Constructive Approximation
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CONSTRUCTIVE APPROXIMATION

Set-Valued α-Fractal Functions

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Abstract

In this paper, we introduce the concept of the α-fractal function and fractal approximation for a set-valued continuous map defined on a closed and bounded interval of real numbers. Also, we study some properties of such fractal functions. Further, we estimate the perturbation error between the given continuous function and its α-fractal function. Additionally, we define a new graph of a set-valued function different from the standard graph introduced in the literature and establish some bounds on the fractal dimension of the newly defined graph of some special classes of set-valued functions. Also, we explain the need to define this new graph with examples. In the sequel, we prove that this new graph of an α-fractal function is an attractor of an iterated function system.

Keywords Set-valued function · Fractal function · Hausdorff metric · Hölder space · Bounded variation · Fractal dimension

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1 Introduction

Approximation theory has gained appreciable attention in the literature. Fractal theory embraced the approximation theory in 1986 by Barnsley [6] through his paper "Fractal functions and interpolations". Following this pioneering work of Barnsley, Navascués [22, 24] studied a parameterized class of fractal interpolation function, known as α-fractal function, associated with the continuous function defined on a real compact interval. After that, several theories have been developed concerning fractal interpolation functions. For example, in [17, 36] concept of α-fractal function has been studied, and in [7, 11, 20, 25], a generalized C^r fractal interpolation function has been studied.

In this paper, we have extended the notion of α-fractal function in the case of set-valued maps. The significance of set-valued maps can be found in many essential areas, such as optimization theory, game theory, control theory, etc. One may refer [3] to understand the properties and nature of set-valued maps. The algebra of sets is different from this of numbers. There are different types of sums have been given for sets. For instance, in [2], the binary metric average of sets is defined. In [13], the Minkowski sum of two sets is used, and in [9], the notion of the sum specified in [2] has been extended, which is known as a metric linear combination of sets. In this paper, we have taken the Minkowski sum of sets.

Approximation of set-valued maps is one of the celebrated topics in the literature. Several theories have been given regarding the classical approximation of set-valued maps. See, for instance, [18] where the notion of univariate data interpolation function has been given. Initially, approximation theory mainly focused on set-valued maps with convex images, known as convex set-valued maps. For example, in [37], Vitale explored the approximation of convex set-valued maps with set-valued Bernstein polynomials. One may refer [4, 10, 13] for more research on the approximation of convex set-valued maps. The Minkowski sum of the two sets has been taken in all those researches. In [2], Arstein studied the approximation of set-valued maps having compact images, known as compact set-valued maps. Instead of Minkowski's sum of sets, he used the set of a sum of special pairs of elements, later known as "metric pairs". One may refer [9, 14] for more research on the approximation of compact set-valued maps. In this paper, we have studied the fractal approximation of set-valued maps.

Like fractal approximation, estimating the fractal dimension is also a fascinating area in fractal theory. It provides the statistical ratio of complexity with the details of how a fractal pattern changes with the scale it is measured. Several notions of fractal dimension have been introduced so far in the literature. For example, Hausdorff dimension, box dimension, packing dimension [7, 15, 20], etc. In this paper, we have worked on Hausdorff and box dimensions.

1.1 Motivation and Work Done

The concept of α-fractal function, fractal approximation, and fractal dimension have been studied for different types of single-valued maps. For instance, in [17, 24] α-fractal function for univariate single-valued maps and fractal dimension of the graph of some classes of univariate single-valued maps have been discussed. In [36], α-fractal function and dimensional results for bivariate single-valued maps have been explored. In [27, 28], the existence of α-fractal function corresponding to continuous multivariate functions is given. Agrawal et al.,[1] worked on the L_p approximation using fractal functions on the Sierpiński gasket. Sahu and Priyadarshi [30] have studied the box dimension of the graph of a harmonic function on the Sierpiński gasket. Persuaded by these researches, we have extended the classical approximation of set-valued maps to the fractal approximation of set-valued maps. We have introduced the notion of α-fractal function for set-valued maps. Still, unlike in the case of single-valued maps, we noticed that, in general, the set-valued α-fractal function is not interpolatory in nature. Further, we have worked on estimating the fractal dimension of the graph of some special classes of set-valued maps.

1.2 Delineation

The proposed paper is assembled as follows. The next section is reserved for notations and preliminaries required for our study. Section 3 is devoted to the development of fractal functions and to explore their properties. In Sect. 4, we have studied the fractal approximations and constrained approximations of the set-valued map. Further, in Sect. 5, we have defined a new definition of the graph of set-valued maps and provided some dimensional results for this new graph. Also, we have explained the need to define this new graph. Moreover, we proved that there exists an iterated function system whose attractor is this new graph of the set-valued α-fractal function. We have concluded our paper in Sect. 6.

2 Notations and Preliminaries

The following are the notations that we have used in our paper:

- $\mathbb{N}$: Collection of all natural numbers
- $\mathbb{R}$: Collection of all real numbers
- I: Closed and bounded interval of $\mathbb{R}$
- $\mathcal{K}(\mathbb{R}) = \{A \subset \mathbb{R} : A \text{ is a compact subset of } \mathbb{R}\}$
- $\mathcal{K}_c(\mathbb{R}) = \{A \in \mathcal{K}(\mathbb{R}) : A \text{ is a convex subset of } \mathbb{R}\}$
- $H_d(A, B) = \max\left\{\sup_{a\in A}\inf_{b\in B}|a-b|, \sup_{b\in B}\inf_{a\in A}|b-a|\right\}$ be the metric defined on $\mathcal{K}(\mathbb{R})$. It is broadly known as the Hausdorff metric
- $\mathcal{C}(I, \mathcal{K}(\mathbb{R}))$: Collection of all the continuous maps from I to $\mathcal{K}(\mathbb{R})$
- $\mathcal{L}ip(I, \mathcal{K}(\mathbb{R}))$: Collection of all the Lipschitz maps from I to $\mathcal{K}(\mathbb{R})$
- σ-$\mathcal{HC}$: Collection of all the Hölder continuous maps from I to $\mathcal{K}(\mathbb{R})$ with exponent σ.

Definition 1 [15] Let $V \subseteq \mathbb{R}$ be a subset of $\mathbb{R}$, then the diameter of V is defined as

$$|V| = \sup\{|u-w| : u, w \in V\}.$$

Let $E \subset \mathbb{R}$ be a subset of $\mathbb{R}$. A countable collection (or finite) of sets, V_i is said to be η-cover of E if it satisfies

$$E \subseteq \bigcup_i V_i \text{ such that } |V_i| \leq \eta \text{ for all } i.$$

Definition 2 [15] For a non-negative number t and $\eta > 0$, define

$$H_\eta^t(E) = \inf\left\{\sum_{i=1}^{\infty} |V_i|^t : \{V_i\} \text{ is a } \eta\text{-cover of } E\right\}.$$

Then, t-dimensional Hausdorff measure of E is defined as

$$H^t(E) = \lim_{\eta \to 0} H_\eta^t(E).$$

Definition 3 [15] Consider $E \subseteq \mathbb{R}$ and $t \geq 0$. The Hausdorff dimension of E is defined as,

$$\dim_H(E) = \sup\{t : H^t(E) = \infty\} = \inf\{t : H^t(E) = 0\}.$$

Definition 4 [15] Assume E be a non-empty and bounded subset of $\mathbb{R}$ and $N_\eta(E)$ be the lowest count of sets having at most η diameter, which can cover E. The upper box dimension and lower box dimension of E are defined as

$$\overline{\dim}_B(E) = \overline{\lim_{\eta \to 0}} \frac{\log N_\eta(E)}{-\log \eta} \text{ and } \underline{\dim}_B(E) = \underline{\lim_{\eta \to 0}} \frac{\log N_\eta(E)}{-\log \eta}, \text{ respectively .}$$

If $\overline{\dim}_B(E) = \underline{\dim}_B(E)$, then it is called as the box dimension of E, and it is defined as, $\dim_B(E) = \lim_{\eta \to 0} \frac{\log N_\eta(E)}{-\log \eta}$.

Definition 5 [3] Let X and Y be metric spaces and $F : X \rightrightarrows Y$ be a set-valued map from X to Y, then the graph of F is defined as,

$$G_F = \{(u, w) \in X \times Y : w \in F(u)\}. \tag{1}$$

$F(u)$ is the image (or) the value of F at u. If there is at least one element $u \in X$ such that $F(u)$ is non-empty, then F is considered nontrivial. If $F(u)$ is non-empty for each $u \in X$, then F is known to be strict. The domain and range of F are defined as

$$\text{Dom}(F) := \{u \in X : F(u) \neq \emptyset\} \text{ and } \text{Im}(F) := \bigcup_{u \in X} F(u), \text{ respectively.}$$

Definition 6 Let $F, G : I \rightrightarrows \mathbb{R}$ be set-valued maps. Then, $F \leq G$ if and only if $F(u) \subseteq G(u)$ for all $u \in I$.

Remark 1 If $F(u)$ is closed (convex, compact, bounded), then F is said to be closed (convex, compact, bounded).

Definition 7 [3] Let $F : X \rightrightarrows Y$ be a set-valued mapping and $u \in \text{Dom}(F)$ such that for every neighborhood $\mathfrak{U}$ of $F(u)$,

$$\text{there exists } \eta > 0 \text{ such that } F(u') \subset \mathfrak{U} \text{ for all } u' \in B_X(u, \eta). \tag{2}$$

Then, F is characterized as upper semicontinuous at u. If it satisfies (2) for each $u \in \text{Dom}(F)$, then F is known as an upper semicontinuous function.

If for every $w \in F(u)$ and every sequence $\{u_n\} \subset \text{Dom}(F)$ converges to u, there exists a sequence of elements $w_n \in F(u_n)$ converges to w, then F is characterized as lower semicontinuous at u. If it is lower semicontinuous at each $u \in \text{Dom}(F)$, then F is said to be the lower semicontinuous function.

Lemma 1 [3] *A set-valued map F is said to be convex if and only if for all $u_1, u_2 \in$ Dom(F), $\lambda \in [0, 1]$, we have*

$$\lambda F(u_1) + (1-\lambda)F(u_2) \subset F(\lambda u_1 + (1-\lambda)u_2).$$

Definition 8 [7] Consider (X, d) be a complete metric space and $\mathcal{K}(X)$ be the collection of all non-empty compact subsets of X and H_d be the Hausdorff metric defined on $\mathcal{K}(X)$ and m be a positive integer such that $w_i : X \to X$ be a contraction map for each $i \in \{1, \ldots, m\}$, then the system $\{(X, d) : w_1, \ldots, w_m\}$ is known as Iterated Function System (IFS).

In the Definition 8, IFS satisfies the Banach contraction principle. One can construct the fractal using those IFSs, which satisfies other contractions. For instance, in [29], the construction of fractal using ϕ-contraction has been introduced.

Definition 9 [5] An IFS, $\{(X, d) : \omega_1, \ldots \omega_m\}$ is said to satisfy Open Set Condition (OSC), if there exists a non-empty open set $V \subset \mathbb{R}$ such that

$$\bigcup_{i=1}^{m} \omega_i(V) \subset V \;\text{ such that }\; \omega_i(V) \cap \omega_j(V) = \emptyset \;\text{ for }\; i \neq j.$$

Moreover, if A is an attractor of IFS such that $V \cap A \neq \emptyset$, then the IFS is said to satisfy the Strong Open Set Condition (SOSC).

3 Fractal Functions in $C(I, \mathcal{K}(\mathbb{R}))$

We know that space $\mathcal{C}(I, \mathcal{K}(\mathbb{R}))$ when endowed with metric $d_{\mathcal{C}}$ is a complete metric space, where

$$d_{\mathcal{C}}(F, G) = \|F - G\|_\infty = \sup_{u \in I} H_d(F(u), G(u)).$$

Note 1 [16, Proposition 1.17] Recall some properties of the Hausdorff metric as follows.

1. Let X be a normed space. Then, For $B, C, D, E \in \mathcal{K}(X)$, we have

$$H_d(B+C, D+E) \leq H_d(B, D) + H_d(C, E),$$

   where $Y + Z := \{y + z : y \in Y \in \mathcal{K}(X), z \in Z \in \mathcal{K}(X)\}$ is known as Minkowski sum of Y and Z.
2. For any $\lambda \in \mathbb{R}$, $H_d(\lambda B, \lambda D) = |\lambda| H_d(B, D)$.

Theorem 1 *Assume $F \in \mathcal{C}(I, \mathcal{K}(\mathbb{R}))$. Let $\Delta := \{(u_1, \dots, u_N) : u_1 < \cdots < u_N\}$ be a given data point such that it forms a partition of I, and let $I_n = [u_n, u_{n+1}]$. Let $L_n : I \to I_n$ be contractive homeomorphism such that $L_n(u_1) = u_n$ and $L_n(u_N) = u_{n+1}$ or $L_n(u_1) = u_{n+1}$ and $L_n(u_N) = u_n$. Further, assume that the base function $S \in \mathcal{C}(I, \mathcal{K}(\mathbb{R}))$ satisfies*

$$S(u_1) - F(u_1) = S(u_N) - F(u_N),$$

where $Y - Z = \{y - z : y \in Y \in \mathcal{K}(\mathbb{R})$ and $z \in Z \in \mathcal{K}(\mathbb{R})\}$ and scaling factor $\alpha \in \mathbb{R}$. If $|\alpha| < 1$, then there exists a unique function $F^{\alpha}_{\Delta,S} \in \mathcal{C}(I, \mathcal{K}(\mathbb{R}))$ satisfying the following self-referential equation

$$F^{\alpha}_{\Delta,S}(u) = F(u) + \alpha[F^{\alpha}_{\Delta,S}(L_n^{-1}(u)) - S(L_n^{-1}(u))] \text{ for every } u \in I_n, \tag{3}$$

where $n \in J = \{1, \dots, N-1\}$.

Proof Let $\mathcal{C}_F(I, \mathcal{K}(\mathbb{R})) = \{G \in \mathcal{C}(I, \mathcal{K}(\mathbb{R})) : G(u_1) - S(u_1) = G(u_N) - S(u_N)\}$. It is elementary to observe that $\mathcal{C}_F(I, \mathcal{K}(\mathbb{R}))$ is a closed subset of $\mathcal{C}(I, \mathcal{K}(\mathbb{R}))$, hence $(\mathcal{C}_F(I, \mathcal{K}(\mathbb{R})), d_{\mathcal{C}})$ is a complete metric space. Define*Read-Bajraktarević* (RB) operator $\Phi : \mathcal{C}_F(I, \mathcal{K}(\mathbb{R})) \to \mathcal{C}_F(I, \mathcal{K}(\mathbb{R}))$ by

$$(\Phi G)(u) = F(u) + \alpha[G(L_n^{-1}(u)) - S(L_n^{-1}(u))]$$

for every $u \in I_n$ and $n \in J$. Well-definedness of Φ can be observed using the assumptions we have taken for F, S, and α. With the reference to Note 1, we get

$$\begin{aligned} H_d((\Phi G)(u), (\Phi H)(u)) &= H_d\Big(F(u) + \alpha[G(L_n^{-1}(u)) - S(L_n^{-1}(u))], F(u) \\ &\qquad + \alpha[H(L_n^{-1}(u)) - S(L_n^{-1}(u))]\Big) \\ &\leq H_d\Big(\alpha G(L_n^{-1}(u)), \alpha H(L_n^{-1}(u))\Big) \\ &= |\alpha| H_d\Big(G(L_n^{-1}(u)), H(L_n^{-1}(u))\Big) \\ &\leq |\alpha| \sup_{u \in I} H_d(G(u), H(u)) \\ &= |\alpha| \|G - H\|_\infty. \end{aligned}$$

Since $|\alpha|\|G-H\|_\infty$ is independent of u, hence we have

$$\|\Phi G-\Phi H\|_\infty \le |\alpha|\|G-H\|_\infty.$$

Because $|\alpha|<1$, Φ is a contraction on $\mathcal{C}(I,\mathcal{K}(\mathbb{R}))$. Hence, Φ has a fixed point in $\mathcal{C}(I,\mathcal{K}(\mathbb{R}))$. Let $F^\alpha_{\Delta,S}$ be that fixed point, and then it satisfies the self-referential equation,

$$F^\alpha_{\Delta,S}(u)=F(u)+\alpha[F^\alpha_{\Delta,S}(L_n^{-1}(u))-S(L_n^{-1}(u))]$$

for every $u\in I_n$ and $n\in J$. □

Note 2 Throughout the paper,

- we denote $F^\alpha_{\Delta,S}$ as F^α if there is no ambiguity.
- we take Δ, S, and J as the same as it is in Theorem 1, unless specified.
- we have used the set difference as follows

$$Y-Z=\{y-z : y\in Y\in\mathcal{K}(\mathbb{R}) \text{ and } z\in Z\in\mathcal{K}(\mathbb{R})\}.$$

Remark 2 In the context of (3), we get

$$F^\alpha(u_i)=F(u_i)+\alpha F^\alpha(u_1)-\alpha S(u_1)=F(u_i)+\alpha F^\alpha(u_N)-\alpha S(u_N) \text{ for every } u_i\in\Delta,$$

where $i=1,\dots,N$. Further, if F^α and S are single-valued at the end points such that $F^\alpha(u_1)-S(u_1)=F^\alpha(u_N)-S(u_N)=\{0\}$, then $F^\alpha(u_i)=F(u_i)$ for each $i=1,\dots,N$, this implies that F^α is a set-valued fractal interpolation function.

Note 3 The above remark hints at the following: in case F and S are single-valued at the end points such that $F(u_1)-S(u_1)=F(u_N)-S(u_N)=\{0\}$, then the set

$$\mathcal{C}_F(I,\mathcal{K}(\mathbb{R}))=\Big\{G\in\mathcal{C}(I,\mathcal{K}(\mathbb{R})) : G(u_1)-S(u_1)=G(u_N)-S(u_N)=\{0\}\Big\}$$

is a complete metric space, and the RB operator $\Phi:\mathcal{C}_F(I,\mathcal{K}(\mathbb{R}))\to\mathcal{C}_F(I,\mathcal{K}(\mathbb{R}))$ as defined in Theorem 1 is well-defined and a contraction mapping. Therefore, we have a unique fixed point F^α of Φ satisfying $F^\alpha(u_i)=F(u_i)$ for all $i=1,\dots,N$, this shows that F^α is a set-valued fractal interpolation function.

Here we give some examples of base functions $S\in\mathcal{C}(I,\mathcal{K}(\mathbb{R}))$ satisfying $S(u_1)-F(u_1)=S(u_N)-F(u_N)$:

(i). $S(u)=F(t(u))+(u-u_1)(F(u_1)-F(u_1)+(u_N-u)(F(u_N)-F(u_N))$, where $t:I\to I$ be a continuous function which satisfies $t(u_1)=u_1,\ t(u_N)=u_N$.

(ii). $S(u)=t(u)F(u)+(u-u_1)(F(u_1)-F(u_1)+(u_N-u)(F(u_N)-F(u_N))$, where $t:I\to\mathbb{R}$ be a continuous function which satisfies $t(u_1)=1$ and $t(u_N)=1$.

The Hölder space is defined as follows:

$$\mathcal{HC}^{\sigma}(I, \mathcal{K}_c(\mathbb{R})) := \{G : I \to \mathcal{K}_c(\mathbb{R}) : \ G \in \sigma\text{-}\mathcal{HC}\},$$

Let us recall [21] that if we endow the space $\mathcal{HC}^{\sigma}(I, \mathcal{K}_c(\mathbb{R}))$ with metric

$$H_{\sigma}^{(1)}(G, H) = \sup_{u\in I} H_d(G(u), H(u)) + \sup_{u,w\in I} \frac{H_d\big(G(u) + H(w), H(u) + G(w)\big)}{|u - w|^{\sigma}}.$$

Then, by [21, Proposition 1], it forms a complete metric space.

Note 4 Throughout the paper, unless specified, take $L_n : I \to I_n$ as affine maps, such that $L_n(u) = a_n u + b_n$ for all $n \in J$, where $a_n = \frac{u_{n+1}-u_n}{u_N-u_1}$ and $b_n = \frac{u_n u_N - u_1 u_{n+1}}{u_N - u_1}$.

Theorem 2 *Consider $F, S \in \mathcal{HC}^{\sigma}(I, \mathcal{K}_c(\mathbb{R}))$ such that $S(u_1) - F(u_1) = S(u_N) - F(u_N)$, and let $\alpha \in (-1, 1)$. Then, $F^{\alpha} \in \sigma\text{-}\mathcal{HC}$ provided $\frac{(N-1)|\alpha|}{a^{\sigma}} < 1$, where $a := \min\{a_j : j \in J\}$.*

Proof Consider $\mathcal{HC}^{\sigma}_F(I, \mathcal{K}_c(\mathbb{R})) = \{G \in \mathcal{HC}^{\sigma}(I, \mathcal{K}_c(\mathbb{R})) : G(u_1) - S(u_1) = G(u_N) - S(u_N)\}$. It is easy to notice that $\mathcal{HC}^{\sigma}_F(I, \mathcal{K}_c(\mathbb{R}))$ is a closed subset of $\mathcal{HC}^{\sigma}(I, \mathcal{K}_c(\mathbb{R}))$, and hence complete with respect to the metric $H_{\sigma}^{(1)}$. Define a map $\Phi : \mathcal{HC}^{\sigma}_F(I, \mathcal{K}_c(\mathbb{R})) \to \mathcal{HC}^{\sigma}_F(I, \mathcal{K}_c(\mathbb{R}))$ as

$$(\Phi G)(u) = F(u) + \alpha\ (G - S)(L_j^{-1}(u))$$

for each $u \in I_j$, where $j \in J$. Clearly, Φ is well-defined. Now for $G, H \in \mathcal{HC}^{\sigma}_F(I, \mathcal{K}_c(\mathbb{R}))$, we have

$$\begin{aligned}
H_{\sigma}^{(1)}(\Phi(G), \Phi(H)) &\le \sup_{u\in I} H_d(\Phi(G)(u), \Phi(H)(u)) \\
&\quad + (N-1)\max_{j\in J} \sup_{u\neq w, u,w\in I_j} \frac{H_d\big(\Phi(G)(u) + \Phi(H)(w), \Phi(H)(u) + \Phi(G)(w)\big)}{|u - w|^{\sigma}} \\
&\le\ |\alpha| \sup_{u\in I} H_d(G(u), H(u)) \\
&\quad + (N-1)\max_{j\in J} \sup_{u\neq w, u,w\in I_j} \frac{H_d\Big(\alpha G(L_j^{-1}(u)) + \alpha H(L_j^{-1}(w)), \alpha H(L_j^{-1}(u)) + \alpha G(L_j^{-1}(w))\Big)}{|u - w|^{\sigma}} \\
&\le |\alpha| \sup_{u\in I} H_d(G(u), H(u)) \\
&\quad + (N-1)|\alpha| \max_{j\in J} \sup_{u\neq w, u,w\in I_j} \frac{H_d\Big(G(L_j^{-1}(u)) + H(L_j^{-1}(w)), H(L_j^{-1}(u)) + G(L_j^{-1}(w))\Big)}{|a_j|^{\sigma} |L_j^{-1}(u) - L_j^{-1}(w)|^{\sigma}} \\
&\le |\alpha| \sup_{u\in I} H_d(G(u), H(u)) \\
&\quad + \frac{(N-1)|\alpha|}{a^{\sigma}} \max_{j\in J} \sup_{u\neq w, u,w\in I_j} \frac{H_d\Big(G(L_j^{-1}(u)) + H(L_j^{-1}(w)), H(L_j^{-1}(u)) + G(L_j^{-1}(w))\Big)}{|L_j^{-1}(u) - L_j^{-1}(w)|^{\sigma}}
\end{aligned}$$

$$\leq \frac{(N-1)|\alpha|}{a^\sigma}\left[\sup_{u\in I} H_d(G(u),H(u)) + \sup_{u\neq w,u,w\in I}\frac{H_d\Big(G(u)+H(w),H(u)+G(w)\Big)}{|u-w|^\sigma}\right]$$
$$\leq \frac{(N-1)|\alpha|}{a^\sigma} H_\sigma^{(1)}(G,H).$$

Since $\frac{(N-1)|\alpha|}{a^\sigma} < 1$, which implies Φ is a contraction map on $\mathcal{HC}_F^\sigma(I,\mathcal{K}_c(\mathbb{R}))$. Now, the Banach contraction principle ensures that a unique fixed point of Φ exists. This completes the proof. □

Definition 10 Assume $F : I \to \mathcal{K}(\mathbb{R})$ be a set-valued map. For every partition $P := \{(t_0,\ldots,t_m) : t_0 < \cdots < t_m\}$ of I, define

$$V(F,I) = \sup_P \sum_{i=1}^{m} H_d(F(t_i),F(t_{i-1})),$$

where the supremum runs over all partitions, P of I.
We set $\|F\|_{\mathcal{BV}} := \|F\|_\infty + V(F,I)$, where $\|F\|_\infty := \sup_{u\in I}\|F(u)\| = \sup_{u\in I} H_d(F(u),\{0\})$. Then, F will be characterized as a bounded variation function if $\|F\|_{\mathcal{BV}} < \infty$. $\mathcal{BV}(I,\mathcal{K}(\mathbb{R}))$ will be denoted as the collection of all bounded variation functions on I.

Remark 3 It is interesting to write the following small observation: define functions $F,T : [0,1] \to \mathcal{K}(\mathbb{R})$ as follows $F(x) = \begin{cases}\sin\left(\frac{1}{x}\right), & \text{when } x\neq 0\\ 0, & \text{otherwise}\end{cases}$ and $T(x) = [-1,1]$. Here $F(x) \subset T(x)$ for each $x \in [0,1]$, such that $T \in \mathcal{BV}(I,\mathcal{K}(\mathbb{R}))$ while $F \notin \mathcal{BV}(I,\mathcal{K}(\mathbb{R}))$. This example shows that for set-valued mappings satisfying $F \leq T$ does not imply $\|F\|_{\mathcal{BV}} \leq \|T\|_{\mathcal{BV}}$.

As a prelude to our next result, we note the following lemma.

Lemma 2 *Consider $\{F_n\}$ is a sequence of set-valued continuous maps which uniformly converges to $F : I \to \mathcal{K}(\mathbb{R})$. Then, for a given partition $P = \{(y_0,\ldots,y_m) : y_0 < \cdots < y_m\}$ of I, we have*

$$\sum_{i=1}^{m} H_d(F_n(y_i),F_n(y_{i-1})) \to \sum_{i=1}^{m} H_d(F(y_i),F(y_{i-1})).$$

Moreover,

$$\sup_P \sum_{i=1}^{m} H_d(F(y_i),F(y_{i-1})) \leq \liminf_{n\to\infty}\sup_P \sum_{i=1}^{m} H_d(F_n(y_i),F_n(y_{i-1})).$$

Proof Let $P = \{(y_0, \ldots, y_m) : y_0 < \cdots < y_m\}$ be a partition of I. The uniform convergence of $\{F_n\}$ implies

$$\lim_{n\to\infty} \sum_{i=1}^{m} H_d(F_n(y_i), F_n(y_{i-1})) = \sum_{i=1}^{m} H_d(F(y_i), F(y_{i-1})).$$

Now for a given partition $P = \{(y_0, \ldots, y_m) : y_0 < \cdots < y_m\}$ of I, we get

$$\begin{aligned}\sum_{i=1}^{m} H_d(F(y_i), F(y_{i-1})) &= \sum_{i=1}^{m} H_d(\lim_{n\to\infty} F_n(y_i), \lim_{n\to\infty} F_n(y_{i-1}))\\ &= \lim_{n\to\infty} \sum_{i=1}^{m} H_d(F_n(y_i), F_n(y_{i-1}))\\ &\leq \liminf_{n\to\infty} \sup_P \sum_{i=1}^{m} H_d(F_n(y_i), F_n(y_{i-1})),\end{aligned}$$

completing the proof. □

Theorem 3 *The space* $\left(\mathcal{BV}(I, \mathcal{K}_c(\mathbb{R})), H_{\mathcal{BV}}\right)$ *is a complete metric space, where*

$$H_{\mathcal{BV}}(G, H) := \|G - H\|_\infty + \sup_P \sum_{i=1}^{m} H_d\Big(G(y_i) + H(y_{i-1}), H(y_i) + G(y_{i-1})\Big).$$

Proof Assume that $\{F_n\}$ is a Cauchy sequence in $\mathcal{BV}(I, \mathcal{K}_c(\mathbb{R}))$ with respect to $H_{\mathcal{BV}}$. Equivalently, for $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$H_{\mathcal{BV}}(F_n, F_k) < \epsilon \text{ for all } n, k \geq n_0.$$

Using the definition of $H_{\mathcal{BV}}$, we obtain $\|F_n - F_k\|_\infty < \epsilon$ for all $n, k \geq n_0$. Since $(\mathcal{C}(I, \mathcal{K}_c(\mathbb{R})), \|.\|_\infty)$ is a complete metric space, there exists a continuous function F with $\|F_n - F\|_\infty \to 0$ as $n \to \infty$. We claim that $F \in \mathcal{BV}(I, \mathcal{K}_c(\mathbb{R}))$ and $H_{\mathcal{BV}}(F_n, F) \to 0$ as $n \to \infty$. Let $P = \{(y_0, \ldots, y_m) : y_0 < \cdots < y_m\}$ be a partition of I and $n \geq n_0$. From the reference to Lemma 2, we get

$$\begin{aligned}H_{\mathcal{BV}}(F_n, F) &= \|F_n - F\|_\infty + \sum_{i=1}^{m} H_d\Big(F_n(y_i) + F(y_{i-1}), F(y_i) + F_n(y_{i-1})\Big)\\ &= \lim_{k\to\infty} \left(\|F_n - F_k\|_\infty + \sum_{i=1}^{m} H_d\Big(F_n(y_i) + F_k(y_{i-1}), F_k(y_i) + F_n(y_{i-1})\Big)\right)\\ &\leq \lim_{k\to\infty} \left(\|F_n - F_k\|_\infty + \sup_P \sum_{i=1}^{m} H_d\Big(F_n(y_i) + F_k(y_{i-1}), F_k(y_i) + F_n(y_{i-1})\Big)\right)\end{aligned}$$

$$\le \sup_{k\ge n_0}\left(\|F_n - F_k\|_\infty + \sup_P \sum_{i=1}^m H_d\Big(F_n(y_i)+F_k(y_{i-1}), F_k(y_i)+F_n(y_{i-1})\Big)\right)$$
$$\le \sup_{k\ge n_0} H_{\mathcal{BV}}(F_n, F_k) < \epsilon.$$

Since P was arbitrary, therefore we have $H_{\mathcal{BV}}(F_n, F) < \epsilon$ for all $n \ge n_0$.
It remains to show that $F \in \mathcal{BV}(I, \mathcal{K}_c(\mathbb{R}))$. Now by using $H_d(B + D, C + D) = H_d(B, C)$ for every $B, C, D \in \mathcal{K}_c(\mathbb{R})$ (see, for instance, [16]), we have

$$\begin{aligned}\sum_{i=1}^m H_d(F(y_i), F(y_{i-1})) &= \sum_{i=1}^m H_d(F(y_i)+F_n(y_{i-1}), F(y_{i-1})+F_n(y_{i-1}))\\ &\le \sum_{i=1}^m H_d(F(y_i)+F_n(y_{i-1}), F(y_{i-1})+F_n(y_i))\\ &\quad + \sum_{i=1}^m H_d(F_n(y_i)+F(y_{i-1}), F(y_{i-1})+F_n(y_{i-1}))\\ &\le \sum_{i=1}^m H_d(F(y_i)+F_n(y_{i-1}), F(y_{i-1})+F_n(y_i))\\ &\quad + \sum_{i=1}^m H_d(F_n(y_i), F_n(y_{i-1})) \le H_{\mathcal{BV}}(F_n, F) + \|F_n\|_{\mathcal{BV}}.\end{aligned}$$

Since $H_{\mathcal{BV}}(F_n, F) < \epsilon$ and $F_n \in \mathcal{BV}(I, \mathcal{K}_c(\mathbb{R}))$, the above inequality yields that $F \in \mathcal{BV}(I, \mathcal{K}_c(\mathbb{R}))$. This completes the proof. □

Theorem 4 *Consider $F \in \mathcal{BV}(I, \mathcal{K}_c(\mathbb{R}))$, Δ as defined in Theorem 1, $S \in \mathcal{BV}(I, \mathcal{K}_c(\mathbb{R}))$ such that $S(u_1) - F(u_1) = S(u_N) - F(u_N)$, and $\alpha \in (-1, 1)$ with $|\alpha| < \frac{1}{N-1}$. Then, α-fractal function, F^α corresponding to F is of bounded variation on I.*

Proof Consider $\mathcal{BV}_*(I, \mathcal{K}_c(\mathbb{R})) = \{G \in \mathcal{BV}(I, \mathcal{K}_c(\mathbb{R})) : G(u_1)-S(u_1) = G(u_N)-S(u_N)\}$. It is easy to prove that $\mathcal{BV}_*(I, \mathcal{K}_c(\mathbb{R}))$ is a closed subset of $\mathcal{BV}(I, \mathcal{K}_c(\mathbb{R}))$, hence complete with respect to metric $H_{\mathcal{BV}}$. Define RB operator $\Phi : \mathcal{BV}_*(I, \mathcal{K}_c(\mathbb{R})) \to \mathcal{BV}_*(I, \mathcal{K}_c(\mathbb{R}))$ by

$$(\Phi G)(u) = F(u) + \alpha\big[G\big(L_j^{-1}(u)\big) - S\big(L_j^{-1}(u)\big)\big]$$

for each $u \in I_j$ and $j \in J$. It is easy to observe the well-definedness of Φ. For $m \in \mathbb{N}$, assume $P^j = \{(t_0^j, \dots, t_m^j) : t_0^j < \cdots < t_m^j\}$ is a partition of I_j and $j \in J$. For $i \in \{1, \dots, m\}$, we have

$$H_d\Big(\Phi(G)(t_i^j) + \Phi(H)(t_{i-1}^j), \Phi(H)(t_i^j) + \Phi(G)(t_{i-1}^j)\Big)$$
$$\le H_d\Big(\alpha G\big(L_j^{-1}(t_i^j)\big) + \alpha H\big(L_j^{-1}(t_{i-1}^j)\big), \alpha H\big(L_j^{-1}(t_i^j)\big) + \alpha G\big(L_j^{-1}(t_{i-1}^j)\big)\Big)$$

$$\leq |\alpha| H_d\Big(G\big(L_j^{-1}(t_i^j)\big) + H\big(L_j^{-1}(t_{i-1}^j)\big), H\big(L_j^{-1}(t_i^j)\big) + G\big(L_j^{-1}(t_{i-1}^j)\big)\Big).$$

Summing over $i = 1$ to m, we have

$$\begin{aligned}
&\sum_{i=1}^{m} H_d\Big(\Phi(G)(t_i^j) + \Phi(H)(t_{i-1}^j), \Phi(H)(t_i^j) + \Phi(G)(t_{i-1}^j)\Big) \\
&\leq |\alpha| \sum_{i=1}^{m} H_d\Big(G\big(L_j^{-1}(t_i^j)\big) + H\big(L_j^{-1}(t_{i-1}^j)\big), H\big(L_j^{-1}(t_i^j)\big) + G\big(L_j^{-1}(t_{i-1}^j)\big)\Big) \\
&\leq |\alpha| \sup_P \sum_{i=1}^{m} H_d\Big(G(t_i) + H(t_{i-1}), H(t_i) + G(t_{i-1})\Big),
\end{aligned}$$

since $P := \{(L_j^{-1}(t_0^j), \ldots, L_j^{-1}(t_m^j)) : L_j^{-1}(t_0^j) < \cdots < L_j^{-1}(t_m^j)\}$ is a partition of I (without loss of generality), and the supremum is taken over all partitions $P = \{(t_0, \ldots, t_m) : t_0 < \cdots < t_m\}$ of I. The above inequality is true for any partition P^j of I_j. Hence, we get

$$\begin{aligned}
H_{\mathcal{BV}}(\Phi(G), \Phi(H)) &= \sup_{u \in I} H_d(\Phi(G)(u), \Phi(H)(u)) \\
&\quad + \sup_P \sum_{i=1}^{m} H_d\Big(\Phi G(t_i) + \Phi H(t_{i-1}), \Phi H(t_i) + \Phi G(t_{i-1})\Big) \\
&\leq |\alpha| \sup_{u \in I} H_d(G(u), H(u)) \\
&\quad + \max_{j \in J} \sup_{P^j} \sum_{i=1}^{m} H_d\Big(\Phi(G)(t_i^j) + \Phi(H)(t_{i-1}^j), \\
&\quad \Phi(H)(t_i^j) + \Phi(G)(t_{i-1}^j)\Big) \\
&\leq |\alpha| \sup_{u \in I} H_d(G(u), H(u)) \\
&\quad + (N-1)|\alpha| \sup_P \sum_{i=1}^{m} H_d\Big(G(t_i) + H(t_{i-1}), H(t_i) + G(t_{i-1})\Big) \\
&\leq (N-1)|\alpha| H_{\mathcal{BV}}(G, H).
\end{aligned}$$

As $|\alpha| < \frac{1}{N-1}$, Φ is a contraction map. Then, the Banach fixed point theorem ensures that Φ has a unique fixed point, say F^α. Further, this fixed point will satisfy the following self-referential equation,

$$F^\alpha(u) = F(u) + \alpha\big[F^\alpha\big(L_j^{-1}(u)\big) - S\big(L_j^{-1}(u)\big)\big] \text{ for each } u \in I_j, \text{ where } j \in J.$$

□

Notice that function F^α is a parametric function depending on parameters, the base function S, scaling function α, partition Δ, and the function F itself. To observe collective behavior of F^α depending on some such parameters, we define a set-valued map, $\mathcal{F}_S^\alpha : \mathcal{C}(I, \mathcal{K}(\mathbb{R})) \to \mathcal{C}(I, \mathcal{K}(\mathbb{R}))$ such that

$$\mathcal{F}_S^\alpha(F) = F^\alpha, \text{ where } \alpha \in (-1, 1). \tag{4}$$

This map is known as a fractal operator.

Remark 4 The notion of the fractal operator has already been studied extensively for single-valued maps. See, for instance, in [22, 24] fractal operator has been defined for univariate single-valued maps. In [35], the fractal operator for bivariate single-valued maps has been studied. Here, we have given the notion of the set-valued fractal operator.

Theorem 5 *$\mathcal{F}_S^\alpha$ defined in (4) is a continuous map.*

Proof Let $\{F_k\}$ be a sequence in $\mathcal{C}(I, \mathcal{K}(\mathbb{R}))$ such that $F_k \to F$, then to prove $\mathcal{F}_S^\alpha$ is a continuous function, it is sufficient to prove that $F_n^\alpha \to F_\alpha$. Since $F_n \to F$, then for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that,

$$\begin{aligned} &\|F_n - F\|_\infty < \epsilon(1 - |\alpha|) \text{ for all } n \geq n_0,\\ &\text{equivalently, } \sup_{u \in I} H_d(F_n(u), F(u)) \qquad < \epsilon(1 - |\alpha|) \text{ for all } n \geq n_0. \end{aligned}$$

Now, we have

$$\begin{aligned} H_d(F_n^\alpha(x), F^\alpha(x)) &= H_d\Big(F_n(x) + \alpha\left[F_n^\alpha(L_j^{-1}(x)) - S(L_j^{-1}(x))\right], F(x)\\ &\quad + \alpha\left[F^\alpha(L_j^{-1}(x)) - S(L_j^{-1}(x))\right]\Big)\\ &\leq H_d(F_n(x), F(x)) + |\alpha| H_d(F_n^\alpha(L_j^{-1}(x)), F^\alpha(L_j^{-1}(x))). \end{aligned}$$

This implies,

$$\begin{aligned} &\sup_{x \in I} H_d(F_n^\alpha(x), F^\alpha(x)) \leq \frac{1}{1 - |\alpha|} \sup_{x \in I} H_d(F_n(x), F(x)),\\ &\quad \text{that is } \|F_n^\alpha(x) - F^\alpha(x)\|_\infty < \epsilon \text{ for all } n \geq n_0. \end{aligned}$$

This completes the proof. ◻

Theorem 6 *For a fixed partition Δ, the mapping $\mathcal{T}_S^\Delta : \mathcal{C}(I, \mathcal{K}(\mathbb{R})) \rightrightarrows \mathcal{C}(I, \mathcal{K}(\mathbb{R}))$ defined as,*

$$\mathcal{T}_S^\Delta(F) = \left\{F^\alpha : \alpha \in (-1, 1)\right\}$$

is lower semi-continuous.

Proof Let $F \in \mathcal{C}(I, \mathcal{K}(\mathbb{R}))$ and let $F^\alpha \in \mathcal{T}_S^\Delta(F)$ and a sequence $F_k \in \mathcal{C}(I, \mathcal{K}(\mathbb{R}))$ such that $F_k \to F$. Using Theorem 5, we have $F_k^\alpha \to F^\alpha$, then clearly $F_k^\alpha \in \mathcal{T}_S^\Delta(F_k)$, establishing the result. □

4 Approximation of Set-Valued Functions

In Sect. 3, we observe that F^α satisfies the following self-referential equation:

$$F^\alpha(u) = F(u) + \alpha\left[F^\alpha\left(L_j^{-1}(u)\right) - S\left(L_j^{-1}(u)\right)\right]$$

for every $u \in I_j$, where $j \in J$.
The following proposition will give the perturbation error between the map F and its α-fractal function F^α. We shall use this proposition as a prelude to our next theorem.

Proposition 1 *Between F and F^α, the following perturbation error will be obtained:*

$$\|F^\alpha - F\|_\infty \leq \frac{|\alpha|}{1-|\alpha|}\|F - S\|_\infty + \frac{2|\alpha|}{1-|\alpha|}\|F\|_\infty.$$

Proof Using the self-referential equation and Note 1, we get

$$\begin{aligned}
H_d(F^\alpha(u), F(u)) &= H_d\Big(F(u) + \alpha\left[F^\alpha\left(L_j^{-1}(u)\right) - S\left(L_j^{-1}(u)\right)\right],\ F(u)\Big)\\
&\leq H_d\Big(\alpha\left[F^\alpha\left(L_j^{-1}(u)\right) - S\left(L_j^{-1}(u)\right)\right],\ \{0\}\Big)\\
&= |\alpha| H_d\Big(F^\alpha\left(L_j^{-1}(u)\right) - S\left(L_j^{-1}(u)\right),\ \{0\}\Big)\\
&\leq |\alpha| H_d\Big(F^\alpha\left(L_j^{-1}(u)\right) - S\left(L_j^{-1}(u)\right),\\
&\quad F\left(L_j^{-1}(u)\right) - F\left(L_j^{-1}(u)\right)\Big)\\
&\quad + |\alpha| H_d\Big(F\left(L_j^{-1}(u)\right) - F\left(L_j^{-1}(u)\right), \{0\}\Big)\\
&\leq |\alpha| H_d\Big(F^\alpha\left(L_j^{-1}(u)\right), F\left(L_j^{-1}(u)\right)\Big)\\
&\quad + |\alpha| H_d\Big(-S\left(L_j^{-1}(u)\right), -F\left(L_j^{-1}(u)\right)\Big)\\
&\quad + 2|\alpha| H_d\Big(F\left(L_j^{-1}(u)\right), \{0\}\Big)\\
&\leq |\alpha| \sup_{u \in I_j, j \in J} H_d\Big(F^\alpha\left(L_j^{-1}(u)\right), F\left(L_j^{-1}(u)\right)\Big)
\end{aligned}$$

$$+ |\alpha| \sup_{u\in I_j, j\in J} H_d\Big(S\left(L_j^{-1}(u)\right), F\left(L_j^{-1}(u)\right)\Big)$$
$$+ 2|\alpha| \sup_{u\in I_j, j\in J} H_d\left(F\left(L_j^{-1}(u)\right), \{0\}\right)$$
$$\le |\alpha| \|F^\alpha - F\|_\infty + |\alpha| \|F - S\|_\infty + 2|\alpha| \|F\|_\infty.$$

This in turn yields $\|F^\alpha - F\|_\infty \le |\alpha|\|F^\alpha - F\|_\infty + |\alpha|\|F - S\|_\infty + 2|\alpha|\|F\|_\infty$. This establishes the proof. □

Remark 5 Perturbation error between single-valued maps and its corresponding α-fractal function have already been studied in the literature. For instance, in [23], a perturbation error between a univariate single-valued map and its corresponding α-fractal function has been given. In [35], a perturbation error between a bivariate single-valued map and its corresponding α-fractal function has been studied. Here, we have studied the perturbation error between a set-valued map and its corresponding α-fractal function.

Definition 11 Let $\mathcal{P} \in \mathcal{C}(I, \mathcal{K}(\mathbb{R}))$ be a set-valued polynomial function, then α-fractal function $\mathcal{P}^\alpha$ corresponding to $\mathcal{P}$ is defined as set-valued fractal polynomial.

Theorem 7 *Consider $F \in \mathcal{C}(I, \mathcal{K}_c(\mathbb{R}))$. For any $\epsilon > 0$, there is a set-valued fractal polynomial $\mathcal{P}^\alpha$ such that*

$$\|F - \mathcal{P}^\alpha\|_\infty < \epsilon.$$

Proof For $\epsilon > 0$ using [37], there is a set-valued polynomial function $\mathcal{P}$ such that

$$\|F - \mathcal{P}\|_\infty < \frac{\epsilon}{3}.$$

Choose a partition $\Delta_{\mathcal{P}} = \{y_0, \dots, y_M\}$ of I and a continuous function $S_{\mathcal{P}}$ satisfying $S_{\mathcal{P}}(y_0) - \mathcal{P}(y_0) = S_{\mathcal{P}}(y_M) - \mathcal{P}(y_M)$, and $\alpha \in (-1, 1)$ such that

$$|\alpha| < \min\left\{ \frac{\frac{\epsilon}{3}}{\frac{\epsilon}{3} + \|\mathcal{P} - S_{\mathcal{P}}\|_\infty}, \frac{\frac{\epsilon}{3}}{\frac{\epsilon}{3} + 2\|\mathcal{P}\|_\infty} \right\}.$$

Then, we get

$$\begin{aligned}
\|F - \mathcal{P}^\alpha\|_\infty &\le \|F - \mathcal{P}\|_\infty + \|\mathcal{P} - \mathcal{P}^\alpha\|_\infty \ \text{(using triangle inequality)} \\
&\le \|F - \mathcal{P}\|_\infty + \frac{|\alpha|}{1-|\alpha|}\|\mathcal{P} - S_{\mathcal{P}}\|_\infty + \frac{2|\alpha|}{1-|\alpha|}\|\mathcal{P}\|_\infty \ \text{(using Proposition 1)} \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&= \epsilon.
\end{aligned}$$

□

Remark 6 We took $\alpha \in \mathbb{R}$ in the above proof, such that

$$|\alpha| < \min\left\{\frac{\frac{\epsilon}{3}}{\frac{\epsilon}{3} + \|\mathcal{P} - S_{\mathcal{P}}\|_\infty}, \frac{\frac{\epsilon}{3}}{\frac{\epsilon}{3} + 2\|\mathcal{P}\|_\infty}\right\}.$$

In this situation, α may be "close" to 0, $\mathcal{P}^\alpha$ may not be self-referential, and it may behave as a classical polynomial. In alter, if we fix $\alpha \in (-1, 1)$ such that $|\alpha| < 1$, but otherwise arbitrary and choose a polynomial $\mathcal{P}$ and a function $S_{\mathcal{P}} \in \mathcal{C}(I, \mathcal{K}_c(\mathbb{R}))$ satisfying $S_{\mathcal{P}}(y_0) - \mathcal{P}(y_0) = S_{\mathcal{P}}(y_M) - \mathcal{P}(y_M)$ and

$$\|\mathcal{P} - S_{\mathcal{P}}\| < \frac{(1 - |\alpha|)\epsilon}{3|\alpha|} \text{ and } \|\mathcal{P}\|_\infty < \frac{(1 - |\alpha|)\epsilon}{6|\alpha|}.$$

This forces F to be a zero set function. Hence, the analog of [35, Remark 5.2] cannot be established in set-valued mappings. In particular, the recently developed notion of Bernstein fractal functions will not be useful in approximating set-valued functions.

With the reference to Theorem 7, we have

Theorem 8 *The set of set-valued fractal polynomials with a non-zero scale vector is dense in* $\mathcal{C}(I, \mathcal{K}_c(\mathbb{R}))$.

4.1 Constrained Approximation

Here we target to study some constrained approximation aspects of fractal functions. Before proving the next theorem, let us recall a result and prove a lemma as a prelude.

Result 1 Consider X, Y are topological spaces, $f : X \to Y$ is a continuous function, and S is a dense subset in X. If $f(u) \leq 0$ $(f(u) \geq 0)$ for each $u \in S$, then $f(u) \leq 0$ $(f(u) \geq 0)$ for each $u \in X$.

Lemma 3 *The set* $C = \bigcup_{n\in\mathbb{N}} \left(\bigcup_{1\leq i_1,\ldots,i_n\leq N} L_{i_1\ldots i_n}\big(\{u_1, \ldots, u_N\}\big) \right)$ *is dense in interval* $I = [0, 1]$, *where* $L_{i_1\ldots i_n}(u) = L_{i_1}(L_{i_2}(\ldots(L_{i_n}(u))))$ *and* $n \in \mathbb{N}$.

Proof Let $u \in I$ be any point. Observe that for some $w \in \{u_1, \ldots, u_N\}$, we have $|u - w| \leq \max_{i\in J}\left\{\frac{u_i - u_{i-1}}{2}\right\}$. Since each L_i is a contraction mapping with contraction coefficient a_i. Choose $a = \max_{i\in J}\{a_i\}$, then for each $u \in I$ and for each $\epsilon > 0$ we can choose $w \in L_{i_1\ldots i_n}\big(\{u_1, \ldots, u_N\}\big)$ for some $n \in \mathbb{N}$ such that,

$$|u - w| \leq a^n \max_{i\in J}\left\{\frac{u_i - u_{i-1}}{2}\right\} < \epsilon.$$

This completes the proof. □

Theorem 9 *Let $F, G \in \mathcal{C}(I, \mathcal{K}(\mathbb{R}))$ and Δ as defined in Theorem 1, and $F(u_1)$, $F(u_N)$, $G(u_1)$, $G(u_N)$ are single-valued. If $F \le G$, then $F^\alpha \le G^\alpha$ provided $S_F, S_G \in \mathcal{C}(I, \mathcal{K}(\mathbb{R}))$ satisfying $S_F \le S_G$ and $S_F(u_1) = F(u_1)$, $S_F(u_N) = F(u_N)$, $S_G(u_1) = G(u_1)$, $S_G(u_N) = G(u_N)$.*

Proof Let $S_F, S_G \in \mathcal{C}(I, \mathcal{K}(\mathbb{R}))$ such that $S_F \le S_G$ and $S_F(u_1) = F(u_1)$, $S_F(u_N) = F(u_N)$, $S_G(u_1) = G(u_1)$, $S_G(u_N) = G(u_N)$. Using Note 3, we have

$$F^\alpha(u_i) = F(u_i),\ G^\alpha(u_i) = G(u_i) \text{ for each } i = 1, \dots, N.$$

From the self-referential equation,

$$\begin{aligned} F^\alpha\left(L_j(u)\right) &= F\left(L_j(u)\right) + \alpha\left[F^\alpha(u) - S_F(u)\right], \text{ and } G^\alpha\left(L_j(u)\right) \\ &= G\left(L_j(u)\right) + \alpha\left[G^\alpha(u) - S_G(u)\right] \end{aligned}$$

for each $u \in I_j$, where $j \in J$. For $u \in \Delta$, we deduce

$$F^\alpha\big(L_j(u)\big) \subset G^\alpha\big(L_j(u)\big) \text{ for any } j \in J.$$

Applying the process repeatedly, we get

$$F^\alpha\left(L_{i_1\dots i_n}(u)\right) \subset G^\alpha\left(L_{i_1\dots i_n}(u)\right) \text{ for any } i_1, \dots, i_n \in J,\ u \in \{u_1, \dots, u_N\},$$

where $L_{i_1\dots i_n}(u) = L_{i_1}(L_{i_2}(\dots(L_{i_n}(u))))$ and $n \in \mathbb{N}$.
This implies that $F^\alpha(u) \subset G^\alpha(u)$ for each $u \in \bigcup_{n\in\mathbb{N}} \left(\bigcup_{1\le i_1,\dots,i_n \le N} L_{i_1\dots i_n}\left(\{u_1, \dots, u_N\}\right)\right)$.
Now using Lemma 3 and Result 1, we are done. □

5 Dimensional Results

To move further in this section, we shall first observe some examples to understand the motivation behind this section.

Example 1 Let $F_1 : [0, 1] \rightrightarrows \mathbb{R}$ be a set-valued map defined as $F_1(u) = \{0\}$, then according to (1) graph of this function will be a line segment in $\mathbb{R}^2$, and hence $\dim_H(G_{F_1}) = 1$.

Example 2 Let $F_2 : [0, 1] \rightrightarrows \mathbb{R}$ be a set-valued map defined as $F_2(u) = [-1, 1]$, then by (1), we have $G_{F_2} = [0, 1] \times [-1, 1]$, and hence $\dim_H(G_{F_2}) = 2$.

Example 3 Let $F_3 : [0, 1] \rightrightarrows \mathbb{R}$ be a set-valued map defined as $F_3(u) = C$, where C is Cantor set. Then, by (1) we have $G_{F_3} = [0, 1] \times C$, and hence $\dim_H(G_{F_2}) = 1 + \frac{\log 2}{\log 3}$.

Notice that F_1, F_2, and F_3 are constant maps. Therefore, these are Lipschitz and bounded variation maps as well. Unlike the case of a single-valued map, here we witness that the Hausdorff dimension of the graph of a set-valued Lipschitz map is other than 1, and the same observation holds for the graph of a set-valued bounded variation map also. One can always find a set-valued Lipschitz map or set-valued bounded variation map whose graph has dimension β for any $1 \leq \beta \leq 2$. We observe that with the definition of the graph as in (1), we could not find any fascinating dimensional result. Therefore, we give a new definition of the graph of a set-valued map and study some dimensional results for this new definition of the graph.

Definition 12 Let $F : [0,1] \to \mathcal{K}(\mathbb{R})$ be a set-valued map, then a graph of F is defined as;

$$\mathcal{G}(F) = \{(u, F(u)) : F(u) \in \mathcal{K}(\mathbb{R})\} \subset [0,1] \times \mathcal{K}(\mathbb{R}). \tag{5}$$

Define a metric on this graph,

$$D_{\mathcal{G}}((u, F(u)), (w, F(w))) = |u - w| + H_d(F(u), F(w)).$$

Next, we prove the graph of F^α defined in (5) is an attractor of an IFS defined on $I \times \mathcal{K}_c(\mathbb{R})$.

Let us note the following lemma as a prelude. The motivation for this following lemma comes from [8, Proposition 1].

Lemma 4 *Let F be a set-valued continuous map and F^α be its corresponding α-fractal function. Define a function $\mathfrak{d} : I \times \mathcal{K}_c(\mathbb{R}) \to [0, \infty)$ as*

$$\mathfrak{d}\big((u, A), (w, B)\big) = |u - w| + H_d\big(A + F^\alpha(w), B + F^\alpha(u)\big).$$

Then, $I \times \mathcal{K}_c(\mathbb{R})$ with respect to $\mathfrak{d}$ is a complete metric space.

Proof Clearly, $\mathfrak{d}\big((u, A), (w, B)\big) = \mathfrak{d}\big((w, B), (u, A)\big) \geq 0$. Suppose that $\mathfrak{d}\big((u, A), (w, B)\big) = 0$, then

$$\begin{aligned}
&|u - w| + H_d\big(A + F^\alpha(w), B + F^\alpha(u)\big) = 0 \\
&\quad \text{i.e., } |u - w| = 0 \text{ and } H_d\big(A + F^\alpha(w), B + F^\alpha(u)\big) = 0 \\
&\quad \text{i.e., } u = w \text{ and } H_d\big(A + F^\alpha(w), B + F^\alpha(u)\big) = H_d(A, B) = 0 \\
&\quad \text{i.e., } u = w \text{ and } A = B \\
&\quad \text{i.e., } (u, A) = (w, B).
\end{aligned}$$

Now to prove that $\mathfrak{d}$ satisfies the triangle inequality. Take $(u_i, A_i) \in I \times \mathcal{K}_c(\mathbb{R})$ for $i = 1, 2, 3$. Then, we have

$$\begin{aligned}
\mathfrak{d}\big((u_1, A_1), (u_2, A_2)\big) &= |u_1 - u_2| + H_d\big(A_1 + F^\alpha(u_2), A_2 + F^\alpha(u_1)\big) \\
&= |u_1 - u_2| + H_d\big(A_1 + F^\alpha(u_2) + A_3 + F^\alpha(u_3), A_2 \\
&\quad + F^\alpha(u_1) + A_3 + F^\alpha(u_3)\big) \\
&\leq \big\{|u_1 - u_3| + |u_3 - u_2|\big\} + \big\{H_d\big(A_1 + F^\alpha(u_3), A_3 + F^\alpha(u_1)\big) \\
&\quad + H_d\big(A_3 + F^\alpha(u_2), A_2 + F^\alpha(u_3)\big)\big\}.
\end{aligned}$$

Hence,

$$\mathfrak{d}\Big((u_1, A_1), (u_2, A_2)\Big) \le \mathfrak{d}\Big((u_1, A_1), (u_3, A_3)\Big) + \mathfrak{d}\Big((u_3, A_3), (u_2, A_2)\Big).$$

To prove completeness, let $\{(u_n, A_n)\}$ is a Cauchy sequence in $I \times \mathcal{K}_c(\mathbb{R})$. For $\epsilon > 0$ there is an integer $N(\epsilon)$ such that

$$|u_n - u_m| + H_d\big(A_n + F^\alpha(u_n), A_m + F^\alpha(u_m)\big) < \epsilon, \text{ whenever } m, n \ge N(\epsilon).$$

This shows $\{u_n\}$ is a Cauchy sequence of I. Hence, it converges to, say, $u^* \in I$. Since F^α is a uniformly continuous map, consequently $\{F^\alpha(u_n)\}$ will also be a Cauchy sequence with respect to Hausdorff metric, and hence converges to $F^\alpha(u^*) \in \mathcal{K}_c(\mathbb{R})$. Then,

$$\begin{aligned} H_d(A_n, A_m) &= H_d\big(A_n + F^\alpha(u_n), A_m + F^\alpha(u_n)\big) \\ &\le H_d\big(A_n + F^\alpha(u_n), A_n + F^\alpha(u_m)\big) + H_d\big(A_n + F^\alpha(u_m) + A_m + F^\alpha(u_n)\big) \\ &= H_d\big(F^\alpha(u_n), F^\alpha(u_m)\big) + H_d\big(A_n + F^\alpha(u_m) + A_m + F^\alpha(u_n)\big) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

This implies, $\{A_n\}$ is a Cauchy sequence of $\mathcal{K}_c(\mathbb{R})$ and so it converges to, say $A^* \in \mathcal{K}_c(\mathbb{R})$. Hence, $\{(u_n, A_n)\}$ converges to $(u^*, A^*) \in I \times \mathcal{K}_c(\mathbb{R})$. This completes the proof. □

Proposition 2 *Let $F \in \mathcal{C}(I, \mathcal{K}_c(\mathbb{R}))$ be a set-valued continuous map and $S \in \mathcal{C}(I, \mathcal{K}_c(\mathbb{R}))$ be the base function. Define $W_j : I \times \mathcal{K}_c(\mathbb{R}) \to I \times \mathcal{K}_c(\mathbb{R})$ for each $j \in J$ such that*

$$W_j(u, A) = \big(L_j(u), \alpha A + F(L_j(u)) - \alpha S(u)\big)\,.$$

Then, each W_j is a contraction map with respect to the metric defined in Lemma 4, provided $\max\{|\alpha|, a_j\} < 1$ *for each $j \in J$.*

Proof Let $(u, A), (w, B) \in I \times \mathcal{K}_c(\mathbb{R})$, then for each $j \in J$, we have

$$\begin{aligned} &\mathfrak{d}\big(W_j(u, A), W_j(w, B)\big) = \mathfrak{d}\Big(\big(L_j(u), \alpha A + F(L_j(u)) - \alpha S(u)\big), \\ &\big(L_j(w), \alpha B + F(L_j(w)) + \alpha S(u)\big)\Big) \\ &\quad = \big|L_j(u) - L_j(w)\big| + H_d\Big(\alpha A + F(L_j(u)) - \alpha S(u) + F^\alpha(L_j(w)), \\ &\quad\ \alpha B + F(L_j(w)) - \alpha S(w) + F^\alpha(L_j(u))\Big) \\ &\quad = \big|L_j(u) - L_j(w)\big| + H_d\Big(\alpha A \\ &\quad\ + F(L_j(u)) - \alpha S(u) + F(L_j(w)) + \alpha F^\alpha(w) - \alpha S(w), \end{aligned}$$

$$\alpha B + F(L_j(w)) - \alpha S(w) + F(L_j(u)) + \alpha F^\alpha(u) - \alpha S(u)\Big)$$
$$= a_j\big|u - w\big| + H_d\Big(\alpha A + \alpha F^\alpha(w), \alpha B + \alpha F^\alpha(u)\Big)$$
$$= a_j\big|u - w\big| + |\alpha| H_d\Big(A + F^\alpha(w), B + F^\alpha(u)\Big)$$
$$\leq \max\{|\alpha|, a_j\}\Big(\big|u - w\big| + H_d\Big(A + F^\alpha(w), B + F^\alpha(u)\Big)\Big)$$
$$= \max\{|\alpha|, a_j\}\mathfrak{d}\big((u, A), (w, B)\big).$$

Since $\max\{|\alpha|, a_j\} < 1$, each W_j is a contraction mapping. □

Now, to prove the next theorem, we first note the following basic results. Their proofs can be found in the literature, but we decided to include them here for the sake of completeness.

Lemma 5 *The space $(\mathcal{K}_c(\mathbb{R}), H_d)$ is a complete metric space.*

Proof Let $\{A_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{K}_c(\mathbb{R})$. This implies that $\{A_n\}_{n\in\mathbb{N}}$ is Cauchy in $\mathcal{K}(\mathbb{R})$. Then, by the completeness of the space $(\mathcal{K}(\mathbb{R}), H_d)$, there exists $A^* \in \mathcal{K}(\mathbb{R})$ such that $A_n \to A^*$ with respect to the Hausdorff metric H_d. It is well-known that for $x^* \in A^*$, there exists a sequence $\{x_n\}_n$, where $x_n \in A_n$ for each $n \in \mathbb{N}$, such that $x_n \to x^*$ as $n \to \infty$.
Now it remains to prove that A^* is a convex set. For this let $x, y \in A^*$, then there exist sequences $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}}$, where $x_n, y_n \in A_n$ for each $n \in \mathbb{N}$ such that $x_n \to x$ and $y_n \to y$. Since $x_n,\ y_n \in A_n$ and A_n is convex, therefore $\lambda x_n + (1-\lambda)y_n \in A_n$ for all $\lambda \in [0, 1]$. This implies that $\lim_{n\to\infty}(\lambda x_n + (1-\lambda)y_n) = \lambda x + (1-\lambda)y \in A^*$. This completes the proof. □

Lemma 6 *The space $(\mathcal{C}(I, \mathcal{K}_c(\mathbb{R})), d_{\mathcal{C}}$ is a complete metric space.*

Proof To prove this, it is sufficient to show that $\mathcal{C}(I, \mathcal{K}_c(\mathbb{R}))$ is a closed subset of $\mathcal{C}(I, \mathcal{K}(\mathbb{R}))$. For this, let F^* be a limit point of $\mathcal{C}(I, \mathcal{K}_c(\mathbb{R}))$. Then, there exists a sequence $\{F_n\}_{n\in\mathbb{N}}$ of $\mathcal{C}(I, \mathcal{K}_c(\mathbb{R}))$ such that $F_n \to F^*$ with respect to the metric $d_{\mathcal{C}}$. This implies that $F_n(x) \to F^*(x)$ for all $x \in I$ with respect to the Hausdorff metric. Since $F_n(x) \in \mathcal{K}_c(\mathbb{R})$ for each $x \in I$, hence using Lemma 5, $F^*(x) \in \mathcal{K}_c(\mathbb{R})$ for each $x \in I$. This completes the proof. □

Theorem 10 *For each $j \in J$, let $W_j : I \times \mathcal{K}_c(\mathbb{R}) \to I \times \mathcal{K}_c(\mathbb{R})$ be the map defined in Proposition 2. Then, by Definition 12, the graph of F^α will be an attractor of the IFS, $\{(I \times \mathcal{K}_c(\mathbb{R}), \mathfrak{d})\,;\, W_1, \ldots, W_{N-1}\}$.*

Proof First we establish that $F^\alpha \in \mathcal{C}(I, \mathcal{K}_c(\mathbb{R}))$. But this can be observed by using Lemma 6 and Theorem 1. Now since $I = \bigcup_{j\in J} L_j(I)$. Then, from (3), we have

$$\bigcup_{j\in J} W_j(\mathcal{G}(F^\alpha)) = \bigcup_{j\in J}\big\{W_j(u, F^\alpha(u)) : u \in I\big\}$$

$$
\begin{aligned}
&= \bigcup_{j \in J} \Big\{ \big(L_j(u), \alpha F^{\alpha}(u) + F(L_j(u)) - \alpha S(u) \big) : u \in I \Big\} \\
&= \bigcup_{j \in J} \Big\{ \big(L_j(u), F^{\alpha}(L_j(u)) \big) : u \in I \Big\} \\
&= \bigcup_{j \in J} \Big\{ (u, F^{\alpha}(u)) : u \in L_j(I) \Big\} \\
&= \mathcal{G}(F^{\alpha}).
\end{aligned}
$$

This completes the proof. □

Schief [31] noted that the dimensional results for Euclidean spaces do not have simple generalizations to complete metric spaces. Following his work, Nussbaum et al. [26] proved a more general result in the setting of a complete metric space. Answering a question raised in [26], Verma [33] has shown the Hausdorff dimension of the invariant set under the SOSC. He explores several dimensional aspects of sets in complete metric space. In his book [15], Falconer studied the dimensional results of sets in Euclidean spaces. Given [33], we may assure the reader that some results, which we will use, also hold in a general complete metric space.

Theorem 11 *Let $\mathcal{I} = \{I \times \mathcal{K}_c(\mathbb{R}); W_1, \ldots W_{N-1}\}$ be the IFS defined in Theorem 10 such that*

$$
r_i D_{\mathcal{G}}\big((u, A), (w, B)\big) \le D_{\mathcal{G}}\big(W_i(u, A), W_i(w, B)\big) \le R_i D_{\mathcal{G}}\big((u, A), (w, B)\big)
$$

for every $(u, A), (w, B) \in I \times \mathcal{K}_c(\mathbb{R})$, where $0 < r_i \le R_i < 1$ for all $i \in J$. Then, $t_* \le \dim_H(\mathcal{G}(F^{\alpha})) \le t^*$, where t_* and t^* are characterized by $\sum_{i=1}^{N} r_i^{t^*} = 1$ and $\sum_{i=1}^{N} R_i^{t^*} = 1$, respectively.

Proof For purposed upper bound one can refer [15, Proposition 9.6](see also, [33, Theorem 2.12]). For the lower bound of the Hausdorff dimension of $\mathcal{G}(F^{\alpha})$, we progress as follows.
Set $V = (u_1, u_N) \times \mathcal{K}_c(\mathbb{R})$, an open set in $I \times \mathcal{K}_c(\mathbb{R})$. Since for each $i, j \in J$ with $i \neq j$, we have

$$
L_j\big((u_1, u_N)\big) = (u_j, u_{j+1}) \text{ and } L_i\big((u_1, u_N)\big) \cap L_j\big((u_1, u_N)\big) = \emptyset,
$$

hence for each $i, j \in J$ and $i \neq j$, we have

$$
W_j(V) = (u_j, u_{j+1}) \times \mathcal{K}_c(\mathbb{R}) \text{ and } W_i(V) \cap W_j(V) = \emptyset.
$$

Therefore,

$$
\bigcup_{i=1}^{N-1} W_i(V) = \bigcup_{i=1}^{N-1} \big\{ (u_i, u_{i+1}) \times \mathcal{K}_c(\mathbb{R}) \big\} \subseteq V \text{ and } W_i(V) \cap W_j(V) = \emptyset.
$$

Then, by using Definition 9 IFS satisfies OSC. We have $V \cap \mathcal{G}(F^\alpha) \neq \emptyset$ this implies that IFS is satisfying SOSC. Since $V \cap \mathcal{G}(F^\alpha) \neq \emptyset$, we have an $i \in J^*$ such that $\mathcal{G}(F^\alpha)_i \subset V$, where $J^* = \underset{m\in\mathbb{N}}{\cup}\{1, \dots, N-1\}^m$, collection of all finite sequences whose terms are in J and

$$\mathcal{G}(F^\alpha)_i = W_i(\mathcal{G}(F^\alpha)) := W_{i_1} \circ W_{i_2} \circ \cdots \circ W_{i_m}(\mathcal{G}(F^\alpha))$$

for $i \in J^m = J \times \cdots \times J$ (m-times) and $m \in \mathbb{N}$. Observe that for any $j \in J^m$ and $k \in \mathbb{N}$, the sets, $\mathcal{G}(F^\alpha)_{j_i}$, are disjoint. Further, the IFS $\{W_{j_i} : j \in J^k\}$ satisfies the hypothesis of [15, Proposition 9.7] (see also, [33, Theorem 2.35]). Therefore, with the notation $r_j = r_{j_1} r_{j_2} \cdots r_{j_k}$ we have $t_k \leq \dim_H(G^*)$, where G^* is an attractor of the IFS and $\sum\limits_{j\in J^k} r_{j_i}^{t_k} = 1$. Since $G^* \subset \mathcal{G}(F^\alpha)$, $t_k \leq \dim_H(G^*) \leq \dim_H(\mathcal{G}(F^\alpha))$. Let if possible $\dim_H(\mathcal{G}(F^\alpha)) < t_*$, where $\sum\limits_{i=1}^{N} r_i^{t_*} = 1$. Then, $t_k < t_*$. Now, we have

$$\begin{aligned} r_i^{-t_k} &= \sum_{j\in J^k} r_j^{t_k} \geq \sum_{j\in J^k} r_j^{\dim_H(\mathcal{G}(F^\alpha))} = \sum_{j\in J^k} r_j^{t_*} r_j^{\dim_H(\mathcal{G}(F^\alpha))-t_*} \\ &\geq \sum_{j\in J^k} r_j^{t_*} r_{max}^{k(\dim_H(\mathcal{G}(F^\alpha))-t_*)} \\ &= r_{max}^{k(\dim_H(\mathcal{G}(F^\alpha))-t_*)}, \end{aligned}$$

where $r_{\max} = \max\{r_1, r_2, \dots, r_N\}$. Since $r_{\max} < 1$, $r_{max}^{k(\dim_H(\mathcal{G}(F^\alpha))-t_*)}$ tends to infinity as k tends to infinity, and therefore $r_i^{-t_k}$ is unbounded, which is a contradiction. Hence, $\dim_H(\mathcal{G}(F^\alpha)) \geq t_*$, which is the required result. □

The following theorem is an immediate application of the Theorem 11.

Theorem 12 *Consider $F : I \to \mathcal{K}_c(\mathbb{R})$ is a set-valued map. If $|\alpha| < \min\{a_i : i \in J\}$, then $\dim_H(\mathcal{G}(F^\alpha)) = 1$.*

Proof Using Proposition 2 for every pair $(u, A), (w, B) \in I \times \mathcal{K}_c(\mathbb{R})$, we have

$$D_{\mathcal{G}}(W_i(u, A), W_i(w, B)) \leq a_i D_{\mathcal{G}}((u, A), (w, B)) \text{ for } i \in J.$$

Since $\sum\limits_{i=1}^{N-1} a_i = 1$, then by Theorem 11, $\dim_H(\mathcal{G}(F^\alpha)) \leq 1$. This concludes the proof. □

Theorem 13 *If $F : [0, 1] \to \mathcal{K}(\mathbb{R})$ is a set-valued Lipschitz map having Lipschitz constant l and the graph of F is as defined in (5), then $\dim_H(\mathcal{G}(F)) = 1$.*

Proof To prove this theorem, it will be sufficient to define a bi-Lipschitz map between $[0, 1]$ and $\mathcal{G}(F)$. Define $T : [0, 1] \to \mathcal{G}(F)$ such that $T(u) = (u, F(u))$. Then, we have

$$D_{\mathcal{G}}(Tu, Tw) = D_{\mathcal{G}}((u, F(u)), (w, F(w)))$$

$$
\begin{aligned}
&= |u - w| + H_d(Fu, Fw) \\
&\le |u - w| + l|u - w| \\
&\le (1 + l)|u - w|, \\
\text{that is, } D_{\mathcal{G}}(Tu, Tw) &\le (1 + l)|u - w| \qquad (6)
\end{aligned}
$$

and

$$
\begin{aligned}
D_{\mathcal{G}}(Tu, Tw) &= D_{\mathcal{G}}((u, Fu), (w, Fw)) \\
&= |u - w| + H_d(Fu, Fw) \\
\text{that is, } D_{\mathcal{G}}(Tu, Tw) &\ge \frac{1}{2}|u - w|. \qquad (7)
\end{aligned}
$$

Equations (6) and (7) will prove the bi-Lipschitz nature of T.
Hence, $\dim_H(\mathcal{G}(F)) = 1$. □

Theorem 14 *Let $F, S \in \mathcal{C}(I, \mathcal{K}_c(\mathbb{R}))$ are Lipschitz functions such that $S(u_1) - F(u_1) = S(u_N) - F(u_N)$, and let $\alpha \in (-1, 1)$. Then, $\dim_H(\mathcal{G}(F^{\alpha})) = 1$ provided that $|\alpha| < a := \min\{a_j : j \in J\}$.*

Proof In view of Theorems 13 and 2, the proof follows; hence we omit. □

Lemma 7 *Let $F, T : [0, 1] \to \mathcal{K}(\mathbb{R})$ be set-valued Lipschitz maps with Lipschitz constant l, then $\dim_H(\mathcal{G}(F+T)) = \dim_H(\mathcal{G}(T))$, where $(F+T)(u) := F(u)+T(u)$ and $F(u) + T(u)$ denotes the Minkowski sum of $F(u)$ and $T(u)$.*

Proof To establish the proof of this lemma, it will be sufficient to show the existence of a Lipschitz map from $\mathcal{G}(T)$ to $\mathcal{G}(F + T)$. Define $\Phi : \mathcal{G}(T) \to \mathcal{G}(F + T)$ such that $\Phi(u, T(u)) = (u, F(u) + T(u))$. It is easy to see that Φ is well defined and onto. Now to get its Lipschitz behavior, we have

$$
\begin{aligned}
D_{\mathcal{G}}(\Phi(u, T(u)), \Phi(w, T(w))) &= D_{\mathcal{G}}((u, F(u) + T(u)), (w, F(w) + T(w))) \\
&= |u - w| + H_d(F(u) + T(u), F(w) + T(w)) \\
&\le |u - w| + H_d(F(u), F(w)) + H_d(T(u), T(w)) \\
&\le |u - w| + l|u - w| + H_d(T(u), T(w)) \\
&\le (1 + l) \{|u - w| + H_d(T(u), T(w))\}.
\end{aligned}
$$

That is, $D_{\mathcal{G}}(\Phi(u, T(u)), \Phi(w, T(w))) \le (1 + l)D_{\mathcal{G}}((u, T(u)), (w, T(w)))$. Hence, Φ being a Lipschitz map implies that

$$
\dim_H(\mathcal{G}(F + T)) \le \dim_H(\mathcal{G}(T)) \text{ and } \dim_B(\mathcal{G}(F + T)) \le \dim_B(\mathcal{G}(T)).
$$

For another side of the inequality, let $t > \dim_H(\mathcal{G}(F + T))$. Then, by definition of the Hausdorff dimension, we have the following.

For each $\epsilon > 0$ and for each $\eta > 0$, there is an open cover $\{U_n : n \in \mathbb{N}\}$ of $\mathcal{G}(F+T))$ such that $|U_n| < \eta$ and $\sum_{n\in\mathbb{N}} |U_n|^s < \epsilon$. Note that

$$\begin{aligned} \mathcal{G}(T) &= \{(u, T(u)) : u \in I\} \\ &\subseteq \{(u, T(u) + F(u) - F(u)) : u \in I\} \\ &\subseteq \{(u, T(u) + F(u)) : u \in I\} + \{(0, -F(u)) : u \in I\} \\ &= \mathcal{G}(F+T)) + \{(0, -F(u)) : u \in I\}. \end{aligned} \tag{8}$$

Define $V_n = U_n + \{(0, -F(u)) : u \in I$ such that $(u, T(u) + F(u)) \in U_n\}$. Observe that each V_n is open and $\mathcal{G}(T) \subseteq \bigcup_{n\in\mathbb{N}} V_n$. Further, $|V_n| \le (1+l)|U_n|$ and $|V_n| < (1+l)\eta$. Then,

$$\sum_{n\in\mathbb{N}} |V_n|^t \le (1+l)^t \sum_{n\in\mathbb{N}} |U_n|^t \le (1+l)^t \epsilon.$$

This gives $\mathcal{H}^t_\eta(\mathcal{G}(T)) = 0$, that is, the t-dimensional Hausdorff measure, $\mathcal{H}^t(\mathcal{G}(T)) = 0$. Therefore, we have $\dim_H(\mathcal{G}(T)) \le \dim_H(\mathcal{G}(F+T))$, proving $\dim_H(\mathcal{G}(T)) = \dim_H(\mathcal{G}(F+T))$. Next, using (8), we obtain

$$\begin{aligned} \overline{\dim}_B(\mathcal{G}(T)) &= \overline{\lim_{\delta\to 0}} \frac{\log N_{(1+l)\eta}(\mathcal{G}(T))}{-\log((1+l)\eta)} \\ &\le \overline{\lim_{\eta\to 0}} \frac{\log N_\eta(\mathcal{G}(F+T))}{-\log((1+l)\eta)} = \overline{\lim_{\eta\to 0}} \frac{\log N_\eta(\mathcal{G}(F+T))}{-\log(\eta)} = \overline{\dim}_B(\mathcal{G}(F+T)), \end{aligned}$$

as desired. □

Remark 7 The above lemma holds for single-valued maps also (see, for instance, [34, Lemma 3.2]), but proof of this is neither straightforward nor just a simple extension of the single-valued map. Because Hausdorff metric does not satisfy the parallelogram law while in [34, Lemma 3.2] metric is the usual metric defined on $\mathbb{R}^n$ which satisfies the parallelogram law and gives the privilege to enjoy the bi-Lipschitz property to T defined in [34, Lemma 3.2] (Φ in our case).

In view of the Lipschitz invariance property of dimension, one may conclude that the upcoming theorem holds for all aforementioned dimensions.

Theorem 15 *Consider* $1 \le \beta$. *Then, set* $\mathcal{S}_\beta := \{F \in \mathcal{C}(I, \mathcal{K}(\mathbb{R})) : \dim_H(\mathcal{G}(F)) = \beta\}$ *is dense in* $\mathcal{C}(I, \mathcal{K}(\mathbb{R}))$.

Proof Let $F \in \mathcal{C}(I, \mathcal{K}(\mathbb{R}))$ and $\epsilon > 0$. Using the density of $\mathcal{L}ip(I, \mathcal{K}(\mathbb{R}))$ in $\mathcal{C}(I, \mathcal{K}(\mathbb{R}))$, there exists G in $\mathcal{L}ip(I, \mathcal{K}(\mathbb{R}))$ such that

$$\|F - G\|_\infty < \frac{\epsilon}{2}.$$

Further, we consider a non-vanishing function $H \in \mathcal{S}_\beta$. Let $H_* = G + \frac{\epsilon}{2\|H\|_\infty} H$, which immediately gives

$$\|G - H_*\|_\infty \leq \frac{\epsilon}{2}.$$

This together with Lemma 7 implies that $\dim(Gr(H_*)) = \dim(Gr(H)) = \beta$. Hence, we have $H_* \in \mathcal{S}_\beta$ and

$$\|F - H_*\|_\infty \leq \|F - G\|_\infty + \|G - H_*\|_\infty < \epsilon.$$

This completes the proof. □

Before proving our next result, let us note the following lemma as a prelude.

Lemma 8 *Consider A, B, C are compact subsets of $\mathbb{R}$. Then,*

$$H_d(AB, CB) \leq \sup_{b \in B} |b| H_d(A, C),$$

where $YZ = \{yz : y \in Y \in \mathcal{K}(\mathbb{R}),\ z \in Z \in \mathcal{K}(\mathbb{R})\}$.

Proof We have

$$\begin{aligned}
H_d(AB, CB) &= \max\left\{\sup_{ab \in AB} \inf_{cb' \in CB} |ab - cb'|, \sup_{cb' \in CB} \inf_{ab \in AB} |cb' - ab|\right\} \\
&\leq \max\left\{\sup_{ab \in AB} \inf_{cb \in Cb} |ab - cb|, \sup_{cb' \in CB} \inf_{ab' \in Ab'} |cb' - ab'|\right\} \\
&\leq \max\left\{\sup_{ab \in AB} \inf_{cb \in Cb} |b||a - c|, \sup_{cb' \in CB} \inf_{ab' \in Ab'} |b'||c - a|\right\} \\
&\leq \max\left\{\sup_{a \in A, b \in B} (|b| \inf_{cb' \in CB} |a - c|), \sup_{c \in C, b' \in B} (|b'| \inf_{ab' \in Ab'} |c - a|)\right\} \\
&\leq \sup_{b \in B} |b| \max\left\{\sup_{a \in A} \inf_{c \in C} |a - c|, \sup_{c \in C} \inf_{a \in A} |c - a|\right\} \\
&= \sup_{b \in B} |b| H_d(A, C),
\end{aligned}$$

proving the assertion. □

Next we define the multiplication of set-valued maps $F, L : W \subseteq \mathbb{R} \rightrightarrows \mathbb{R}$ by $(FT)(w) = F(w)T(w)$.

Lemma 9 *Consider $F, T : [0, 1] \to \mathcal{K}(\mathbb{R})$ to be set-valued Lipschitz maps with Lipschitz constant l. Then,*

$$\dim_H(\mathcal{G}(FT)) \leq \dim_H(\mathcal{G}(T)).$$

Proof Define $\Phi : \mathcal{G}(T) \to \mathcal{G}(FT)$ such that

$$\Phi\big((u, T(u))\big) = \big(u, F(u)T(u)\big).$$

Choose $M = \max\{1 + l \sup_{z \in \bigcup_{u\in[0,1]} Tu} |z|, \sup_{v \in \bigcup_{w\in[0,1]} Fw} |v|\}$.

Notice that Φ is well-defined and surjective. To prove our lemma, it is enough to prove Φ is a Lipschitz map. For this,

$$\begin{aligned} D_{\mathcal{G}}(\Phi(u, Tu), \Phi(w, Tw)) &= D_{\mathcal{G}}((u, FuTu), (w, FwTw)) \\ &= |u-w| + H_d(FuTu, FwTw) \\ &\leq |u-w| + H_d(FuTu, FwTu) + H_d(FwTu, FwTw) \\ &\leq |u-w| + \sup_{z\in Tu} |z| H_d(Fu, Fw) + \sup_{v\in Fw} |v| H_d(Tu, Tw) \\ &\leq |u-w| + \sup_{z\in Tu} |z| l |u-w| + \sup_{v\in Fw} |v| H_d(Tu, Tw) \\ &\leq M\left\{|u-w| + H_d(Tu, Tw)\right\}. \end{aligned}$$

Hence, $D_{\mathcal{G}}(\Phi(u, Tu), \Phi(w, Tw)) \leq M D_{\mathcal{G}}((u, Tu), (w, Tw))$.
This completes the proof. □

Remark 8 In the Lemma 9, equality may not generally hold. For instance, consider T to be a Weierstrass function whose Hausdorff dimension is strictly greater than 1 (refer [32]) and F to be the zero function. Then, we obtain $1 = \dim_H(\mathcal{G}(FT)) < \dim_H(\mathcal{G}(T))$.

Definition 13 Consider W be a bounded and closed interval of $\mathbb{R}$ and $F : W \rightrightarrows \mathbb{R}$ is a set-valued map. The maximum range of F over the rectangle W is defined as

$$R_F[W] = \sup_{x,y\in W} \sup_{w,z\in F(x)\cup F(y)} |w - z|.$$

As indicated in the introductory section, next, we shall provide a set-valued analog of [15, Proposition 11.1].

Proposition 3 *Assume $F : [w, u] \rightrightarrows \mathbb{R}$ be a set-valued continuous map, $0 < \eta < u - w$, and $\frac{u-w}{\eta} \leq m \leq 1 + \frac{u-w}{\eta}$ for some $m \in \mathbb{N}$. If $N_\eta(G_F)$ is the number of η-boxes that intersect the graph of F, then*

$$\frac{1}{\eta}\sum_{i=1}^{m} R_F[W_i] \leq N_\eta(G_F) \leq 2m + \frac{1}{\eta}\sum_{i=1}^{m} R_F[W_i],$$

where $W_i = [i\eta, (i+1)\eta]$.

Proof The count of squares having η side length in the part above W_i intersecting the graph of F is at least $\frac{R_F[W_i]}{\eta}$ and at most $2 + \frac{R_F[W_i]}{\eta}$, using the continuity of F. Taking sum over all such parts yield the required bounds. □

Example 4 Consider $F : [0, 1] \rightrightarrows \mathbb{R}$ is a set-valued map defined as $F(x) = [-1, 1]$. By Proposition 3, we have

$$\begin{aligned}\overline{\dim}_B(G_F) = \overline{\lim_{\eta\to 0}} \frac{\log N_\eta(G_F)}{-\log(\eta)} &\leq \overline{\lim_{\eta\to 0}} \frac{\log\left(2m + \frac{1}{\eta}\sum_{i=1}^{m} R_F[W_i]\right)}{-\log(\eta)} \\ &\leq \overline{\lim_{\eta\to 0}} \frac{\log\left(2m + \frac{1}{\eta}\sum_{i=1}^{m} 2\right)}{-\log(\eta)} = 2,\end{aligned}$$

because $R_F[W_i] = 2$ for each $i = 1, \ldots, m$ and $W_i = [i\eta, (i+1)\eta]$. Similarly,

$$\underline{\dim}_B(G_F) = \underline{\lim_{\eta\to 0}} \frac{\log N_\eta(G_F)}{-\log(\eta)} \geq \underline{\lim_{\eta\to 0}} \frac{\log\left(\frac{1}{\eta}\sum_{i=1}^{m} R_F[W_i]\right)}{-\log(\eta)} = \underline{\lim_{\eta\to 0}} \frac{\log\left(\frac{1}{\eta}\sum_{i=1}^{m} 2\right)}{-\log(\eta)} = 2.$$

Therefore, $\dim_B(G_F) = 2$. This shows that Proposition 3 will be very useful in estimating or finding box dimensions of set-valued functions.

6 Conclusion and Future Direction

In this paper, the term α-fractal function has been introduced (Theorem 1), corresponding to set-valued maps. Next, we noticed that, unlike a single-valued α-fractal function, a set-valued α-fractal function is generally not interpolatory. Still, under certain conditions, it is interpolatory in nature (Remark 2, Note 3). Also, some properties of this fractal function have been observed (Theorems 2, 4). After that, the existence of a fractal polynomial, which approximates the convex set-valued map, was established (Theorem 7). Also, the concept of constrained approximation for set-valued maps is introduced (Theorem 9). Further, we added the definition of a graph of a set-valued map (Definition 12) and calculated the fractal dimension of this graph for some class of set-valued maps (Theorem 13, Lemmas 7, and 9).

In this paper, we have taken Minkowski's sum of two sets. In the future, we may study the fractal functions using the metric linear sum of two sets introduced by Dyn and her group [9]. Here most of the results are available for convex set-valued maps, but using the metric linear sum of two sets, we may try to establish these results for the compact set-valued map.

Further, fractional calculus for the single-valued map has been widely explored. See, for instance, [12, 19]. In the future, we may try to extend this concept of fractional calculus for set-valued maps and estimate some dimensional results for the graph of the fractional integral and fractional differentiation of set-valued maps.

Another future direction of work is in the selection of set-valued maps. The following remark can work as a motivation.

Remark 9 Consider $F : I \to \mathcal{K}(\mathbb{R})$ to be a set-valued function, then a function $f : I \to \mathbb{R}$ will be characterized as a selection of F if $f(x) \in F(x)$ for all $x \in I$. It is an interesting fact to note that for any $1 \leq \beta \leq 2$, we are getting a selection $f_\beta : I \to \mathbb{R}$ of the map F_2 of Example 2 such that $\dim_H(Gr(f_\beta)) = \beta$. This motivates us to ask a natural question of whether such a selection respecting dimension exists or not.

Moreover, we have worked in this paper by taking the same scaling factor $\alpha_n = \alpha$ for all n. One may also try to generalize the result for different scaling factors, and also one may try to generalize the result for non-constant scaling factors.

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Declaration

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