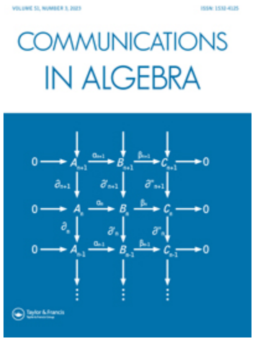

Communications in Algebra

ISSN: (Print) (Online) Journal homepage: www.tandfonline.com/journals/lagb20

On the relative logarithmic connections and relative residue formula

Snehajit Misra & Anoop Singh

COMMUNICATIONS IN ALGEBRA®
2023, VOL. 51, NO. 3, 1217–1228
https://doi.org/10.1080/00927872.2022.2133132

Taylor & Francis
Taylor & Francis Group

On the relative logarithmic connections and relative residue formula

Snehajit Misra[a] and Anoop Singh[b]

[a]Chennai Mathematical Institute, Chennai, India; [b]Department of Mathematical Sciences, Indian Institute of Technology (BHU), Varanasi, India

ABSTRACT
We investigate the relative logarithmic connections on a holomorphic vector bundle over a complex analytic family. We give a sufficient condition for the existence of a relative logarithmic connection on a holomorphic vector bundle singular over a relative simple normal crossing divisor. We define the relative residue of relative logarithmic connection and express relative Chern classes of a holomorphic vector bundle in terms of relative residues.

ARTICLE HISTORY
Received 08 January 2022
Revised 21 July 2022
Communicated by Manuel Reyes

KEYWORDS
Relative Chern class; relative logarithmic connection; relative residue

2020 MATHEMATICS SUBJECT CLASSIFICATION
32C38; 14F10; 53C05; 14H15

1. Introduction

In view of [1, Theorem 4, p. 192], we have that not every holomorphic vector bundle on a compact Kähler manifold admits a holomorphic connection. On the other hand, *Atiyah* [1]–*Weil* [9] *criterion*, says that a holomorphic vector bundle over a *compact Riemann surface* admits a holomorphic connection if and only if the degree of each of its indecomposable component is zero. This criterion over compact Riemann surface has been generalized in the logarithmic set up [4], that is, a necessary and sufficient condition is given for a holomorphic vector bundle on a compact Riemann surface X to admit a logarithmic connection singular along a fixed reduced effective divisor D on X with prescribed rigid residues along D.

More generally, one can ask when does a holomorphic vector bundle over a compact Kähler manifold admit a meromorphic connection?

Simplest case of meromorphic connection is logarithmic connection. So it is natural to ask when a given holomorphic bundle on a admits a logarithmic connection singular along a given divisor with prescribed residues. To the best of our knowledge, no such criterion for the existence of a logarithmic connection on a holomorphic bundle on a compact Kähler manifold with prescribed residues along a given reduced effective divisor is known. Moreover, this seems a difficult problem to answer. In this article, we work in relative set up, that is, we consider a complex analytic family of compact Kähler manifolds and study the relative logarithmic connections over it.

In [5], the relative holomorphic connections on a holomorphic vector bundle over a complex analytic family has been introduced, and a sufficient condition is given for the existence of relative holomorphic connections. Further, there is a well-studied notion of relative logarithmic connection on a holomorphic vector bundle [6]. In this article, we reconsider the relative logarithmic connections over a complex analytic family and explore it further. Our aim is to give a sufficient condition for the existence of it, and establish a formula between relative Chern classes and relative residues.

Let $\pi : X \longrightarrow S$ be a complex analytic family of compact connected complex manifolds of fixed relative dimension l. Let $\dim(X) = m$ and $\dim(S) = n$ so that $m = n + l$. We fix simple normal crossing

CONTACT Snehajit Misra misra08@gmail.com Chennai Mathematical Institute, Chennai 603103, India.

(SNC) divisors Y on X and T on S such that $\pi^{-1}(T) \subset Y$ set-theoratically. We say Y/T a **relative SNC divisor** if the quotient sheaf

$$\Omega^1_{X/S}(\log Y) := \frac{\Omega^1_X(\log Y)}{\pi^*\Omega^1_S(\log T)},$$

is locally free sheaf of rank $l = m - n$ on X, where $\Omega^1_X(\log Y)$ and $\Omega^1_S(\log T)$ are defined in Section 2.1 (for more details see [3]).

In this article we try to answer the following question:

Question 1.1. *Let $\pi : X \longrightarrow S$ be a surjective holomorphic proper submersion with connected fibers, and let $\varpi : E \longrightarrow X$ a holomorphic vector bundle. We fix a relative SNC divisor Y/T over X. Is there a good criterion for existence of a relative logarithmic connection on E singular along Y/T?*

For each fiber $\pi^{-1}(s) = X_s$, $s \in S$, we set $Y_s := X_s \cap Y$.

In order to answer the question, we have studied relative logarithmic connection and relative logarithmic Atiyah bundle in Section 2, and we observe the following:

Proposition 1.2 (Proposition 2.1). *Let $\pi : X \longrightarrow S$ be a surjective holomorphic proper submersion with connected fibers and $\varpi : E \longrightarrow X$ a holomorphic vector bundle. Let Y/T is a relative SNC divisor over X. Let ∇ be a relative logarithmic connection on E. Then we have a family $\{\nabla_s \mid s \in S\}$ which consists of logarithmic and holomorphic connections on the holomorphic family of vector bundles $\{E_s \longrightarrow X_s \mid s \in S\}$ depending on $Y_s \neq \emptyset$ or not. In particular, for every $s \in T$, we have a logarithmic connection ∇_s on the holomorphic vector bundle $E_s \longrightarrow X_s$.*

We also give a sufficient condition for existence of relative logarithmic connection on a holomorphic vector bundle. More specifically we prove the following:

Theorem 1.3 (Theorem 3.2). *Let $\pi : X \longrightarrow S$ be a surjective holomorphic proper submersion of complex manifolds with connected fibers and E be a holomorphic vector bundle on X. Let Y/T be a relative SNC divisor over X. Suppose that the vector bundle $E_s := E|_{X_s}$ admits a logarithmic connection singular along Y_s for each $s \in S$, and*

$$\mathrm{H}^1(S, \pi_*(\Omega^1_{X/S}(\log Y) \otimes \mathcal{E}nd_{\mathcal{O}_X}(E))) = 0.$$

Then, E admits a relative logarithmic connection singular along Y.

In the final section, we introduce the notion of relative residue, and motivated by a result due to Ohtsuki [8, Theorem 3], we prove the following result in the relative context. For the notations in the following theorem see Section 4.

Theorem 1.4 (Theorem 4.3). *Let $\pi : X \longrightarrow S$ be a surjective holomorphic proper submersion of complex manifolds with connected fibers and E be a holomorphic vector bundle on X. Assume that X is compact. Let Y/T be a relative SNC divisor over X. Let D be a relative logarithmic connection on E singular over Y/T. Then, we have following relation in $\mathcal{H}^{2k}_{dR}(X/S)(S)$*

$$C^S_k(E) = (-1)^k\{\sum_{I\in J^k}\sum_{\alpha} \mathrm{Res}_{X/S}(D, Y_{I^*})^{k,\alpha} C^S_p(Y^\alpha_{I^*})\} \prod_{m=1}^{p} C^S_1(Y_{i^*_m})^{a_m-1}, \tag{1.1}$$

where $C^S_k(E)$ denote the k-th relative Chern class of E.

2. Preliminaries

2.1. Logarithmic forms

Let X be a connected smooth complex manifold of dimension at least 1. An effective divisor D on X is said to be a simple normal crossing or SNC in short, if D is reduced, each irreducible component of D is smooth, and for each point $x \in X$, there exists a local system $(U, z_1, z_2, \dots, z_n)$ around $x \in U \subset X$ such that $D \cap U$ is given by the equation $z_1 z_2 \cdots z_r = 0$ for some integer r with $1 \leq r \leq n$. This means that the irreducible components of D passing through x are given by the equations $z_i = 0$ for $i = 1, 2, \dots, r$, and the these components intersect each other transversally.

For an integer $k \geq 0$ and for an SNC divisor D on X, a section of

$$\Omega_X^k(D) := \Omega_X^k \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)$$

is called a meromorphic k-form on X. A meromorphic k-form $\alpha \in \Omega_X^k(U)$ on an open set $U \subset X$ is said to have logarithmic pole along D if it satisfies the following conditions:

(1) α is holomorphic on $U \setminus (U \cap D)$ and α has pole of order at most one along each irreducible component of D.
(2) The condition (2.1) should also holds for $d\alpha$, where d is the holomorphic exterior differential operator.

We denote the sheaf of meromorphic k-forms on X having logarithmic pole along D by $\Omega_X^k(\log D)$, and call it sheaf of **logarithmic** k-**forms** on X singular over D.

2.2. Complex analytic families

Let $(S, \mathcal{O}_S)$ be a complex manifold of dimension n. For each $t \in S$, let there be given a compact connected complex manifold X_t of fixed dimension l. We say that the set $\{X_t : t \in S\}$ of compact connected complex manifolds is called a *complex analytic family of compact connected complex manifolds*, if there is a complex manifold $(X, \mathcal{O}_X)$ and a surjective holomorphic map $\pi : X \to S$ of complex manifolds such that the followings hold;

(1) $\pi^{-1}(t) = X_t$, for all $t \in S$,
(2) $\pi^{-1}(t)$ is a compact connected complex submanifold of X, for all $t \in S$,
(3) the rank of the Jacobian matrix of π is equal to n at each point of X.

Note that, if such a π exists, then $\pi : X \longrightarrow S$ is a surjective holomorphic proper submersion such that each fiber $\pi^{-1}(s) = X_s$ is connected for every $s \in S$.
Let $d\pi_S : TX \longrightarrow \pi^* T_S$ be the differential of π. Then the sheaf of holomorphic sections of the subbundle $T(X/S) := \mathrm{Ker}(d\pi_S) \subset TX$ is called the relative tangent sheaf of π, denoted by $\mathcal{T}_{X/S}$.

We have the following short exact sequence

$$0 \longrightarrow \mathcal{T}_{X/S} \longrightarrow \mathcal{T}_X \xrightarrow{d\pi_S} \pi^* \mathcal{T}_S \longrightarrow 0.$$

of locally free $\mathcal{O}_X$-modules.

The dual $\mathcal{T}_{X/S}^*$ is called the relative cotangent sheaf of π and it is denoted by $\Omega_{X/S}^1$. Dualizing the above short exact sequence we get

$$0 \longrightarrow \pi^* \Omega_S^1 \longrightarrow \Omega_X^1 \longrightarrow \Omega_{X/S}^1 \longrightarrow 0.$$

Note that both the relative tangent sheaf $\mathcal{T}_{X/S}$ and the relative cotangent sheaf $\Omega_{X/S}^1$ are locally free $\mathcal{O}_X$-modules of rank l.

2.3. Relative logarithmic connection and Atiyah Bundle

The notion of relative logarithmic connection was introduced by P. Deligne in [6]. For more details on logarithmic and meromorphic connections we refer [2, 6]. We recall the definition of relative logarithmic connection on a holomorphic vector bundle.

Let E be a holomorphic vector bundle of rank r over X. A **relative logarithmic connection** on E singular along Y is a first order holomorphic differential operator

$$D : E \longrightarrow E \otimes \Omega^1_{X/S}(\log Y)$$

which satifies the Leibniz property

$$D(fs) = fD(s) + \mathrm{d}_{X/S}(f) \otimes s$$

where s and f are local sections of E and $\mathcal{O}_X$, respectively.

In [5, section 2], the notions of S-derivation, S-connection and S-differential operators have been introduced in the relative set up.

For a proper submersion $\pi : X \longrightarrow S$ as above, and for a vector bundle E on X, we recall the following symbol exact sequence from [5, Proposition 4.2],

$$0 \longrightarrow \mathcal{E}nd_{\mathcal{O}_X}(E) \xrightarrow{\iota} \mathcal{D}iff^1_S(E, E) \xrightarrow{\sigma_1} \mathcal{T}_{X/S} \otimes \mathcal{E}nd_{\mathcal{O}_X}(E) \longrightarrow 0\,,$$

where σ_1 is the symbol morphism, and $\mathcal{D}iff^1_S(E, E)$ is the sheaf of first order S-differential operators on E. Define a bundle

$$\mathcal{A}t_S(E) := \sigma_1^{-1}(\mathcal{T}_{X/S} \otimes \mathbf{1}_E),$$

which is known as **relative Atiyah bundle** of E and fits in to the following **Atiyah exact sequence**

$$0 \longrightarrow \mathcal{E}nd_{\mathcal{O}_X}(E) \xrightarrow{\iota} \mathcal{A}t_S(E) \xrightarrow{\sigma_1} \mathcal{T}_{X/S} \longrightarrow 0. \tag{2.1}$$

Further, we define

$$\mathcal{A}t_S(E)(-\log Y) := \sigma_1^{-1}\big(\mathbf{1}_E \otimes \mathcal{T}_{X/S} \otimes \mathcal{O}_X(-\log Y)\big).$$

So, we have the following short exact sequence

$$0 \longrightarrow \mathcal{E}nd_{\mathcal{O}_X}(E) \xrightarrow{\iota} \mathcal{A}t_S(E)(-\log Y) \xrightarrow{\sigma_1} \mathcal{T}_{X/S}(-\log Y) \longrightarrow 0, \tag{2.2}$$

which we call **relative logarithmic Atiyah exact sequence.**

The extension class of the logarithmic Atiyah exact sequence (2.2) of a holomorphic vector bundle E over X is an element of cohomology group

$$\mathrm{H}^1(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{T}_{X/S}(-\log Y), \mathcal{E}nd_{\mathcal{O}_X}(E))).$$

This extension class is called the **relative logarithmic Atiyah class** of E, and it is denoted by $\mathrm{at}_S(E)(\log Y)$. Note that

$$\mathrm{H}^1(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{T}_{X/S}(-\log Y), \mathcal{E}nd_{\mathcal{O}_X}(E))) \cong \mathrm{H}^1(X, \Omega^1_{X/S}(\log Y) \otimes \mathcal{E}nd_{\mathcal{O}_X}(E))\,,$$

therefore, we have

$$\mathrm{at}_S(E)(\log Y) \in \mathrm{H}^1(X, \Omega^1_{X/S}(\log Y) \otimes \mathcal{E}nd_{\mathcal{O}_X}(E)).$$

2.4. Family of logarithmic connections

Now, we describe that given a relative logarithmic connection gives a family of logarithmic connections.

Let $\varpi : E \longrightarrow X$ be a holomorphic vector bundle. For every $s \in S$, the restriction of E to $X_s = \pi^{-1}(s)$ is denoted by E_s. Let U be an open subset of X and $\alpha : U \longrightarrow E$ a holomorphic section. We denote by $r_s(\alpha)$ the restriction of α to $X_s \cap U$, whenever $U \cap X_s \neq \emptyset$. Clearly, $r_s(\alpha)$ is a holomorphic section of E_s over $U \cap X_s$. The map $r_s : \alpha \longmapsto r_s(\alpha)$ induces, therefore, a homomorphism of $\mathbb{C}$-vector spaces from E to E_s, which is denoted by the same symbol r_s. Also, X_s is a complex submanifold of X, so $\mathcal{O}_X|_{X_s} = \mathcal{O}_{X_s}$. We also have the restriction map $r_s : \mathcal{E}nd_{\mathcal{O}_X}(E) \longrightarrow \mathcal{E}nd_{\mathcal{O}_{X_s}}(E_s)$.

Similarly, if $P : E \longrightarrow F$ is a first order S-differential operator, where F is a holomorphic vector bundle over X, then the restriction map $r_s : E_s \longrightarrow F_s$ gives rise to a first order differential operator $P_s : E_s \longrightarrow F_s$ for every $s \in S$. Thus, we have the restriction map $r_s : \mathcal{D}iff^1_S(E, F) \longrightarrow \mathcal{D}iff^1_{\mathbb{C}}(E_s, F_s)$. In particular, for $E = F$, we have the restriction map $r_s : \mathcal{D}iff^1_S(E, E) \longrightarrow \mathcal{D}iff^1_{\mathbb{C}}(E_s, E_s)$ for every $s \in S$. Since, the restriction of the relative tangent bundle $T(X/S)$ to each fiber X_s of π is the tangent bundle $T(X_s)$ of the fiber X_s, we have the restriction map $r_s : \mathcal{T}_{X/S}(-\log Y) \longrightarrow \mathcal{T}_{X_s}(-\log Y_s)$.

Now, for every $s \in S$, the restriction maps gives a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{E}nd_{\mathcal{O}_X}(E) & \longrightarrow & \mathcal{A}t_S(E)(-\log Y) & \xrightarrow{\sigma_1} & \mathcal{T}_{X/S}(-\log Y) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle r_s} & & \downarrow{\scriptstyle r_s} & & \downarrow{\scriptstyle r_s} & & \\ 0 & \longrightarrow & \mathcal{E}nd_{\mathcal{O}_{X_s}}(E_s) & \longrightarrow & \mathcal{A}t(E_s)(-\log Y_s) & \xrightarrow{\sigma_{1s}} & \mathcal{T}_{X_s}(-\log Y_s) & \longrightarrow & 0 \end{array} \tag{2.3}$$

where the bottom sequence is the logarithmic Atiyah sequence of the holomorphic vector bundle E_s over X_s singular along Y_s and σ_{1s} is the restriction of the symbol map σ_1.

Suppose that E admits a relative logarithmic connection, which is equivalent to saying that the relative logarithmic Atiyah sequence in (2.2) splits holomorphically. If

$$\nabla : \mathcal{T}_{X/S}(-\log Y) \longrightarrow \mathcal{A}t_S(E)(-\log Y)$$

is a holomorphic splitting of the relative logarithmic Atiyah sequence in (2.2), then for every $s \in T$, the restriction of ∇ to $\mathcal{T}_{X_s}(-\log Y_s)$ gives an $\mathcal{O}_{X_s}$-module homomorphism

$$\nabla_s : \mathcal{T}_{X_s}(-\log Y_s) \longrightarrow \mathcal{A}t(E_s)(-\log Y_s).$$

Now, ∇_s is a holomorphic splitting of the logarithmic Atiyah sequence of the holomorphic vector bundle E_s, which follows from the fact that the restriction maps r_s defined above are surjective. Note that if $Y_s = \emptyset$, then ∇_s is nothing but the holomorphic connection in E_s.

Thus, we have the following:

Proposition 2.1. *Let $\pi : X \longrightarrow S$ be a surjective holomorphic proper submersion with connected fibers and $\varpi : E \longrightarrow X$ a holomorphic vector bundle. Let Y/T be the relative SNC divisor over X. Let ∇ be a relative logarithmic connection on E. Then we have a family $\{\nabla_s \mid s \in S\}$ which consists of logarithmic and holomorphic connections on the holomorphic family of vector bundles $\{E_s \longrightarrow X_s \mid s \in S\}$ depending on $Y_s \neq \emptyset$ or not. In particular, for every $s \in T$, we have a logarithmic connection ∇_s on the holomorphic vector bundle $E_s \longrightarrow X_s$.*

3. A sufficient condition for existence of logarithmic connections

In this section, we prove the equivalent assertions for a holomorphic vector bundle to admit a relative logarithmic connections. Further, we give a sufficient condition for existence of relative logarithmic connections.

Theorem 3.1. *Let $\pi : X \longrightarrow S$ be a surjective holomorphic proper submersion of complex manifolds with connected fibers and E be a holomorphic vector bundle on X. Let Y/T be the relative SNC divisor over X. Then the followings are equivalent:*

(1) The exact sequence

$$0 \longrightarrow \mathcal{E}nd_{\mathcal{O}_X}(E) \xrightarrow{\iota} \mathcal{A}t_S(E)(-\log Y) \xrightarrow{\sigma_1} \mathcal{T}_{X/S}(-\log Y) \longrightarrow 0,$$

splits holomorphically.

(2) E admits a relative logarithmic connection singular along Y.
(3) The extension class $\mathrm{at}_S(E)(\log Y) \in \mathrm{H}^1(X, \Omega^1_{X/S}(\log Y) \otimes \mathcal{E}nd_{\mathcal{O}_X}(E))$ *is zero.*

Proof. $(i) \iff (ii)$ Suppose the Atiyah exact sequence splits holomorphically, i.e. there exists an $\mathcal{O}_X$-module homomophism

$$\nabla : \mathcal{T}_{X/S}(-\log Y) \longrightarrow \mathcal{A}t_S(E)(-\log Y)$$

such that $\sigma_1 \circ \nabla = \mathbf{1}_{\mathcal{T}_{X/S}(-\log Y)}$. For any open set $U \subset X$, for every $\xi \in \mathcal{T}_{X/S}(-\log Y)(U)$ and for every $a \in \mathcal{O}_X(-Y)(U)$, we then have

$$\sigma_1(\nabla_U(\xi))(a) = [\nabla_U(\xi), a] = \xi(a)\mathbf{1}_E,$$

see [5, Proposition 3.1] for the symbol map σ_1. This in particular implies that

$$\nabla_U(\xi)(as) = a\nabla_U(\xi)(s) + \xi(a)s.$$

This shows that ∇ satisfies the Leibniz condition. Since $\mathcal{A}t_S(E)(-\log Y)$ is an $\mathcal{O}_X$ submodule of $\mathcal{E}nd_{\mathcal{O}_S}(E)(-\log Y)$, we conclude that ∇ indeed defines a relative logarithmic connection singular along Y.

Conversely, any relative logarithmic connection singular along Y satisfies Leibniz property. In particular, this will give a holomorphic splitting of the Atiyah exact sequence.

$(i) \iff (iii)$ The splitting of the exact sequence

$$0 \longrightarrow \mathcal{E}nd_{\mathcal{O}_X}(E) \xrightarrow{\iota} \mathcal{A}t_S(E)(-\log Y) \xrightarrow{\sigma_1} \mathcal{T}_{X/S}(-\log Y) \longrightarrow 0,$$

is given by the vanishing of the extension class

$$\mathrm{at}_S(E)(\log Y) \in \mathrm{Ext}^1(\mathcal{T}_{X/S}(-\log Y), \mathcal{E}nd_{\mathcal{O}_X}(E)).$$

Note that

$$\mathrm{Ext}^1(\mathcal{T}_{X/S}(-\log Y), \mathcal{E}nd_{\mathcal{O}_X}(E)) = \mathrm{H}^1(X, \Omega^1_{X/S}(\log Y) \otimes \mathcal{E}nd_{\mathcal{O}_X}(E)).$$

This proves the theorem. □

Theorem 3.2. *Let* $\pi : X \longrightarrow S$ *be a surjective holomorphic proper submersion of complex manifolds with connected fibers and E be a holomorphic vector bundle on X. Let* Y/T *be the relative SNC divisor over X. Suppose that the vector bundle* $E_s := E|_{X_s}$ *admits a logarithmic connection singular along* Y_s *for each* $s \in S$*, and*

$$\mathrm{H}^1(S, \pi_*(\Omega^1_{X/S}(\log Y) \otimes \mathcal{E}nd_{\mathcal{O}_X}(E))) = 0.$$

Then, E admits a relative logarithmic connection singular along Y.

Proof. Consider the relative logarithmic Atiyah exact sequence in (2.2). Now, tensoring it by $\Omega^1_{X/S}(\log Y)$ gives the following exact sequence

$$0 \longrightarrow \Omega^1_{X/S}(\log Y) \otimes \mathcal{E}nd_{\mathcal{O}_X}(E) \longrightarrow \Omega^1_{X/S}(\log Y) \otimes \mathcal{A}t_S(E)(-\log Y)$$

$$\xrightarrow{q} \Omega^1_{X/S}(\log Y) \otimes \mathcal{T}_{X/S}(-\log Y) \longrightarrow 0\,. \tag{3.1}$$

We have $\mathcal{O}_X \cdot \mathrm{Id} \subset \mathrm{End}(\mathcal{T}_{X/S}(-\log Y)) = \Omega^1_{X/S}(\log Y) \otimes \mathcal{T}_{X/S}(-\log Y)$. Define

$$\Omega^1_{X/S}(\log Y)\big(\mathcal{A}t'_S(E)\big) := q^{-1}(\mathcal{O}_X \cdot \mathrm{Id}) \subset \Omega^1_{X/S}(\log Y) \otimes \mathcal{A}t_S(E)(-\log Y),$$

where q is the projection in (3.1). So we have the short exact sequence of sheaves

$$0 \longrightarrow \Omega^1_{X/S}(\log Y) \otimes \mathcal{E}nd_{\mathcal{O}_X}(E) \longrightarrow \Omega^1_{X/S}(\log Y)\big(\mathcal{A}t'_S(E)\big) \xrightarrow{q} \mathcal{O}_X \longrightarrow 0 \tag{3.2}$$

on X, where $\Omega^1_{X/S}(\log Y)(\mathcal{A}t'_S(E))$ is constructed above. Let

$$\Phi : \mathbb{C} \subset \mathrm{H}^0(X, \mathcal{O}_X \cdot \mathrm{Id}) \longrightarrow \mathrm{H}^1\big(X, \Omega^1_{X/S}(\log Y) \otimes \mathcal{E}nd_{\mathcal{O}_X}(E)\big) \tag{3.3}$$

be the homomorphism in the long exact sequence of cohomologies associated to the exact sequence in (3.2). The relative Atiyah class $\mathrm{at}_S(E)(\log Y)$ (see Theorem 3.1) coincides with $\Phi(1) \in \mathrm{H}^1\big(X, \Omega^1_{X/S}(\log Y) \otimes \mathcal{E}nd_{\mathcal{O}_X}(E)\big)$. Therefore, from Theorem 3.1, it follows that E admits a relative logarithmic connection if and only if

$$\Phi(1) = 0. \tag{3.4}$$

Note that $\mathrm{H}^1\big(X, \Omega^1_{X/S}(\log Y) \otimes \mathcal{E}nd_{\mathcal{O}_X}(E)\big)$ fits in the following exact sequence

$$\mathrm{H}^1\big(S, \pi_*\big(\Omega^1_{X/S}(\log Y) \otimes \mathcal{E}nd_{\mathcal{O}_X}(E)\big)\big) \xrightarrow{\beta_1} \mathrm{H}^1\big(X, \Omega^1_{X/S}(\log Y) \otimes \mathcal{E}nd_{\mathcal{O}_X}(E)\big)$$

$$\xrightarrow{q_1} \mathrm{H}^0\big(S, R^1\pi_*\big(\Omega^1_{X/S}(\log Y) \otimes \mathcal{E}nd_{\mathcal{O}_X}(E)\big)\big), \tag{3.5}$$

where π is the projection of X to S.

The given condition that for every $s \in S$, there is a logarithmic connection on the holomorphic vector bundle $\varpi|_{E_s} : E_s \longrightarrow X_s$, implies that

$$q_1(\Phi(1)) = 0,$$

where q_1 is the homomorphism in (3.5). Therefore, from the exact sequence in (3.5) we conclude that

$$\Phi(1) \in \beta_1\big(\mathrm{H}^1\big(S, \pi_*\big(\Omega^1_{X/S}(\log Y) \otimes \mathcal{E}nd_{\mathcal{O}_X}(E)\big)\big)\big).$$

Finally, the given condition that $\mathrm{H}^1\big(S, \pi_*\big(\Omega^1_{X/S}(\log Y) \otimes \mathcal{E}nd_{\mathcal{O}_X}(E)\big)\big) = 0$ implies that $\Phi(1) = 0$. Since (3.4) holds, the vector bundle E admits a relative logarithmic connection. □

4. Relative Chern classes in terms of relative residue

In this section, we express the relative Chern classes in terms of relative residues which generalizes [8, Theorem 3] due to Ohtsuki in the relative context.

4.1. Relative residue:

We define the relative residues of a relative logarithmic connection D on E. Let

$$Y = \bigcup_{j \in J} Y_j$$

be the decomposition of Y into it's irreducible components, and

$$\tau_j : Y_j \longrightarrow X$$

the inclusion map for every $j \in J$. Since Y is a normal crossing divisor on X, we can choose a fine open cover $\{U_\lambda : \lambda \in \Lambda\}$ of X such that for each $\lambda \in \Lambda$, we have the following:

(1) each $E|_{U_\lambda}$ is trivial,
(2) for each irreducible component Y_j of Y with $Y_j \cap U_\lambda \neq \emptyset$, we can choose a local coordinate function $f_{\lambda j} \in \mathcal{O}_X(U_\lambda)$ for a local coordinate system on U_λ, such that $f_{\lambda j}$ is a defining equation of $Y_j \cap U_\lambda$. If $Y_j \cap U_\lambda = \emptyset$, then we take $f_{\lambda j} = 1$.

Let $e_\lambda = (e_{1\lambda}, \ldots, e_{r\lambda})$ be the local frame of E, and ω_λ the relative connection matrix of D with respect to a holomorphic local frame e_λ for E on U_λ, that is, we have

$$D(e_\lambda) = \omega_\lambda \otimes e_\lambda,$$

where ω_λ is the $r \times r$ matrix whose entries are holomorphic sections of $\Omega^1_{X/S}(\log Y)$ over U_λ. For each Y_j, the matrix ω_λ can be written as

$$\omega_\lambda = R_{\lambda j}\frac{df_{\lambda j}}{f_{\lambda j}} + S_{\lambda j},$$

where $R_{\lambda j}$ is an $r \times r$ matrix with entries in $\mathcal{O}_X(U_\lambda)$ and $S_{\lambda j}$ is a $r \times r$ matrix with entries in $\big(\Omega^1_{X/S}(\log Y)\big)(U_\lambda)$ with simple pole along $\bigcup_{i\neq j} Y_i$.

Then

$$\mathrm{Res}_{X/S}(\omega_\lambda, Y_j) := R_{\lambda j}|_{U_\lambda \cap Y_j}$$

is an $r \times r$ matrix whose entries are holomorphic functions on $U_\lambda \cap Y_j$ and it is independent of choice of local defning equation $f_{\lambda j}$ for Y_j. Then $\{\mathrm{Res}_{X/S}(\omega_\lambda, Y_j)\}_{\lambda\in\Lambda}$ defnes a holomorphic global section

$$\mathrm{Res}_{X/S}(D, Y_j) \in \mathrm{H}^0(Y_j, \mathcal{E}nd_{\mathcal{O}_X}(E)|_{Y_j}) \tag{4.1}$$

called the **relative residue** of the relative connection D along Y_j.

For every $s \in T$, we have the decomposition

$$Y_s = Y \cap \pi^{-1}(s) = \bigcup_{j\in J}(Y_j \cap \pi^{-1}(s))$$

of Y_s into its irreducible components. We denote $Y_j \cap \pi^{-1}(s)$ by Y_{js}.

Recall that for a given relative logarithmic connection D on E singular over Y/T, we get a logarithmic connections D_s on $E_s := E|_{X_s}$ singular over Y_s for every $s \in T$. Then we have residue (see [8]) of D_s over each irreducible component Y_{js} of Y_s denoted as

$$\mathrm{Res}_{X_s}(D_s, Y_{js}) \in \mathrm{H}^0(Y_{js}, \mathcal{E}nd_{\mathcal{O}_{X_s}}(E_s)|_{Y_{js}}), \tag{4.2}$$

for every $s \in T$.

4.2. Relative Chern class

We recall the definition of the relative Chern classes of a holomorphic vector bundle over $\pi : X \to S$, for more details see [5, Section 4.6].

Let E be a hermitian holomorphic vector bundle on X, that is, E is a holomorphic vector bundle with Hermitian metric on it. Then there exists canonical (smooth) connection ∇ on E compatible with the Hermitian metric.

Let $\mathcal{A}^r_{X/S}$ denote the sheaf of *complex valued smooth relative r-form* on X over S. Then we have relative de Rham complex

$$0 \longrightarrow \pi^{-1}\mathcal{C}^\infty_S \longrightarrow \mathcal{C}^\infty_X \xrightarrow{d_{X/S}} \mathcal{A}^1_{X/S} \xrightarrow{d_{X/S}} \cdots \xrightarrow{d_{X/S}} \mathcal{A}^{2l}_{X/S} \longrightarrow 0$$

of $\mathcal{C}^\infty_X$-module and S-linear maps, which we denote by the pair $(\mathcal{A}^\bullet_{X/S}, d_{X/S})$.

Moreover, because of the following short exact sequence

$$0 \longrightarrow \pi^*\mathcal{A}^1_S \longrightarrow \mathcal{A}^1_X \longrightarrow \mathcal{A}^1_{X/S} \longrightarrow 0,$$

we get a relative smooth connection D on E induced from ∇.

Given a relative smooth connection D on E. Let (U_α, h_α) be a trivialization of E over $U_\alpha \subset X$. Let R be the S-curvature (relative curvature) form for D, and let $\Omega_\alpha = (\Omega_{ij\alpha})$ be the curvature matrix of D over U_α, so $\Omega_{ij\alpha} \in \mathcal{A}^2_{X/S}(U_\alpha)$. We have $\Omega_\beta = g^{-1}_{\alpha\beta}\Omega_\alpha g_{\alpha\beta}$, where $g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{GL}_r(\mathbb{C})$ is the change of frame matrix (transition function), which is a smooth map.

Consider the adjoint action of $\mathrm{GL}_r(\mathbb{C})$ on it Lie algebra $\mathfrak{gl}_r(\mathbb{C}) = \mathrm{M}_r(\mathbb{C})$. Let f be a $\mathrm{GL}_r(\mathbb{C})$-invariant homogeneous polynomial on $\mathfrak{gl}_r(\mathbb{C})$ of degree p. Then, we can associate a unique p-multilinear symmetric map $\widetilde{f}$ on $\mathfrak{gl}_r(\mathbb{C})$ such that $f(X) = \widetilde{f}(X, \dots, X)$, for all $X \in \mathfrak{gl}_r(\mathbb{C})$. Define

$$\gamma_\alpha = \widetilde{f}(\Omega_\alpha, \dots, \Omega_\alpha) = f(\Omega_\alpha) \in \mathcal{A}^{2p}_{X/S}(U_\alpha).$$

Since f is $\mathrm{GL}_r(\mathbb{C})$-invariant, it follows that γ_α is independent of the choice of frame, and hence it defines a global smooth relative differential form of degree $2p$, which we denote by the symbol $\gamma \in \mathcal{A}^{2p}_{X/S}(X)$.

Theorem 4.1. [5, Theorem 4.9] *Let $\pi : X \longrightarrow S$ be a surjective holomorphic proper submersion of complex manifolds with connected fibers and $\varpi : E \longrightarrow X$ a differentiable family of complex vector bundle. Let D be a relative smooth connection on E. Suppose that f is a $\mathrm{GL}_r(\mathbb{C})$-invariant polynomial function on $\mathfrak{gl}_r(\mathbb{C})$ of degree p. Then the following hold:*

(1) $\gamma = f(\Omega_\alpha)$ is $d_{X/S}$-closed, that is, $d_{X/S}(\gamma) = 0$.
(2) The image $[\gamma]$ of γ in the relative de Rham cohomology group

$$\mathrm{H}^{2p}(\Gamma(X, \mathcal{A}^\bullet_{X/S})) = \mathrm{H}^{2p}(X, \pi^{-1}\mathcal{C}^\infty_S)$$

is independent of the relative smooth connection D on E.

Define homogeneous polynomials f_p on $\mathfrak{gl}_r(\mathbb{C})$, of degree $p = 1, 2, \ldots, r$, to be the coefficient of λ^p in the following expression:

$$\det(\lambda \mathrm{I} + \frac{\sqrt{-1}}{2\pi}A) = \Sigma_{j=0}^{r}\lambda^{r-j}f_j(\frac{\sqrt{-1}}{2\pi}A), \tag{4.3}$$

where $f_0(\frac{\sqrt{-1}}{2\pi}A) = 1$ while $f_r(\frac{\sqrt{-1}}{2\pi}A)$ is the coefficient of λ^0. These polynomials $f_1, \ldots, f_r$ are $GL_r(\mathbb{C})$-invariant, and they generate the algebra of $\mathrm{GL}_r(\mathbb{C})$-invariant polynomials on $\mathfrak{gl}_r(\mathbb{C})$. We now define the *p-th cohomology class* as follows:

$$c_p^S(E) = [f_p(\frac{\sqrt{-1}}{2\pi}\Omega)] \in \mathrm{H}^{2p}(\Gamma(X, \mathcal{A}^\bullet_{X/S}))$$

for $p = 0, 1, \ldots, r$.

The relative de Rham cohomology sheaf $\mathcal{H}^p_{dR}(X/S) \cong R^p\pi_*(\pi^{-1}\mathcal{C}^\infty_S)$ on S is by definition the sheafification of the presheaf $V \longmapsto \mathrm{H}^p(\pi^{-1}(V), \pi^{-1}\mathcal{C}^\infty_S|_{\pi^{-1}(V)})$, for open subset $V \subset S$. Therefore, we have a natural homomorphism

$$\rho : \mathrm{H}^{2p}(X, \pi^{-1}\mathcal{C}^\infty_S) \longrightarrow \mathcal{H}^{2p}_{dR}(X/S)(S) \tag{4.4}$$

which maps $c_p^S(E)$ to $\rho(c_p^S(E)) \in \mathcal{H}^{2p}_{dR}(X/S)(S)$.

Define $C_p^S(E) = \rho(c_p^S(E))$. We call $C_p^S(E)$ the *p-th relative Chern class of E over S*. Let

$$C^S(E) = \sum_{p\geq 0} C_p^S(E) \in \mathcal{H}^*_{dR}(X/S)(S) = \oplus_{k\geq 0}\mathcal{H}^k_{dR}(X/S)(S)$$

be the *total relative Chern class* of E.

4.3. Relative Chern classes in terms of relative residue

We follow the notations as above. Let $J^k := J \times \cdots \times J$ be the k-fold product of J. Let $I = (i_1, \ldots, i_k) \in J^k$. If there are p-different indices among $i_1, \ldots, i_k$, we denote them by $i_1^*, \ldots, i_p^*$, tuple is denoted by $I^* = (i_1^*, \ldots, i_p^*)$. Let a_m be the number of i_m^* appearing in I, then we have

$$\sum_{m=1}^{p} a_m = k.$$

For given $I \in J^k$, we define

$$Y_{I^*} = \bigcap_{m=1}^{p} Y_{i_m^*}. \tag{4.5}$$

Then either $Y_{I^*} = \emptyset$ or a submanifold of X of codimension p. Further, Y_{I^*} need not be connected. Let

$$Y_{I^*} = \bigcup_{\alpha} Y_{I^*}^{\alpha} \tag{4.6}$$

be the disjoint union of connected components of Y_{I^*}. Then each $Y_{I^*}^{\alpha}$ is a submanifold of codimension p.

Let $\widetilde{f_k}$ be the unique k-multilinear symmetric map on $\mathfrak{gl}_r(\mathbb{C})$ such that

$$f_k(A) = \widetilde{f_k}(A, \cdots, A),$$

for all $A \in \mathfrak{gl}_r(\mathbb{C})$, where f_k is defined in (4.3) for every $k = 0, \ldots, r$. Now onwards we assume that X is compact.

Lemma 4.2. *Let $\pi : X \longrightarrow S$ be a surjective holomorphic proper submersion of complex manifolds with connected fibers and E be a holomorphic vector bundle on X. Assume that X is compact. Let Y/T be the relative SNC divisor over X. Let D be a relative logarithmic connection on E singular over Y/T. Then for any $I = (i_1, \ldots, i_k) \in J^k$, the following polynomial*

$$\widetilde{f_k}(\mathrm{Res}_{X/S}(D, Y_{i_1}), \mathrm{Res}_{X/S}(D, Yi_2), \ldots, \mathrm{Res}_{X/S}(D, Y_{i_k}))$$

is constant on each connected component $Y_{I^*}^{\alpha}$ of Y_{I^*} described in (4.6).

Proof. Since X is a compact complex manifold, each connected component $Y_{I^*}^{\alpha}$ is a compact complex submanifold of X. Hence proof follows from the fact that $\widetilde{f_k}$ is a polynomial function. □

For the simplicity of the notation, we denote the constant number

$$\widetilde{f_k}(\mathrm{Res}_{X/S}(D, Y_{i_1}), \mathrm{Res}_{X/S}(D, Y_{i_2}), \ldots, \mathrm{Res}_{X/S}(D, Y_{i_k}))$$

on each $Y_{I^*}^{\alpha}$ in the above Lemma 4.2 by $\mathrm{Res}_{X/S}(D, Y_{I^*})^{k,\alpha}$.

Let W be a submanifold of X of codimension p. Then, we get a cohomology class $[W] \in \mathrm{H}^{2p}(X, \mathbb{C})$. Because of the following inclusion of sheaves

$$\mathbb{C} \hookrightarrow \pi^{-1}\mathcal{C}_S^{\infty},$$

we get a homomorphism

$$\gamma : \mathrm{H}^{2p}(X, \mathbb{C}) \longrightarrow \mathrm{H}^{2p}(X, \pi^{-1}\mathcal{C}_S^{\infty}) \tag{4.7}$$

on cohomology groups. Further using the natural homomorphism

$$\rho \ : \ \mathrm{H}^{2p}(X, \pi^{-1}\mathcal{C}_S^{\infty}) \ \longrightarrow \ \mathcal{H}_{dR}^{2p}(X/S)(S)$$

in (4.4), we define

$$C_p^S(W) := \rho(\gamma([W])) \tag{4.8}$$

call it the p-th relative Chern classes associated to W.

Theorem 4.3. *Let $\pi : X \longrightarrow S$ be a surjective holomorphic proper submersion of complex manifolds with connected fibers and E be a holomorphic vector bundle on X. Assume that X is compact. Let Y/T be the relative SNC divisor over X. Let D be a relative logarithmic connection on E singular over Y/T. Then, we have following relation in $\mathcal{H}_{dR}^{2k}(X/S)(S)$*

$$C_k^S(E) = (-1)^k \{\sum_{I \in J^k} \sum_{\alpha} \mathrm{Res}_{X/S}(D, Y_{I^*})^{k,\alpha} C_p^S(Y_{I^*}^{\alpha})\} \prod_{m=1}^{p} C_1^S(Y_{i_m^*})^{a_m - 1}, \tag{4.9}$$

where $C_k^S(E)$ denote the k-th relative Chern class of E.

Proof. It is enough to show the formula (4.9) stalkwise, and in particular stalks at $s \in T$. First note that for every $s \in S$, and inclusion morphism $j : X_s \hookrightarrow X$, we have a natural map (see [5, Corollary 4.11])

$$j^* : \mathcal{H}^{2k}_{dR}(X/S)(S) \longrightarrow \mathrm{H}^{2k}(X_s, \mathbb{C})$$

which maps the k-th relative Chern class of E to the k-th Chern class of the vector bundle $E_s \longrightarrow X_s$, that is, $j^*(C^S_k(E)) = c_k(E_s)$, where $c_k(E_s)$ denote the k-th Chern class of E_s.

Note that $\mathcal{H}^{2k}_{dR}(X/S)$ is a locally free $\mathcal{C}^\infty_S$-module, and using the topological proper base change theorem given in [7, p. 202, Remark 4.17.1] and [6, p. 19, Corollary 2.25], we have a $\mathbb{C}$-vector space isomorphism

$$\eta : \mathcal{H}^{2k}_{dR}(X/S)_s \otimes_{\mathcal{C}^\infty_{S,s}} k(s) \longrightarrow \mathrm{H}^{2k}(X_s, \mathbb{C}) \tag{4.10}$$

for every $s \in S$. In fact, we have the following commutative diagram;

$$\begin{array}{ccc} \mathcal{H}^{2k}_{dR}(X/S)(S) & \longrightarrow & \mathcal{H}^{2k}_{dR}(X/S)_s \otimes_{\mathcal{C}^\infty_{S,s}} k(s) \\ & \searrow^{j^*} & \downarrow \eta \\ & & \mathrm{H}^{2k}(X_s, \mathbb{C}) \end{array}$$

Hence, we get

$$\eta(C^S_k(E)_s \otimes 1) = j^*(C^S_k(E)) = c_k(E_s). \tag{4.11}$$

Let us fix the following notation for $s \in T$;

$$Y_{sI^*} = Y_{I^*} \cap \pi^{-1}(s) = \bigcup_\alpha (Y^\alpha_{I^*} \cap \pi^{-1}(s)),$$

$$Y^\alpha_{sI^*} = Y^\alpha_{I^*} \cap \pi^{-1}(s),$$

and

$$Y_{si^*_m} = Y_{i^*_m} \cap \pi^{-1}(s).$$

Now, note that the germ at $s \in T$ of the right hand side of the formula (4.9) is associated to the logarithmic connection D_s on $E_s \longrightarrow X_s$, that is, we get the following expression

$$(-1)^k \{\sum_{I \in J^k} \sum_\alpha \mathrm{Res}_{X_s}(D_s, Y_{sI^*})^{k,\alpha} c_p(Y^\alpha_{sI^*})\} \prod_{m=1}^{p} c_1(Y_{si^*_m})^{a_m - 1}, \tag{4.12}$$

where $\mathrm{Res}_{X_s}(D_s, Y_{sI^*})^{k,\alpha}$ denote the constant function

$$\widetilde{f}_k(\mathrm{Res}_{X_s}(D_s, Y_{si_1}), \mathrm{Res}_{X_s}(D_s, Y_{si_2}), \ldots, \mathrm{Res}_{X_s}(D_s, Y_{si_k}))$$

on each connected component $Y^\alpha_{sI^*}$ of Y_{sI^*}.

From [8, Theorem 3], we have

$$c_k(E_s) = (-1)^k \{\sum_{I \in J^k} \sum_\alpha \mathrm{Res}_{X_s}(D_s, Y_{sI^*})^{k,\alpha} c_p(Y^\alpha_{sI^*})\} \prod_{m=1}^{p} c_1(Y_{si^*_m})^{a_m - 1}. \tag{4.13}$$

In view of (4.11)–(4.13), proof of the theorem is complete. □

Acknowledgments

The authors would like to thank Prof. Mainak Poddar for a careful reading of the manuscript and pointing out a mistake in an earlier version of the paper. The authors thank Indian Institute of Technology, Bombay (IITB) and Tata Institute of Fundamental Research, Mumbai for the hospitality where most of the work has been done.

Funding

The first author is supported financially by SERB-NPDF fellowship (File no: PDF/2021/00028).

References

[1] Atiyah, M. F. (1957). Complex analytic connections in fibre bundles. *Trans. Amer. Math. Soc.* 85:181–207.
[2] Borel, A., Grivel, P.-P., Kaup, B., Haefliger, A., Malgrange, B., Ehlers, F. (1987). *Algebraic D-modules*. Perspectives in Mathematics, 2. Boston, MA: Academic Press, Inc., pp. xii+355.
[3] Bloch, S., Esnault, H. (1999). *Lectures on Algebro-Geometric Chern Weil and Cheeger-Chern-Simsons Theory for Vector Bundles*. The Arithmetic and Geometry of Algebraic Cycles. NATO Science Series, Vol 548. Dordrecht: Springer.
[4] Biswas, I., Dan, A., Paul, A. (2018). Criterion for logarithmic connections with prescribed residues. *Manuscripta Math.* 155(1–2):77–88.
[5] Biswas, I., Singh, A. (2020). On the relative connections. *Commun. Algebra* 48(4):1452–1475.
[6] Deligne, P. (1970). *Equations différentielles á points singuliers réguliers*. Lecture Notes in Mathematics, Vol. 163. Berlin: Springer.
[7] Godement, R. (1964). *Théorie des faisceaux*. Paris: Hermann.
[8] Ohtsuki, M. (1982). A residue formula for Chern classes associated with logarithmic connections. *Tokyo J. Math.* 5(1):13–21.
[9] Weil, A. (1938). Généralisation des fonctions abéliennes. *J. Math. Pures Appl.* 17:47–87.