have

$$\begin{aligned}
&\mathbf{F}_1(\bar{x}) \preceq \liminf_{x\to\bar{x}} \mathbf{F}_1(x) \text{ and } \mathbf{F}_2(\bar{x}) \preceq \liminf_{x\to\bar{x}} \mathbf{F}_2(x) \\
&\implies \mathbf{F}_1(\bar{x}) \oplus \mathbf{F}_2(\bar{x}) \preceq \liminf_{x\to\bar{x}} \mathbf{F}_1(x) \oplus \liminf_{x\to\bar{x}} \mathbf{F}_2(x), \quad \text{by } (ii) \text{ of Lemma 2.1} \\
&\implies (\mathbf{F}_1 \oplus \mathbf{F}_2)(\bar{x}) \preceq \liminf_{x\to\bar{x}} (\mathbf{F}_1 \oplus \mathbf{F}_2)(x), \quad \text{by } (i)\, of\, Theorem\ 3.4 \\
&\implies \mathbf{F}_1 \oplus \mathbf{F}_2 \text{ is } gH\text{-lsc at } \bar{x}.
\end{aligned}$$

Since $\bar{x}$ is arbitrarily chosen, so $\mathbf{F}_1 \oplus \mathbf{F}_2$ is gH-lsc on $\mathcal{X}$. □

Lemma 3.2 *(*Characterization of lower limits of IVFs*). Let* $\boldsymbol{F}$ *be an extended IVF. Then,*

$$\liminf_{x\to\bar{x}} \boldsymbol{F}(x) = \inf \left\{ \boldsymbol{\alpha} \in \overline{I(\mathbb{R})} : \textit{there exists a sequence } x_k \to \bar{x} \textit{ with } \boldsymbol{F}(x_k) \to \boldsymbol{\alpha} \right\}.$$

Proof Let $\bar{\boldsymbol{\alpha}} = \liminf_{x\to\bar{x}} \mathbf{F}(x)$. Assume that sequence $x_k \to \bar{x}$ with $\mathbf{F}(x_k) \to \boldsymbol{\alpha}$. In the below, we show that $\bar{\boldsymbol{\alpha}} \preceq \boldsymbol{\alpha}$.
Since $x_k \to \bar{x}$, for any $\delta > 0$, there exists $k_\delta \in \mathbb{N}$ such that $x_k \in B_\delta(\bar{x})$ for every $k \geq k_\delta$. Therefore,

$$\begin{aligned}
&\inf\{\mathbf{F}(x) : x \in B_\delta(\bar{x})\} \preceq \mathbf{F}(x_k) \text{ for any } \delta > 0 \\
&\implies \inf\{\mathbf{F}(x) : x \in B_\delta(\bar{x})\} \preceq \lim_{k\to+\infty} \mathbf{F}(x_k) \text{ for any } \delta > 0 \\
&\implies \inf\{\mathbf{F}(x) : x \in B_\delta(\bar{x})\} \preceq \boldsymbol{\alpha} \text{ for any } \delta > 0 \\
&\implies \lim_{\delta\downarrow 0} \inf\{\mathbf{F}(x) : x \in B_\delta(\bar{x})\} \preceq \boldsymbol{\alpha} \\
&\implies \liminf_{x\to\bar{x}} \mathbf{F}(x) = \bar{\boldsymbol{\alpha}} \preceq \boldsymbol{\alpha}.
\end{aligned}$$

Next, we show that there exists a sequence $x_k \to \bar{x}$ with $\mathbf{F}(x_k) \to \bar{\boldsymbol{\alpha}}$.
Consider a nonnegative sequence $\{\delta_k\}$ with $\delta_k \downarrow 0$, and construct a sequence $\bar{\boldsymbol{\alpha}}_k = \inf\{\mathbf{F}(x) : x \in B_{\delta_k}(\bar{x})\}$.
As $\delta_k \downarrow 0$, by Definition 3.3 of lower limit, $\bar{\boldsymbol{\alpha}}_k \to \bar{\boldsymbol{\alpha}}$. Also, by definition of infimum, for a given $\epsilon > 0$ and $k \in \mathbb{N}$, there exists $x_k \in B_{\delta_k}(\bar{x})$ such that $\mathbf{F}(x_k) \preceq \bar{\boldsymbol{\alpha}}_k$. That is, $\bar{\boldsymbol{\alpha}}_k \preceq \mathbf{F}(x_k) \preceq \boldsymbol{\alpha}_k$, where $\boldsymbol{\alpha}_k \to \bar{\boldsymbol{\alpha}}$.

Note that $x_k \in B_{\delta_k}(\bar{x})$ and $\delta_k \downarrow 0$. Therefore, as $k \to +\infty$, $x_k \to \bar{x}$. Also, note that $\mathbf{F}(x_k)$ is a monotonic increasing bounded sequence and therefore, by Lemma 2.3, $\mathbf{F}(x_k)$ converges to $\bar{\boldsymbol{\alpha}}$, and the proof is complete. □

Lemma 3.3 (*Characterization of upper limits of IVFs*). *Let* $\boldsymbol{F}$ *be an extended IVF. Then,*

$$\limsup_{x\to\bar{x}} \boldsymbol{F}(x) = \sup\left\{\boldsymbol{\alpha} \in \overline{I(\mathbb{R})} : \textit{there exists a sequence } x_k \to \bar{x} \textit{ with } \boldsymbol{F}(x_k) \to \boldsymbol{\alpha}\right\}.$$

Proof Similar to the proof of Lemma 3.2. □

Definition 3.6 (*Level set of an IVF*). Let $\mathbf{F}$ be an extended IVF. For an $\boldsymbol{\alpha} \in \overline{I(\mathbb{R})}$, the level set of $\mathbf{F}$, denoted as $\text{lev}_{\boldsymbol{\alpha}\nprec}\mathbf{F}$, is defined by

$$\text{lev}_{\boldsymbol{\alpha}\nprec}\mathbf{F} = \{x \in \mathcal{X} : \boldsymbol{\alpha} \nprec \mathbf{F}(x)\}.$$

Example 3.3 Consider $\mathbf{F} : \mathbb{R}^2 \to \overline{I(\mathbb{R})}$ as $\mathbf{F}(x) = [1, 2] \odot {x_1}^2 \oplus [3, 4] \odot e^{{x_2}^2}$ and $\boldsymbol{\alpha} = [-1, 10]$. Then,

$$\begin{aligned}
\text{lev}_{\boldsymbol{\alpha}\nprec}\mathbf{F} &= \left\{(x_1, x_2) \in \mathbb{R}^2 : [-1, 10] \nprec [1, 2] \odot {x_1}^2 \oplus [3, 4] \odot e^{{x_2}^2}\right\}\\
&= \left\{(x_1, x_2) \in \mathbb{R}^2 : [-1, 10] \nprec \left[{x_1}^2 + 3e^{{x_2}^2}, 2{x_1}^2 + 4e^{{x_2}^2}\right]\right\}\\
&= \left\{(x_1, x_2) \in \mathbb{R}^2 : \left[{x_1}^2 + 3e^{{x_2}^2}, 2{x_1}^2 + 4e^{{x_2}^2}\right] \preceq [-1, 10] \text{ or}\right.\\
&\qquad\left. [-1, 10] \text{ and } \left[{x_1}^2 + 3e^{{x_2}^2}, 2{x_1}^2 + 4e^{{x_2}^2}\right] \text{ are not comparable}\right\}\\
&= \left\{(x_1, x_2) \in \mathbb{R}^2 : [-1, 10] \text{ and } \left[{x_1}^2 + 3e^{{x_2}^2}, 2{x_1}^2 + 4e^{{x_2}^2}\right] \text{ are not comparable}\right\}\\
&= \left\{(x_1, x_2) \in \mathbb{R}^2 : \text{`}{x_1}^2 + 3e^{{x_2}^2} < -1 \text{ and } 2{x_1}^2 + 4e^{{x_2}^2} > 10\text{' or}\right.\\
&\qquad\left. \text{`}{x_1}^2 + 3e^{{x_2}^2} > -1 \text{ and } 2{x_1}^2 + 4e^{{x_2}^2} < 10\text{'}\right\}\\
&= \left\{(x_1, x_2) \in \mathbb{R}^2 : {x_1}^2 + 3e^{{x_2}^2} > -1 \text{ and } 2{x_1}^2 + 4e^{{x_2}^2} < 10\right\}\\
&= \left\{(x_1, x_2) \in \mathbb{R}^2 : 2{x_1}^2 + 4e^{{x_2}^2} < 10\right\}.
\end{aligned}$$

Hence,

$$\text{lev}_{\boldsymbol{\alpha}\nprec}\mathbf{F} = \left\{(x_1, x_2) \in \mathbb{R}^2 : {x_1}^2 + 2e^{{x_2}^2} < 5\right\}.$$

Definition 3.7 (*Level-bounded IVF*). An extended IVF $\mathbf{F}$ is said to be level-bounded if for any $\boldsymbol{\alpha} \in I(\mathbb{R})$, $\text{lev}_{\boldsymbol{\alpha}\nprec}\mathbf{F}$ is bounded.

Lemma 3.4 *Let* $\boldsymbol{F}$ *be an extended IVF and* $\bar{x} \in \mathcal{X}$*. Then,*

$$\inf_{\{x_k\}} (\liminf \boldsymbol{F}(x_k)) \nprec \liminf_{x\to\bar{x}} \boldsymbol{F}(x), \tag{3}$$

where the infimum on the left-hand side is taken over all sequences $x_k \to \bar{x}$.

Proof Let $\mathbf{M} = \liminf_{x\to\bar{x}} \mathbf{F}(x)$ and $\mathbf{L} = \inf_{\{x_k\}} \liminf \mathbf{F}(x_k)$.
If $\mathbf{M} = -\infty$, there is nothing to prove.
Next, let $\mathbf{M} = +\infty$. Let $\{x_k\}$ be an arbitrary sequence converging to $\bar{x}$. We show that $\mathbf{F}(x_k) \to +\infty$. Since $\mathbf{M} = +\infty$, for any given $\alpha > 0$, there exists a $\delta > 0$ such that

$[\alpha, \alpha] \prec \inf_{x \in B_\delta(\bar{x})} \mathbf{F}(x)$. Since $x_k \to \bar{x}$, there exists an integer $m > 0$ such that $x_k \in B_\delta(\bar{x})$ for all $n \geq m$. Thus, $[\alpha, \alpha] \prec \mathbf{F}(x_k)$ for all $n \geq m$, and hence $\mathbf{F}(x_k) \to +\infty$.
Finally, let $[-\infty, -\infty] \prec \mathbf{M} \prec [+\infty, +\infty]$, i.e., $\mathbf{M} \in I(\mathbb{R})$. Suppose that there exists $\epsilon_0 > 0$ such that for all $\delta > 0$, $\inf_{x \in B_\delta(\bar{x})} \mathbf{F}(x) \preceq \mathbf{M} \ominus_{gH} [\epsilon_0, \epsilon_0]$. Then,

$$\begin{aligned} &\lim_{\delta \downarrow 0} \inf_{x \in B_\delta(\bar{x})} \mathbf{F}(x) \preceq \mathbf{M} \ominus_{gH} [\epsilon_0, \epsilon_0] \\ &\implies \liminf_{x \to \bar{x}} \mathbf{F}(x) \preceq \mathbf{M} \ominus_{gH} [\epsilon_0, \epsilon_0] \\ &\text{i.e., } \mathbf{M} \preceq \mathbf{M} \ominus_{gH} [\epsilon_0, \epsilon_0], \end{aligned}$$

which is not true. Thus, for a given $\epsilon > 0$, there exists a $\delta > 0$ such that $\inf_{x \in B_\delta(\bar{x})} \mathbf{F}(x) \npreceq \mathbf{M} \ominus_{gH} [\epsilon, \epsilon]$. This implies $\mathbf{F}(x) \npreceq \mathbf{M} \ominus_{gH} [\epsilon, \epsilon]$ for all $x \in B_\delta(\bar{x})$.
Let $\{x_k\}$ be a sequence converging to $\bar{x}$. Since $x_k \in B_\delta(\bar{x})$ for large enough k, we have $\liminf \mathbf{F}(x_k) \npreceq \mathbf{M} \ominus_{gH} [\epsilon, \epsilon]$ for any $\epsilon > 0$. Thus, $\liminf \mathbf{F}(x_k) \nprec \mathbf{M}$ for any sequence converging to $\bar{x}$, and hence $\mathbf{L} \nprec \mathbf{M}$. Therefore, (3) holds. ◻

Theorem 3.6 *Let $\mathbf{F}$ be an extended IVF. Then, $\mathbf{F}$ is gH-lsc on $\mathcal{X}$ if and only if the level set $\text{lev}_{\boldsymbol{\alpha} \nprec} \mathbf{F}$ is closed for every $\boldsymbol{\alpha} \in I(\mathbb{R})$.*

Proof Let $\mathbf{F}$ be gH-lsc on $\mathcal{X}$. For a fixed $\boldsymbol{\alpha} \in I(\mathbb{R})$, suppose that $\{x_k\} \subseteq \text{lev}_{\boldsymbol{\alpha} \nprec} \mathbf{F}$ such that $x_k \to \bar{x}$. Then,

$$\begin{aligned} &\boldsymbol{\alpha} \nprec \mathbf{F}(x_k) \\ &\implies \boldsymbol{\alpha} \nprec \liminf \mathbf{F}(x_k) \\ &\implies \boldsymbol{\alpha} \nprec \liminf_{x \to \bar{x}} \mathbf{F}(x), \quad \text{by Lemma 3.4} \\ &\implies \boldsymbol{\alpha} \nprec \mathbf{F}(\bar{x}) \text{ since } \mathbf{F} \text{ is gH-lsc at } \bar{x}. \end{aligned}$$

Thus, $\bar{x} \in \text{lev}_{\boldsymbol{\alpha} \nprec} \mathbf{F}$, and hence $\text{lev}_{\boldsymbol{\alpha} \nprec} \mathbf{F}$ is closed.
Since $\boldsymbol{\alpha} \in I(\mathbb{R})$ is arbitrarily chosen, $\text{lev}_{\boldsymbol{\alpha} \nprec} \mathbf{F}$ is closed for every $\boldsymbol{\alpha} \in I(\mathbb{R})$.
Conversely, suppose the level set $\text{lev}_{\boldsymbol{\alpha} \nprec} \mathbf{F}$ is closed for every $\boldsymbol{\alpha} \in I(\mathbb{R})$. Fix an $\bar{x} \in \mathcal{X}$. To prove that $\mathbf{F}$ is gH-lsc at $\bar{x}$, we need to show that

$$\mathbf{F}(\bar{x}) \preceq \liminf_{x \to \bar{x}} \mathbf{F}(x).$$

Let $\bar{\boldsymbol{\alpha}} = \liminf_{x \to \bar{x}} \mathbf{F}(x)$. The case of $\bar{\boldsymbol{\alpha}} = +\infty$ is trivial; so assume $\bar{\boldsymbol{\alpha}} \prec [+\infty, +\infty]$.
By Lemma 3.2, there exists a sequence $x_k \to \bar{x}$ with $\mathbf{F}(x_k) \to \bar{\boldsymbol{\alpha}}$. For any $\boldsymbol{\alpha}$ such that $\bar{\boldsymbol{\alpha}} \prec \boldsymbol{\alpha}$, it will eventually be true that $\boldsymbol{\alpha} \nprec \mathbf{F}(x_k)$, or in other words, that $x_k \in \text{lev}_{\boldsymbol{\alpha} \nprec} \mathbf{F}$. Since $\text{lev}_{\boldsymbol{\alpha} \nprec} \mathbf{F}$ is closed, $\bar{x} \in \text{lev}_{\boldsymbol{\alpha} \nprec} \mathbf{F}$.
Thus, $\boldsymbol{\alpha} \nprec \mathbf{F}(\bar{x})$ for every $\boldsymbol{\alpha}$ such that $\bar{\boldsymbol{\alpha}} \prec \boldsymbol{\alpha}$, then $\bar{\boldsymbol{\alpha}} \nprec \mathbf{F}(\bar{x})$. Therefore, either $\mathbf{F}(\bar{x}) \preceq \bar{\boldsymbol{\alpha}}$ or $\bar{\boldsymbol{\alpha}}$ and $\mathbf{F}(\bar{x})$ are not comparable. But since $\bar{\boldsymbol{\alpha}} = \liminf_{x \to \bar{x}} \mathbf{F}(x)$, so $\bar{\boldsymbol{\alpha}}$ is comparable with $\mathbf{F}(\bar{x})$, and hence $\mathbf{F}(\bar{x}) \preceq \bar{\boldsymbol{\alpha}}$.
Since $\bar{x} \in \mathcal{X}$ is arbitrarily chosen, $\mathbf{F}$ is gH-lsc on $\mathcal{X}$. This completes the proof. ◻

Definition 3.8 (*Indicator function*). Consider a subset $\mathcal{S}$ of $\mathcal{X}$. The indicator function of $\mathcal{S}$, $\delta_{\mathcal{S}}(s) : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\delta_{\mathcal{S}}(s) = \begin{cases} \mathbf{0} & \text{if } s \in \mathcal{S} \\ +\infty & \text{if } s \notin \mathcal{S}. \end{cases}$$

Remark 3.1 (i) It is easy to see that $\delta_{\mathcal{S}}$ is proper if and only if $\mathcal{S}$ is nonempty.

(ii) By Theorem 3.6, $\delta_{\mathcal{S}}$ is gH-lsc if and only if $\mathcal{S}$ is closed.

Definition 3.9 (*Argument minimum of an IVF*). Let $\mathbf{F}$ be an extended IVF. Then, the argument minimum of $\mathbf{F}$, denoted as $\underset{x\in\mathcal{X}}{\operatorname{argmin}}\mathbf{F}(x)$, is defined by

$$\underset{x\in\mathcal{X}}{\operatorname{argmin}}\mathbf{F}(x) = \begin{cases} \left\{x \in \mathcal{X} : \mathbf{F}(x) = \inf\limits_{y\in\mathcal{X}} \mathbf{F}(y)\right\} & \text{if } \inf\limits_{y\in\mathcal{X}} \mathbf{F}(y) \neq +\infty \\ \emptyset & \text{if } \inf\limits_{y\in\mathcal{X}} \mathbf{F}(y) = +\infty. \end{cases}$$

Example 3.4 Consider $\mathbf{F} : \mathbb{R}^2 \to \overline{I(\mathbb{R})}$ as $\mathbf{F}(x_1, x_2) =$

$$\begin{cases} \left[-\frac{1}{|x_1|}, e^{-\frac{1}{|x_1|}+x_2{}^2}\right] & \text{if } x_1 \neq 0 \\ [-\infty, 0] & \text{if } x_1 = 0. \end{cases}$$

Then, $\inf\limits_{(x_1,x_2)\in\mathbb{R}^2} \mathbf{F}(x_1, x_2) = [-\infty, 0]$.

$$\begin{aligned} \underset{x\in\mathbb{R}^2}{\operatorname{argmin}}\mathbf{F}(x) &= \left\{(x_1, x_2) \in \mathbb{R}^2 : \mathbf{F}(x_1, x_2) = \inf_{x\in\mathbb{R}^2} \mathbf{F}(x_1, x_2) = [-\infty, 0]\right\} \\ &= \{(0, x_2) : x_2 \in \mathbb{R}\}. \end{aligned}$$

Therefore, $\underset{x\in\mathbb{R}^2}{\operatorname{argmin}}\mathbf{F}(x) = \{(0, x_2) : x_2 \in \mathbb{R}\}$.

Theorem 3.7 (Minimum attained by an extended IVF). *Let $\mathbf{F}$ be gH-lsc, level-bounded and proper extended IVF. Then, the set* $\operatorname{argmin}_{\mathcal{X}} \mathbf{F}$ *is nonempty and compact.*

Proof Let $\bar{\boldsymbol{\alpha}} = \inf \mathbf{F}$. So, $\bar{\boldsymbol{\alpha}} \prec [+\infty, +\infty]$ because $\mathbf{F}$ is proper.

Note that $\text{lev}_{\boldsymbol{\alpha}\nprec}\mathbf{F} \neq \emptyset$ for any $\boldsymbol{\alpha}$ that satisfies $\bar{\boldsymbol{\alpha}} \prec \boldsymbol{\alpha} \prec [+\infty, +\infty]$. Also, as $\mathbf{F}$ is level-bounded, $\text{lev}_{\boldsymbol{\alpha}\nprec}\mathbf{F}$ is bounded and by Theorem 3.6, it is also closed. Thus, $\text{lev}_{\boldsymbol{\alpha}\nprec}\mathbf{F}$ is nonempty compact for $\bar{\boldsymbol{\alpha}} \prec \boldsymbol{\alpha} \prec [+\infty, +\infty]$ and are nested as $\text{lev}_{\boldsymbol{\alpha}\nprec}\mathbf{F} \subseteq \text{lev}_{\boldsymbol{\beta}\nprec}\mathbf{F}$ when $\boldsymbol{\alpha} \prec \boldsymbol{\beta}$. Therefore,

$$\bigcap_{\bar{\boldsymbol{\alpha}}\prec\boldsymbol{\alpha}\prec+\infty} \text{lev}_{\boldsymbol{\alpha}\nprec}\mathbf{F} = \text{lev}_{\bar{\boldsymbol{\alpha}}\nprec}\mathbf{F} = \operatorname{argmin}_{\mathcal{X}} \mathbf{F}$$

is nonempty and compact. □

Next, we present a theorem which gives a characterization of the argument minimum set of an IVF in terms of gH-Gâteaux differentiability. An IVF $\mathbf{F} : \mathcal{X} \to I(\mathbb{R})$ is said to be gH-Gâteaux differentiable (see Ghosh et al. 2020) at $\bar{x} \in \mathcal{X}$ if the limit

$$\mathbf{F}_{\mathscr{G}}(\bar{x})(h) = \lim_{\lambda\to0+} \frac{1}{\lambda} \odot \left(\mathbf{F}(\bar{x} + \lambda h) \ominus_{gH} \mathbf{F}(\bar{x})\right)$$

exists for all $h \in \mathcal{X}$ and $\mathbf{F}_{\mathscr{G}}(\bar{x})$ is a gH-continuous linear IVF from $\mathcal{X}$ to $I(\mathbb{R})$. Then, we call $\mathbf{F}_{\mathscr{G}}(\bar{x})$ as the gH-Gâteaux derivative of $\mathbf{F}$ at $\bar{x}$.

Theorem 3.8 (Characterization of the set argument minimum of an IVF). *Let $\mathbf{F}$ be an extended IVF and $\bar{x} \in$* $\operatorname{argmin}_{x\in\mathcal{X}}$ *$\mathbf{F}(x)$. If the function $\mathbf{F}$ has a gH-Gâteaux derivative at $\bar{x}$ in every direction $h \in \mathcal{X}$, then*

$$\boldsymbol{F}_{\mathscr{G}}(\bar{x})(h) = \boldsymbol{0} \textit{ for all } h \in \mathcal{X}.$$

Proof Observe that any $\bar{x} \in \operatorname{argmin}_{x\in\mathcal{X}} \mathbf{F}(x)$, is also an efficient point. Then, the proof follows from proof of the Theorem 4.2 in Ghosh et al. (2020). □

4 Ekeland's variational principle and its applications

In this section, we present the main results—Ekeland's variational principle for IVFs along with its application for gH-Gâteaux differentiable IVFs.

Lemma 4.1 *Let $\bar{x} \in \mathcal{X}$ and $\mathbf{A} \in I(\mathbb{R})$. Then, $\{x \in \mathcal{X} : \mathbf{A} \nprec \|x - \bar{x}\|_{\mathcal{X}}\}$ is a bounded set.*

Proof Let $\mathbf{A} = [\underline{a}, \overline{a}]$. Then,

$$\begin{aligned}
&\{x \in \mathcal{X} : \mathbf{A} \nprec \|x - \bar{x}\|_{\mathcal{X}}\}\\
&\quad = \{x \in \mathcal{X} : [\underline{a}, \overline{a}] \nprec \|x - \bar{x}\|_{\mathcal{X}}\}\\
&\quad = \{x \in \mathcal{X} : \|x - \bar{x}\|_{\mathcal{X}} \preceq [\underline{a}, \overline{a}] \text{ or `}[\underline{a}, \overline{a}] \text{ and } \|x - \bar{x}\|_{\mathcal{X}} \text{ are not comparable'}\}\\
&\quad = \{x \in \mathcal{X} : \text{`}\|x - \bar{x}\|_{\mathcal{X}} \leq \underline{a} \text{ and } \|x - \bar{x}\|_{\mathcal{X}} \leq \overline{a}\text{'}\\
&\qquad\qquad \text{or `}[\underline{a}, \overline{a}] \text{ and } \|x - \bar{x}\|_{\mathcal{X}} \text{ are not comparable'}\}\\
&\quad = \{x \in \mathcal{X} : \text{`}\|x - \bar{x}\|_{\mathcal{X}} \leq \underline{a}\text{' or `}\|x - \bar{x}\|_{\mathcal{X}} < \underline{a} \text{ and } \|x - \bar{x}\|_{\mathcal{X}} > \overline{a}\text{'}\\
&\qquad\qquad \text{or `}\|x - \bar{x}\|_{\mathcal{X}} > \underline{a} \text{ and } \|x - \bar{x}\|_{\mathcal{X}} < \overline{a}\text{'}\}\\
&\quad = \{x \in \mathcal{X} : \|x - \bar{x}\|_{\mathcal{X}} \leq \underline{a} \text{ or } \underline{a} < \|x - \bar{x}\|_{\mathcal{X}} < \overline{a}\},
\end{aligned}$$

which is a bounded set.
Hence, for any $\bar{x} \in \mathcal{X}$ and $\mathbf{A} \in I(\mathbb{R})$, $\{x \in \mathcal{X} : \mathbf{A} \nprec \|x - \bar{x}\|_{\mathcal{X}}\}$ is bounded. □

Theorem 4.1 (Ekeland's variational principle for IVFs). *Let $\boldsymbol{F} : \mathcal{X} \to I(\mathbb{R}) \cup \{+\infty\}$ be a gH-lsc extended IVF and $\epsilon > 0$. Assume that*

$$\inf_{\mathcal{X}} \boldsymbol{F} \text{ is finite and } \boldsymbol{F}(\bar{x}) \prec \inf_{\mathcal{X}} \boldsymbol{F} \oplus [\epsilon, \epsilon].$$

Then, for any $\delta > 0$, there exists an $x_0 \in \mathcal{X}$ such that

(i) $\|x_0 - \bar{x}\|_{\mathcal{X}} < \frac{\epsilon}{\delta}$,
(ii) $\boldsymbol{F}(x_0) \preceq \boldsymbol{F}(\bar{x})$, *and*
(iii) $\underset{x \in \mathcal{X}}{\operatorname{argmin}}\{\boldsymbol{F}(x) \oplus \delta\|x - x_0\|_{\mathcal{X}}\} = \{x_0\}$.

Proof Let $\bar{\boldsymbol{\alpha}} = \inf_{\mathcal{X}} \mathbf{F}$ and $\overline{\mathbf{F}}(x) = \mathbf{F}(x) \oplus \delta\|x - \bar{x}\|_{\mathcal{X}}$.
Since $\overline{\mathbf{F}}$ is the sum of two gH-lsc and proper IVFs, $\overline{\mathbf{F}}$ is gH-lsc by Theorem 3.5. Also,

$$\begin{aligned}
\operatorname{lev}_{\boldsymbol{\alpha}\nprec}\overline{\mathbf{F}} &= \left\{x \in \mathcal{X} : \boldsymbol{\alpha} \nprec \overline{\mathbf{F}}(x)\right\}\\
&= \{x \in \mathcal{X} : \boldsymbol{\alpha} \nprec \mathbf{F}(x) \oplus \delta\|x - \bar{x}\|_{\mathcal{X}}\}\\
&\subseteq \{x \in \mathcal{X} : \boldsymbol{\alpha} \nprec \bar{\boldsymbol{\alpha}} \oplus \delta\|x - \bar{x}\|_{\mathcal{X}}\}\\
&= \left\{x \in \mathcal{X} : \frac{\boldsymbol{\alpha} \ominus_{gH} \bar{\boldsymbol{\alpha}}}{\delta} \nprec \|x - \bar{x}\|_{\mathcal{X}}\right\}\\
&= \{x \in \mathcal{X} : \mathbf{A} \nprec \|x - \bar{x}\|_{\mathcal{X}}\}, \text{ where } \mathbf{A} = \frac{\boldsymbol{\alpha} \ominus_{gH} \bar{\boldsymbol{\alpha}}}{\delta}.
\end{aligned}$$

Therefore, by Lemma 4.1, $\overline{\mathbf{F}}$ is level-bounded. Clearly, $\overline{\mathbf{F}}$ is proper. Hence, by Theorem 3.7, $C = \operatorname{argmin}_{\mathcal{X}} \overline{\mathbf{F}}$ is nonempty and compact.
Let us consider the function $\tilde{\mathbf{F}} = \mathbf{F} \oplus \delta_C$ on $\mathcal{X}$. Note that $\tilde{\mathbf{F}}$ is proper and level-bounded. Since C is nonempty and compact, so by Remark 3.1, δ_C is gH-lsc. Thus, by Theorem 3.5, $\tilde{\mathbf{F}}$ is gH-lsc, and hence by Theorem 3.7, $\operatorname{argmin}_{\mathcal{X}} \tilde{\mathbf{F}}$ is nonempty.
Let $x_0 \in \operatorname{argmin}_{\mathcal{X}} \tilde{\mathbf{F}}$. Then, over the set C, $\mathbf{F}$ is minimum at x_0.

Since $x_0 \in C$, $\overline{\mathbf{F}}(x_0) \prec \overline{\mathbf{F}}(x)$ for $x \notin C$. This implies that for any $x \notin C$,

$$\begin{aligned} &\mathbf{F}(x_0) \oplus \delta\|x_0 - \bar{x}\|_{\mathcal{X}} \prec \mathbf{F}(x) \oplus \delta\|x - \bar{x}\|_{\mathcal{X}} \\ &\implies \mathbf{F}(x_0) \prec \mathbf{F}(x) \oplus \delta\|x - \bar{x}\|_{\mathcal{X}} \ominus_{gH} \delta\|x_0 - \bar{x}\|_{\mathcal{X}}. \end{aligned}$$

Hence, $\mathbf{F}(x_0) \prec \mathbf{F}(x) \oplus \delta\|x - x_0\|_{\mathcal{X}}$ for all $x \notin C$ with $x \neq x_0$, and thus $\operatorname{argmin}_{x \in \mathcal{X}}\{\mathbf{F}(x) \oplus \delta\|x - x_0\|_{\mathcal{X}}\} = \{x_0\}$.
Also, as $x_0 \in C$, we have $\overline{\mathbf{F}}(x_0) \preceq \overline{\mathbf{F}}(\bar{x})$, which implies

$$\begin{aligned} &\overline{\mathbf{F}}(x_0) \preceq \mathbf{F}(\bar{x}) \text{ because } \overline{\mathbf{F}}(\bar{x}) = \mathbf{F}(\bar{x}) \\ &\implies \mathbf{F}(x_0) \oplus \delta\|x_0 - \bar{x}\|_{\mathcal{X}} \preceq \mathbf{F}(\bar{x}) \\ &\implies \mathbf{F}(x_0) \preceq \mathbf{F}(\bar{x}) \ominus_{gH} \delta\|x_0 - \bar{x}\|_{\mathcal{X}} \\ &\implies \mathbf{F}(x_0) \prec \bar{\boldsymbol{\alpha}} \oplus [\epsilon, \epsilon] \ominus_{gH} \delta\|x_0 - \bar{x}\|_{\mathcal{X}} \text{ because } \mathbf{F}(\bar{x}) \prec \inf_{\mathcal{X}} \mathbf{F} \oplus [\epsilon, \epsilon] \\ &\implies \delta\|x_0 - \bar{x}\|_{\mathcal{X}} \prec \bar{\boldsymbol{\alpha}} \oplus [\epsilon, \epsilon] \ominus_{gH} \mathbf{F}(x_0) \\ &\implies \delta\|x_0 - \bar{x}\|_{\mathcal{X}} \prec [\epsilon, \epsilon] \text{ because } \bar{\boldsymbol{\alpha}} \ominus_{gH} \mathbf{F}(x_0) \preceq \mathbf{0} \\ &\implies \|x_0 - \bar{x}\|_{\mathcal{X}} < \frac{\epsilon}{\delta}. \end{aligned}$$

This completes the proof. □

Note 4.1 It is to note that if the IVF $\mathbf{F}$ considered in Theorem 4.1 is degenerate IVF, i.e., $\mathbf{F} = \underline{f} = \overline{f}$, then Theorem 4.1 reduces to the conventional Ekeland's variational principle (Theorem 2.1). Hence, Ekeland's variational principle for IVFs (Theorem 4.1) is a true generalization of conventional Ekeland's variational principle (Theorem 2.1).

Example 4.1 In this example, we verify Theorem 4.1 for the IVF $\mathbf{F} : \mathbb{R}^2 \to I(\mathbb{R})$ given by

$$\mathbf{F}(x_1, x_2) = \left[|x_1 x_2|, e^{|x_1 x_2|}\right].$$

It is easy to see that $\underline{f}$ and $\overline{f}$ are lsc, and hence by Note 3.1, $\mathbf{F}$ is gH-lsc. Note that $\mathbf{F}(0, 0) = [0, 1] \prec \mathbf{F}(x_1, x_2)$ for all $x_1, x_2 \in \mathbb{R}^2$. Therefore, by Definition 2.6, $[0, 1]$ is the infimum of $\mathbf{F}$. Let $\bar{x} = (1, 1)$ and $\epsilon = 2$. Note that $\mathbf{F}(\bar{x}) = [1, e] \prec [0, 1] + [\epsilon, \epsilon] = [2, 3]$. Thus, all the hypotheses of Theorem 4.1 are satisfied. We verify Theorem 4.1, by taking $\delta = 4$. For $x_0 = \left(\frac{3}{4}, \frac{3}{4}\right)$ observe the following.

1. $\|x_0 - \bar{x}\| < \frac{\epsilon}{\delta} = \frac{2}{4}$,
2. $\mathbf{F}(x_0) = [\frac{9}{16}, e^{\frac{9}{16}}] \preceq [1, e] = \mathbf{F}(\bar{x})$.
3. $\operatorname{argmin}_{x \in \mathbb{R}^2}[|x_1 x_2|, e^{|x_1 x_2|}] \oplus 4\|x - x_0\| = \{x_0\}$.

Similarly, Theorem 4.1 can be verified for other values of δ.

Next, we give an application of Ekeland's variational principle for IVFs. In order to do that we need the concept of norm of a bounded linear IVF. By a bounded linear IVF (see Ghosh et al. 2020), we mean a linear IVF $\mathbf{G} : \mathcal{X} \to I(\mathbb{R})$ for which there exists a nonnegative real number C such that

$$\|\mathbf{G}(x)\|_{I(\mathbb{R})} \leq C\|x\|_{\mathcal{X}} \text{ for all } x \in \mathcal{X}.$$

In the next lemma, we introduce norm for a bounded linear IVF.

Lemma 4.2 (Norm of a bounded linear IVF). *Let $\mathbf{G} : \mathcal{X} \to I(\mathbb{R})$ be a bounded linear IVF. Then,*

$$\|\mathbf{G}\| = \sup_{\substack{x \in \mathcal{X} \\ \|x\|_{\mathcal{X}}=1}} \|\mathbf{G}(x)\|_{I(\mathbb{R})}$$

is a norm on the set of all bounded linear IVFs on $\mathcal{X}$.

Proof Observe that $\|\mathbf{G}\| \geq 0$ for any bounded linear IVF $\mathbf{G}$ and $\|\mathbf{G}\| = 0$ if and only if $\mathbf{G} = \mathbf{0}$. Let $\gamma \in \mathbb{R}$. We see that

$$\begin{aligned}
&\|\gamma \odot \mathbf{G}\| \\
&= \sup_{\substack{x \in \mathcal{X} \\ \|x\|_{\mathcal{X}}=1}} \|(\gamma \odot \mathbf{G})(x)\|_{I(\mathbb{R})} = \sup_{\substack{x \in \mathcal{X} \\ \|x\|_{\mathcal{X}}=1}} |\gamma| \|\mathbf{G}(x)\|_{I(\mathbb{R})} \\
&= |\gamma| \sup_{\substack{x \in \mathcal{X} \\ \|x\|_{\mathcal{X}}=1}} \|\mathbf{G}(x)\|_{I(\mathbb{R})} = |\gamma| \|\mathbf{G}\|.
\end{aligned}$$

Further,

$$\begin{aligned}
\|\mathbf{G_1} \oplus \mathbf{G_2}\| &= \sup_{\substack{x \in \mathcal{X} \\ \|x\|_{\mathcal{X}}=1}} \|(\mathbf{G_1} \oplus \mathbf{G_2})(x)\|_{I(\mathbb{R})} \\
&= \sup_{\substack{x \in \mathcal{X} \\ \|x\|_{\mathcal{X}}=1}} \|\mathbf{G_1}(x) \oplus \mathbf{G_2}(x)\|_{I(\mathbb{R})} \\
&\leq \sup_{\substack{x \in \mathcal{X} \\ \|x\|_{\mathcal{X}}=1}} (\|\mathbf{G_1}(x)\|_{I(\mathbb{R})} + \|\mathbf{G_2}(x)\|_{I(\mathbb{R})}), \quad \text{by } (i) \text{ of Lemma 2.1} \\
&= \sup_{\substack{x \in \mathcal{X} \\ \|x\|_{\mathcal{X}}=1}} \|\mathbf{G_1}(x)\|_{I(\mathbb{R})} + \sup_{\substack{x \in \mathcal{X} \\ \|x\|_{\mathcal{X}}=1}} \|\mathbf{G_2}(x)\|_{I(\mathbb{R})} \\
&= \|\mathbf{G_1}\| + \|\mathbf{G_2}\|.
\end{aligned}$$

Hence, the result follows. □

Theorem 4.2 *Let $\mathbf{G} : \mathcal{X} \to I(\mathbb{R})$ be a linear IVF. If $\mathbf{G}$ is gH-continuous on $\mathcal{X}$, then $\mathbf{G}$ is a bounded linear IVF.*

Proof By the hypothesis, $\mathbf{G}$ is gH-continuous at the zero vector of $\mathcal{X}$. Therefore, by Lemma 4.2 in Ghosh et al. (2020), $\mathbf{G}$ is a bounded linear IVF. □

As an application of Theorem 4.1, we give a variational principle for gH-Gâteaux differentiable IVFs.

Theorem 4.3 (Variational principle for gH-Gâteaux differentiable IVFs). *Let $\boldsymbol{F} : \mathcal{X} \to I(\mathbb{R}) \cup \{+\infty\}$ be a gH-lsc and gH-Gâteaux differentiable extended IVF, and $\epsilon > 0$. Suppose that*

$$\inf_{\mathcal{X}} \boldsymbol{F} \text{ is finite and } \boldsymbol{F}(\bar{x}) \prec \inf_{\mathcal{X}} \boldsymbol{F} \oplus [\epsilon, \epsilon].$$

Then, for any $\delta > 0$, there exists an $x_0 \in \mathcal{X}$ such that

(i) $\|x_0 - \bar{x}\|_{\mathcal{X}} < \frac{\epsilon}{\delta}$,
(ii) $\boldsymbol{F}(x_0) \preceq \boldsymbol{F}(\bar{x})$, *and*
(iii) $\|\boldsymbol{F}_{\mathscr{G}}(x_0)\| \leq \delta$.

Proof By Theorem 4.1, there exists an $x_0 \in \mathcal{X}$ that satisfies (i) and (ii), and $x_0 \in \operatorname{argmin}_{x \in \mathcal{X}}\{\mathbf{F}(x) \oplus \delta\|x - x_0\|_{\mathcal{X}}\}$. Therefore, $\mathbf{F}(x_0) \preceq \mathbf{F}(x) \oplus \delta\|x - x_0\|_{\mathcal{X}}$ and hence

$$\mathbf{F}(x_0) \ominus_{gH} \delta\|x - x_0\|_{\mathcal{X}} \preceq \mathbf{F}(x). \tag{4}$$

Take any $h \in \mathcal{X}$ and set $x = x_0 + th$ in Eq. (4) with $t > 0$. Then, we get

$$\mathbf{F}(x_0) \ominus_{gH} \delta\|th\|_{\mathcal{X}} \preceq \mathbf{F}(x_0 + th).$$

Thus,

$$-\delta\|h\|_{\mathcal{X}} \preceq \frac{1}{t} \odot \left(\mathbf{F}(x_0 + th) \ominus_{gH} \mathbf{F}(x_0)\right).$$

Letting $t \to 0+$, we get

$$-\delta\|h\|_{\mathcal{X}} \preceq \mathbf{F}_{\mathscr{G}}(x_0)(h).$$

Taking the infimum on both sides over all $h \in \mathcal{X}$ with $\|h\|_{\mathcal{X}} = 1$, we get

$$-\delta \leq -\|\mathbf{F}_{\mathscr{G}}(x_0)\|, \text{ or, } \|\mathbf{F}_{\mathscr{G}}(x_0)\| \leq \delta.$$

This completes the proof. □

The importance of the Theorem 4.3 is that in the absence of points belonging to the set $\operatorname{argmin}_{x \in \mathcal{X}} \mathbf{F}(x)$, we can capture a point x_0 that almost minimizes $\mathbf{F}$. In other words, the equations $\mathbf{F}(x_0) = \inf_{\mathcal{X}} \mathbf{F}$ and $\mathbf{F}_{\mathscr{G}}(x_0) = \mathbf{0}$ can be satisfied to any prescribed accuracy $\delta > 0$.

5 Discussion and conclusion

In this article, the concept of gH-semicontinuity (Definitions 3.3 and 3.4) has been introduced for IVFs. Their interrelation with gH-continuity has been shown (Theorem 3.3). The concept of sequence of intervals is used to give a characterization of lower and upper limits of extended IVFs (Lemmas 3.2 and 3.3). By using a characterization of gH-lower semicontinuity for IVFs (Theorem 3.6), it has been reported that an extended gH-lsc, level-bounded and proper IVF always attains its minimum (Theorem 3.7). A characterization of the set of argument minimum of an IVF has been provided with the help of gH-Gâteaux differentiability (Theorem 3.8). We have further presented Ekeland's variational principle for IVFs (Theorem 4.1). The proposed Ekeland's variational principle has been applied to find variational principle for gH-Gâteaux differentiable IVFs (Theorem 4.3).
In this article, we have considered analyzing closed and bounded intervals and IVFs whose values are closed and bounded intervals. A future study can be performed for other types of intervals. The analysis for other types of intervals is important because if we do not restrict the study for closed and bounded intervals the supremum of a set of closed and bounded intervals may become an open interval. For instance, for $\mathbf{S} = \left\{\left[1 - \frac{1}{n}, 2 - \frac{1}{n}\right] : n \in \mathbb{N}\right\}$, $\sup \mathbf{S} = (1, 2)$.

Immediately in the next step, we shall consider to solve the following two problems as the applications of the proposed study.

Problem 1. The applications of the proposed variational principles in control systems in imprecise or uncertain environment will be shown shortly. Study of a control system in imprecise environment eventually appears due to the incomplete information (e.g., demand for a product) or unpredictable changes (e.g., changes in the climate) in the system. The general control problem in an imprecise or uncertain environment that we shall consider to study is the following:

$$\begin{aligned} \min \quad & \mathbf{G}(x(T)) \\ \text{subject to} \quad & \frac{dx}{dt} = \mathbf{F}(t, x(t), u(t)), \\ & x(0) = x_0 \in C_0,\ x(T) \in C_1, \end{aligned}$$

where C_0 and C_1 are closed subsets of $\mathbb{R}^n$; $x : [0, T] \to \mathbb{R}^n$ and $u : [0, T] \to K$ are state and control variables, respectively, for some metrizable subset K of $\mathbb{R}^n$; $\mathbf{F} : [0, T] \times \mathbb{R}^n \times K \to I(\mathbb{R})$ is a gH-continuous IVF and $\mathbf{G} : \mathbb{R}^n \to I(\mathbb{R})$ is a gH-Fréchet differentiable IVF. To solve this system, the procedure adopted by Clarke in Clarke (1976) may be useful.

Problem 2. We shall attempt to give optimality conditions for the following IOP, where $\mathcal{X}$ and $\mathcal{Y}$ are finite dimensional Banach spaces, C is a nonempty closed subset of $\mathcal{X} \times \mathcal{Y}$, and S is a closed convex subset of $\mathcal{Y}$:

$$\begin{aligned} \min \quad & \mathbf{F}(x, y) \\ \text{subject to} \quad & \mathbf{g}_i(x, y) \preceq \mathbf{0},\ i = 1, 2, \ldots, m, \\ & \mathbf{h}_j(x, y) = \mathbf{0},\ j = 1, 2, \ldots, k, \\ & (x, y) \in C, \\ & y \in S,\ \langle F(x, y), y - z\rangle \le 0 \ \text{ for all } z \in S, \end{aligned}$$

where $\mathbf{F} : \mathcal{X} \times \mathcal{Y} \to I(\mathbb{R})$, $\mathbf{g}_i : \mathcal{X} \times \mathcal{Y} \to I(\mathbb{R}) \cup \{+\infty\}$, $i = 1, 2, \ldots, m$, $\mathbf{h}_j : \mathcal{X} \times \mathcal{Y} \to I(\mathbb{R}) \cup \{+\infty\}$, $j = 1, 2, \ldots, k$, $F : \mathcal{X} \times \mathcal{Y} \to \mathcal{Y}$, and $\langle F(x, y), y - z\rangle$ denotes an inner product of $F(x, y)$ and $y - z$.

Also, with the help of the proposed Ekeland's variational principle, in future, we shall try to investigate the concept of weak sharp minima (Burke and Deng 2002) for IVFs and use it for sensitivity analysis of IOPs.
In parallel to the research proposed on IVFs, the research of fuzzy-valued functions (FVFs) may be another interesting path for future study. We hope that some FVF results would be similarly obtained to this article.

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A proof of Lemma 2.1

Proof of (i) Let $\mathbf{A} = [\underline{a}, \overline{a}]$ and $\mathbf{B} = [\underline{b}, \overline{b}]$. Then,

$$\|\mathbf{A} \oplus \mathbf{B}\|_{I(\mathbb{R})} = \|[\underline{a}, \overline{a}] \oplus [\underline{b}, \overline{b}]\|_{I(\mathbb{R})} = \|[\underline{a} + \underline{b}, \overline{a} + \overline{b}]\|_{I(\mathbb{R})} = \max\{|\underline{a} + \underline{b}|, |\overline{a} + \overline{b}|\}.$$

We now have the following two possible cases.

- **Case** 1. $\|\mathbf{A} \oplus \mathbf{B}\|_{I(\mathbb{R})} = |\underline{a} + \underline{b}|$. Since $|\underline{a} + \underline{b}| \le |\underline{a}| + |\underline{b}| \le \max\{|\underline{a}|, |\overline{a}|\} + \max\{|\underline{b}|, |\overline{b}|\} = \|\mathbf{A}\|_{I(\mathbb{R})} + \|\mathbf{B}\|_{I(\mathbb{R})}$,
  we get $\|\mathbf{A} \oplus \mathbf{B}\|_{I(\mathbb{R})} \le \|\mathbf{A}\|_{I(\mathbb{R})} + \|\mathbf{B}\|_{I(\mathbb{R})}$.

- **Case** 2. $\|\mathbf{A}\oplus\mathbf{B}\|_{I(\mathbb{R})} = |\overline{a}+\overline{b}|$. Since $|\overline{a}+\overline{b}| \leq |\overline{a}|+|\overline{b}| \leq \max\{|\underline{a}|, |\overline{a}|\}+\max\{|\underline{b}|, |\overline{b}|\} = \|\mathbf{A}\|_{I(\mathbb{R})} + \|\mathbf{B}\|_{I(\mathbb{R})}$,
therefore, $\|\mathbf{A} \oplus \mathbf{B}\|_{I(\mathbb{R})} \leq \|\mathbf{A}\|_{I(\mathbb{R})} + \|\mathbf{B}\|_{I(\mathbb{R})}$.

Hence, $\|\mathbf{A} \oplus \mathbf{B}\|_{I(\mathbb{R})} \leq \|\mathbf{A}\|_{I(\mathbb{R})} + \|\mathbf{B}\|_{I(\mathbb{R})}$ for all $\mathbf{A},\ \mathbf{B} \in I(\mathbb{R})$. □

Proof of (ii) Let $\mathbf{A} = [\underline{a}, \overline{a}],\ \mathbf{B} = [\underline{b}, \overline{b}],\ \mathbf{C} = [\underline{c}, \overline{c}]$ and $\mathbf{D} = [\underline{d}, \overline{d}]$. We note that

$$\mathbf{A} \preceq \mathbf{C} \implies [\underline{a}, \overline{a}] \preceq [\underline{c}, \overline{c}] \implies \underline{a} \leq \underline{c} \text{ and } \overline{a} \leq \overline{c}. \tag{5}$$

Also,

$$\mathbf{B} \preceq \mathbf{D} \implies [\underline{b}, \overline{b}] \preceq [\underline{d}, \overline{d}] \implies \underline{b} \leq \underline{d} \text{ and } \overline{b} \leq \overline{d}. \tag{6}$$

From (5) and (6), we have

$$\begin{aligned} &\underline{a} + \underline{b} \leq \underline{c} + \underline{d} \text{ and } \overline{a} + \overline{b} \leq \overline{c} + \overline{d} \\ &\implies [\underline{a} + \underline{b}, \overline{a} + \overline{b}] \preceq [\underline{c} + \underline{d}, \overline{c} + \overline{d}]. \end{aligned}$$

Thus, $\mathbf{A} \oplus \mathbf{B} \preceq \mathbf{C} \oplus \mathbf{D}$. □

Proof of Lemma 2.2

Proof of (i). Let $\mathbf{A} = [\underline{a}, \overline{a}],\ \mathbf{B} = [\underline{b}, \overline{b}]$ and $\epsilon > 0$. $\mathbf{A} \ominus_{gH} \mathbf{B} = [\underline{a}-\underline{b}, \overline{a}-\overline{b}]$ or $[\overline{a}-\overline{b}, \underline{a}-\underline{b}]$. Let us now consider the following four possible cases.

- **Case** 1. $\mathbf{A} \ominus_{gH} \mathbf{B} = [\underline{a} - \underline{b}, \overline{a} - \overline{b}]$ and $\|\mathbf{A} \ominus_{gH} \mathbf{B}\|_{I(\mathbb{R})} = |\underline{a} - \underline{b}|$.
So, we have
$$\underline{a} - \underline{b} \leq \overline{a} - \overline{b} \text{ and } |\overline{a} - \overline{b}| \leq |\underline{a} - \underline{b}|. \tag{7}$$
Let $\|\mathbf{A} \ominus_{gH} \mathbf{B}\|_{I(\mathbb{R})} < \epsilon$. Then,
$$|\underline{a} - \underline{b}| < \epsilon. \tag{8}$$
By eq. (8), we have $-\epsilon < \underline{a} - \underline{b} < \epsilon$, and hence $\underline{b} - \epsilon < \underline{a}$. By eqs. (7) and (8), we have $|\overline{a}-\overline{b}| < \epsilon$. This implies $\overline{b}-\epsilon < \overline{a}$. Therefore, $\mathbf{B}\ominus_{gH}[\epsilon, \epsilon] = [\underline{b}-\epsilon, \overline{b}-\epsilon] \prec [\underline{a}, \overline{a}] = \mathbf{A}$. Note that by eq. (8), $\underline{a} < \underline{b} + \epsilon$. Also, by eqs. (7) and (8), we have $|\overline{a} - \overline{b}| < \epsilon$. This implies $\overline{a} < \overline{b} + \epsilon$. Therefore, $\mathbf{A} = [\underline{a}, \overline{a}] \prec [\underline{b} + \epsilon, \overline{b} + \epsilon] = \mathbf{B} \oplus [\epsilon, \epsilon]$.
- **Case** 2. $\mathbf{A} \ominus_{gH} \mathbf{B} = [\underline{a} - \underline{b}, \overline{a} - \overline{b}]$ and $\|\mathbf{A} \ominus_{gH} \mathbf{B}\|_{I(\mathbb{R})} = |\overline{a} - \overline{b}|$.
So, we have
$$\underline{a} - \underline{b} \leq \overline{a} - \overline{b} \text{ and } |\underline{a} - \underline{b}| \leq |\overline{a} - \overline{b}|. \tag{9}$$
Consider
$$\begin{aligned} &\|\mathbf{A} \ominus_{gH} \mathbf{B}\|_{I(\mathbb{R})} < \epsilon \\ &\implies |\overline{a} - \overline{b}| < \epsilon \end{aligned} \tag{10}$$
By eq. (10), we have
$$\overline{b} - \epsilon < \overline{a}.$$
By eqs. (9) and 10), we have $|\underline{a}-\underline{b}| < \epsilon$. This implies $\underline{b}-\epsilon < \underline{a}$. Therefore, $\mathbf{B}\ominus_{gH}[\epsilon, \epsilon] = [\underline{b} - \epsilon, \overline{b} - \epsilon] \prec [\underline{a}, \overline{a}]$.
Note that by eq. (10), $\overline{a} < \overline{b} + \epsilon$. Also, by eqs. (9) and (10), we have $|\underline{a} - \underline{b}| < \epsilon$. This implies $\underline{a} < \underline{b} + \epsilon$. Therefore, $\mathbf{A} = [\underline{a}, \overline{a}] \prec [\underline{b} + \epsilon, \overline{b} + \epsilon] = \mathbf{B} \oplus [\epsilon, \epsilon]$.
- **Case** 3. $\mathbf{A} \ominus_{gH} \mathbf{B} = [\overline{a} - \overline{b}, \underline{a} - \underline{b}]$ and $\|\mathbf{A} \ominus_{gH} \mathbf{B}\|_{I(\mathbb{R})} = |\underline{a} - \underline{b}|$.
This case can be proved by following the steps similar to **Case** 1.

- **Case** 4. $\mathbf{A} \ominus_{gH} \mathbf{B} = [\overline{a} - \overline{b}, \underline{a} - \underline{b}]$ and $\|\mathbf{A} \ominus_{gH} \mathbf{B}\|_{I(\mathbb{R})} = |\overline{a} - \overline{b}|$.
This case can be proved by following the steps similar to **Case** 2.

Conversely, let $\mathbf{B} \ominus_{gH} [\epsilon, \epsilon] \prec \mathbf{A} \prec \mathbf{B} \oplus [\epsilon, \epsilon]$. Note that

$$\begin{aligned} \mathbf{B} \ominus_{gH} [\epsilon, \epsilon] \prec \mathbf{A} &\implies [\underline{b} - \epsilon, \overline{b} - \epsilon] \prec [\underline{a}, \overline{a}] \\ &\implies \underline{b} - \epsilon < \underline{a} \text{ and } \overline{b} - \epsilon < \overline{a}. \end{aligned} \tag{11}$$

Also,

$$\begin{aligned} \mathbf{A} \prec \mathbf{B} \oplus [\epsilon, \epsilon] &\implies [\underline{a}, \overline{a}] \prec [\underline{b} + \epsilon, \overline{b} + \epsilon] \\ &\implies \underline{a} < \underline{b} + \epsilon \text{ and } \overline{a} < \overline{b} + \epsilon. \end{aligned} \tag{12}$$

From eqs. (11) and (12), we have

$$\begin{aligned} &\underline{b} - \epsilon < \underline{a} < \underline{b} + \epsilon \text{ and } \overline{b} - \epsilon < \overline{a} < \overline{b} + \epsilon \\ &\implies |\underline{a} - \underline{b}| < \epsilon \text{ and } |\overline{a} - \overline{b}| < \epsilon \\ &\implies \max\{|\underline{a} - \underline{b}|, |\overline{a} - \overline{b}|\} < \epsilon \\ &\text{i.e.,} \|\mathbf{A} \ominus_{gH} \mathbf{B}\|_{I(\mathbb{R})} < \epsilon. \end{aligned}$$

This completes the proof of (i). □

Proof of (ii) Let $\mathbf{A} = [\underline{a}, \overline{a}]$, $\mathbf{B} = [\underline{b}, \overline{b}]$ and $\epsilon > 0$.
Consider $\mathbf{A} \ominus_{gH} [\epsilon, \epsilon] \not\prec \mathbf{B}$. This implies $[\underline{a} - \epsilon, \overline{a} - \epsilon] \not\prec [\underline{b}, \overline{b}]$. Thus, '$\underline{b} \leq \underline{a} - \epsilon$ and $\overline{b} \leq \overline{a} - \epsilon$' or '$\underline{b} < \underline{a} - \epsilon$ and $\overline{b} > \overline{a} - \epsilon$' or '$\underline{b} > \underline{a} - \epsilon$ and $\overline{b} < \overline{a} - \epsilon$'. Let us consider all these three possibilities in the following three cases.

- **Case** 1. $\underline{b} \leq \underline{a} - \epsilon$ and $\overline{b} \leq \overline{a} - \epsilon$.
So, we have

$$\begin{aligned} &\underline{a} > \underline{b} \text{ and } \overline{a} > \overline{b}, \text{ because } \epsilon > 0 \\ &\implies \mathbf{B} \prec \mathbf{A} \implies \mathbf{A} \not\prec \mathbf{B}. \end{aligned}$$

- **Case** 2. $\underline{b} < \underline{a} - \epsilon$ and $\overline{b} > \overline{a} - \epsilon$. Since $\underline{b} < \underline{a} - \epsilon$, so $\underline{a} > \underline{b}$, and thus $\mathbf{A} \not\prec \mathbf{B}$.
- **Case** 3. $\underline{b} > \underline{a} - \epsilon$ and $\overline{b} < \overline{a} - \epsilon$. Since $\overline{b} < \overline{a} - \epsilon$, so $\overline{a} > \overline{b}$, and thus $\mathbf{A} \not\prec \mathbf{B}$.

Hence, proof of (ii) is complete. □

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