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WATSON WAVELET TRANSFORM: CONVOLUTION PRODUCT AND TWO-WAVELET MULTIPLIERS

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Abstract

In this paper, utilizing the theory of Watson transform and Watson convolution, we explore the Watson wavelet convolution product and its related properties. The relation between the Watson Wavelet convolution product and Watson convolution is also computed. Watson wavelet transform and its inversion formula are analyzed heuristically. Watson two-wavelet multipliers and its trace class are derived from Watson wavelet convolution product

Keywords: Pseudo-Differential Operators; Watson Transform; Convolution Operator; Continuous Watson Wavelet; Wavelet Multiplier; Unitary Representation; Sobolev Space.

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1. INTRODUCTION

The Wavelet transform is of great importance for the analysis of non-stationary signals and provides information of time-frequency representation at a time. Many researchers exploited the theory of the wavelet transform and applied their research works in the areas of mathematical sciences and engineering. In this connection, we refer to Refs. 1–3 regarding the applications of wavelet transform. The concept of the wavelet transform is heavily dependent on the theory of convolution and each integral transform has its own convolution with its rich calculus. Using the theory of convolutions of different integral transforms, many problems of the wavelet transform have been solved by many mathematicians. Using the convolution theory of the Fourier transform, Wong[4] discussed the boundedness of wavelet multipliers and signals. In 2000, Du and Wong[5] obtained the traces of localization operator on a separable complex Hilbert space and got many important results. In 2001, Du and Wong[6] observed trace formula for wavelet multipliers as a bounded linear operator in the trace class from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$ and were able to compute the trace of n-dimensional Landau Pollak Slepian operators. Wong and Zhang[7] studied the traces of two-wavelet multipliers. In 2002, Wong[8] proved an L^p-boundedness result for the localization operator associated to left regular representations of locally compact and Hausdorff groups and gave an application of the wavelet multiplier. In 2003, Wong and Zhang[9] gave the resolution of the identity formula for a localization operator with two admissible wavelets on a separable and complex Hilbert space and the traces of these operators. Wong *et al.*,[10] examined the boundedness and compactness of the localization operator on various functional spaces in terms of the wavelet multiplier. In the same year, Pinsky[11] developed the Heuristic treatment of the wavelet transform and its inversion formula by exploiting the theory of the Fourier transform.

The motivations of theory of the Hankel transform can be seen in the next paragraph:

For the consideration of the Hankel transform, the theory of Hankel convolution is introduced by Haimo,[12] Hirschmann[13] and many others. Using this theory, Pathak and Dixit[14] introduced the Bessel wavelet transform and studied many properties. Motivated from the results of Refs. 14 and 15 many authors extensively studied the characterizations of Bessel wavelet transform on certain functional spaces and applied this theory in Sobolev spaces and other problems of mathematics. Taking the Hankel transform theory, Pathak *et al.*[16] considered the Bessel wavelet convolution product and its properties. Upadhyay *et al.*[17] found the relation between Bessel wavelet convolution product and Hankel convolution and other results.

The theory of the Watson transform came into the light from the book of Titchmarsh.[18] Motivated by Ref. 18, Watson convolution is defined in Refs. 19 and 20. Later on Brakshma and Schuitman[21] studied Watson transform in more detail. As per book of Schuitman,[22] it was found that Laplace transform, Hankel transform and Fourier transform are examples of the Watson transform. Later on in 2011, exploiting the theory of Watson convolution, Upadhyay and Tripathi[20] introduced continuous Watson wavelet transform and its various properties. Pathak,[15] introduced the idea of wavelet convolution product with the help of the Fourier transform techniques. Using this idea, he defined wavelet convolution and discussed various properties. This theory is useful to find the solution of many problems of wavelet transform.

Motivated from the above results and concepts, our main objective in this paper is to study the relation between Watson wavelet convolution product and Watson convolution by exploiting the theory of Watson transform. Later on, we shall also find the relation between Watson wavelet convolution product and Watson two-wavelet multipliers involving the Watson transform. Using the same technique, the relation between trace class of Watson two-wavelet multipliers and Watson wavelet convolution product will be established.

This paper is organized by the following way:

Section 1 is introductory, which gives the brief history and motivations of the entire research work. Section 2 provides the proper background including definitions, formulae and properties of our present paper. In Sec. 3, Watson wavelet convolution product is formally defined after that authors found the relation between Watson wavelet convolution product and Watson convolution and its various properties. In Sec. 4, the authors used the results of Pinsky[11] and found the Heuristic treatment of the Watson wavelet transform and its inversion formula by exploiting the theory of the Watson transform. In Sec. 5, taking Watson wavelet transform, motivated from the result of Wong,[7] two-wavelet multipliers are introduced and their various properties studied. The relation between Watson wavelet convolution and two-wavelet multipliers is obtained by the authors.

2. PRELIMINARIES

In this section, we discussed definitions of Watson transform, Watson convolution, Watson wavelet transform and its various properties and formulae. The notations and terminologies are taken from the following sources, Refs. 19, 20 and 23, — which are useful for our further investigations.

The Watson transform of function $f \in L^1(\mathbb{R}^+)$ is defined by

$$(Wf)(x) = \int_0^\infty k(xt)f(t)dt. \tag{1}$$

If $f \in L^1(\mathbb{R}^+)$ and $Wf \in L^1(\mathbb{R}^+)$, then inversion of the Watson transform is

$$f(t) = \int_0^\infty k(xt)(Wf)(x)dx, \tag{2}$$

where $k(x)$ is called kernel of the Watson transform and it is also the inverse Mellin transform of K, see Ref. 21 in p. 772, which would be in the following form:

$$(M^{-1}K)(x) = k(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} K(s)x^{-s}ds \tag{3}$$

and $s = c + it$, where $K(s)$ be an analytic function on $\lambda < \text{Res} < \mu$ such that $K(c+it) \in L^1(-\infty,\infty)$ for some c with $\lambda < c < \mu$. Assume further that for every pair (a,b) such that $K(s) = O(s^\gamma)$ as $|s| \to \infty$, uniformly on $a \le \text{Res} \le b$, for $\lambda,\mu \in \mathbb{R}^*, \lambda < \mu$.

We denote $\mathbb{R}^* = \mathbb{R} \cup \{-\infty,\infty\}$, $\mathbb{C}$ is the set of complex numbers and $\mathbb{N}_0 = \{0,1,2,\ldots\}$. Let $\lambda,\mu \in \mathbb{R}^*, \lambda < \mu$. Let $\{\lambda_n\}_{n=0}^\infty$ and $\{\mu_n\}_{n=0}^\infty$ be sequence of real numbers with $\lambda_n \downarrow \lambda, \mu_n \uparrow \mu$ and $\lambda_n < \mu_n$ for all $n \in \mathbb{N}_0$. Then $T(\lambda,\mu)$ is the space of all functions $\phi \in C^\infty(0,\infty)$ with the property that

$$\beta_n(\phi) = \sup_{\substack{t>0 \\ \rho=0,1,2,\ldots,n \\ \lambda_n \le c \le \mu_n}} |t^{c+p}\phi^p(t)| < \infty \quad \forall\, n \in \mathbb{N}_0. \tag{4}$$

The basic function is defined by

$$w(h,g,t) = \int_0^\infty k(h\xi)k(g\xi)k(t\xi)d\xi, \tag{5}$$

provided integral (5) being convergent under the assumption $k \in L^1(0,\infty) \cap L^\infty(0,\infty)$ and assume that $k(0) = 1$ and $w(h,g,t) > 0$ for every $h,g,t \in I = (0,\infty)$.

From Pathak and Dixit[14] in p. 242, and by inversion formula, we have

$$k(h\xi)k(g\xi) = \int_0^\infty w(h,g,t)k(t\xi)dt, \tag{6}$$

for $0 < x,y < \infty$, $0 \le \xi < \infty$.

Setting $\xi = 0$ in (6), we obtain

$$\int_0^\infty w(h,g,t)dt = 1. \tag{7}$$

Using (5)–(7), the Watson translation is given by

$$f(h,g) = (\tau_g f) = \int_0^\infty f(t)w(h,g,t)dt. \tag{8}$$

Let $f \in L^1(\mathbb{R}^+)$ and $\psi \in L^1(\mathbb{R}^+)$. Then the Watson convolution is defined by

$$(f\#\psi)(h) = \int_0^\infty f(h,g)\psi(g)dg \tag{9}$$

and

$$(f\#\psi)(h) = \int_0^\infty\int_0^\infty f(t)\psi(g)w(h,g,t)dtdg. \tag{10}$$

With the help of Ref. 20, we give the definition of Watson wavelet for $\psi \in L^p(\mathbb{R}^+)$,

$$\psi_{b,a}(x) = D_a\tau_b\psi(x) = D_a\psi(b,x) = \frac{1}{a}\psi\left(\frac{b}{a},\frac{x}{a}\right). \tag{11}$$

From (8), we have

$$\psi_{b,a}(x) = \frac{1}{a}\int_0^\infty w\left(\frac{b}{a},\frac{x}{a},z\right)\psi(z)dz, \tag{12}$$

where $b \ge 0$ and $a > 0$.

Using (12), the Watson wavelet transform is defined by

$$\begin{aligned} W(b,a) &= (W_\psi\phi)(b,a) \\ &= \langle \phi(t), \psi_{b,a}(t)\rangle \qquad (13)\\ &= \int_0^\infty \phi(t)\overline{\psi_{b,a}(t)}dt \\ &= \int_0^\infty \phi(t)\left(\frac{1}{a}\int_0^\infty w\left(\frac{b}{a},\frac{t}{a},z\right)\overline{\psi(z)}dz\right)dt \\ &= \frac{1}{a}\int_0^\infty\int_0^\infty \phi(t)w\left(\frac{b}{a},\frac{t}{a},z\right)\overline{\psi(z)}dzdt, \qquad (14)\end{aligned}$$

provided the integral is convergent.

With the help of Ref. 20 in p. 644, we see that above integral is convergent for $\phi \in L^p(0,\infty)$ and $\psi \in L^q(0,\infty)$,

$$|(W_\psi \phi)(b,a)| \leq \|\phi\|_p \|\bar{\psi}_a\|_q < \infty.$$

If ϕ and ψ are in $L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$, then the following Parseval formula holds:

$$\int_0^\infty (W\phi)(t)(W\psi)(t)dt = \int_0^\infty \phi(x)\psi(x)dx. \tag{15}$$

From Ref. 20 in p. 643, the continuous Watson wavelet transform of a function $\phi, \psi \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$,

$$(W_\psi \phi)(b,a) = \int_0^\infty k(wb)(W\phi)(w)\overline{(W\psi)(aw)}dw. \tag{16}$$

For $f \in L^2(\mathbb{R}^+)$ and $g \in L^2(\mathbb{R}^+)$, we state the Parseval formula of the Watson wavelet transform as

$$\int_0^\infty\int_0^\infty \frac{(W_\psi f)(b,a)\overline{(W_\psi g)(b,a)}dadb}{a} = C_\psi\langle f,g\rangle, \tag{17}$$

where C_ψ satisfies the admissible condition for $\psi \in L^2(\mathbb{R}^+)$, which is

$$C_\psi = \int_0^\infty \frac{|(W\psi)(\omega)|^2}{\omega}d\omega. \tag{18}$$

Remark: The Watson transform is generalization of Hankel transform. Exploiting the theory of the Hankel transform, Bessel wavelet convolution product was studied in Refs. 14, 17 and 20. So this paper is more general than Refs. 14, 17 and 20.

3. WATSON WAVELET CONVOLUTION PRODUCT

In this section, Watson wavelet convolution product is introduced and its associated results are obtained by exploiting the theory of the Watson transform. We also find the relation between Watson wavelet convolution product and Watson convolution.

For finding the properties of Watson wavelet convolution product, we formally defined

$$W_\psi(f\otimes g)(b,a) = (W_\psi f)(b,a)(W_\psi g)(b,a). \tag{19}$$

Theorem 1. *Let $f,g \in L^1(\mathbb{R}^+)\cap L^2(\mathbb{R}^+)$ and $(W\psi)(\omega)\neq 0$. Then the Watson wavelet convolution product can be written in the following form:*

$$(f\otimes g)(z) = \int_0^\infty (\tau_{z,a}f)(y)g(y)dy,$$

where

$$(\tau_{z,a}f)(y) = \int_0^\infty f(x)W_a(x,y,z)dx,$$

$$W_a(x,y,z) = \int_0^\infty\int_0^\infty k(y\xi)k(xt)(W\psi)(at)(W\psi)(a\xi) \times L_a(t,\xi,z)dtd\xi,$$

$$L_a(t,\xi,z) = \int_0^\infty k(yt)k(y\xi)Q_a(y,z)dy$$

and

$$Q_a(y,z) = \int_0^\infty \frac{k(\omega z)k(\omega y)}{(W\psi)(a\omega)}d\omega.$$

Proof. From (16), we have

$$W[(W_\psi\phi)(b,a)](\omega) = \overline{(W\psi)}(a\omega)(W\phi)(\omega).$$

Putting $\phi = f\otimes g$, in the above expression

$$W[W_\psi(f\otimes g)(b,a)](\omega) = \overline{(W\psi)}(a\omega)(W(f\otimes g))(\omega).$$

Thus,

$$\overline{(W\psi)}(a\omega)(W(f\otimes g))(\omega) = W[W_\psi(f\otimes g)(b,a)](\omega).$$

Using (19), we have

$$\overline{(W\psi)}(a\omega)(W(f\otimes g))(\omega) = W[W_\psi f(b,a)W_\psi g(b,a)](\omega). \tag{20}$$

In view of (16), we get

$$\overline{(W\psi)}(a\omega)W(f\otimes g)(\omega) = W\{W^{-1}[(Wf)(\omega)\overline{(W\psi)}(a\omega)](b) \times W^{-1}[(Wg)(\omega)\overline{(W\psi)}(a\omega)](b)\}. \tag{21}$$

By the definition of Watson convolution, we get

$$\begin{aligned}\overline{(W\psi)}(a\omega)W(f\otimes g)(\omega) &= W(W^{-1}\{(Wf)(\omega)(\overline{W\psi}(a\omega))\#(Wg)(\omega) \\ &\quad \times (\overline{W\psi}(a\omega))\}(b)) \\ &= (Wf)(\omega)(\overline{W\psi})(a\omega)\#(Wg)(\omega)(\overline{W\psi})(a\omega). \end{aligned} \tag{22}$$

Let $F_a = (Wf)(\omega)\overline{(W\psi)}(a\omega)$ and $G_a = (Wg)(\omega)\overline{(W\psi)}(a\omega)$.

Then we have

$$\begin{aligned}\overline{(W\psi)}(a\omega)W(f\otimes g)(\omega) &= (F_a\#G_a)(\omega) \\ &= \int_0^\infty\int_0^\infty F_a(\eta)G_a(\xi)w(\omega,\xi,\eta)d\xi d\eta.\end{aligned}$$

From (5), we write

$$\begin{aligned}&(W\psi)(a\omega)W(f\otimes g)(\omega) \\ &= (F_a\#G_a)(\omega) \\ &= \int_0^\infty\int_0^\infty F_a(\eta)G_a(\xi)\left(\int_0^\infty k(\omega y)\right. \\ &\quad \left.\times k(\xi y)k(\eta y)dy\right)d\xi d\eta. \\ &= \int_0^\infty\left(\int_0^\infty F_a(\eta)k(\eta y)d\eta\right) \\ &\quad \times\left(\int_0^\infty G_a(\xi)k(\xi y)d\xi\right)k(\omega y)dy. \\ &= \int_0^\infty (WF_a)(y)(WG_a)(y)k(\omega y)dy.\end{aligned}$$

Therefore,

$$\begin{aligned}W(f\otimes g)(\omega) &= \int_0^\infty \frac{1}{(W\psi)(a\omega)}(WF_a)(y) \\ &\quad \times (WG_a)(y)k(\omega y)dy.\end{aligned}$$

By inversion formula of Watson transform

$$\begin{aligned}(f\otimes g)(z) &= \int_0^\infty \frac{k(\omega z)}{(W\psi)(a\omega)}\left(\int_0^\infty (WF_a)(y)\right. \\ &\quad \left.\times (WG_a)(y)k(\omega y)dy\right)d\omega \\ &= \int_0^\infty (WF_a)(y)(WG_a)(y) \\ &\quad \times\left(\int_0^\infty \frac{k(\omega z)k(\omega y)}{(W\psi)(a\omega)}d\omega\right)dy \\ &= \int_0^\infty (WF_a)(y)(WG_a)(y)Q_a(y,z)dy,\end{aligned}$$

where

$$Q_a(y,z) = \int_0^\infty \frac{k(\omega z)k(\omega y)}{(W\psi)(a\omega)}d\omega$$

and $(W\psi)(a\omega)\neq 0$,

$$\begin{aligned}&(f\otimes g)(z) \\ &= \int_0^\infty\left(\int_0^\infty k(yt)(W\psi)(at)(Wf)(t)dt\right) \\ &\quad \times\left(\int_0^\infty k(y\xi)(W\psi)(a\xi)(Wg)(\xi)d\xi\right) \\ &\quad \times Q_a(y,z)dy \\ &= \int_0^\infty\int_0^\infty (W\psi)(at)(W\psi)(a\xi)(Wf)(t) \\ &\quad \times (Wg)(\xi)\left(\int_0^\infty k(yt)k(y\xi)Q_a(y,z)dy\right) \\ &\quad \times dtd\xi \\ &= \int_0^\infty\int_0^\infty (W\psi)(at)(W\psi)(a\xi)(Wf)(t) \\ &\quad \times (Wg)(\xi)L_a(t,\xi,z)dtd\xi,\end{aligned}$$

where $L_a(t,\xi,z) = \int_0^\infty k(yt)k(y\xi)Q_a(y,z)dy$.

Therefore,

$$\begin{aligned}(f\otimes g)(z) &= \int_0^\infty\int_0^\infty (W\psi)(at)(W\psi)(a\xi) \\ &\quad \times\left(\int_0^\infty k(xt)f(x)dx\right) \\ &\quad \times\left(\int_0^\infty k(y\xi)g(y)dy\right)L_a(t,\xi,z)dtd\xi \\ &= \int_0^\infty\int_0^\infty f(x)g(y)\left(\int_0^\infty\int_0^\infty k(y\xi)k(xt)\right. \\ &\quad \left.\times (W\psi)(at)(W\psi)(a\xi)L_a(t,\xi,z)dtd\xi\right) \\ &\quad \times dxdy \\ &= \int_0^\infty\int_0^\infty f(x)g(y)W_a(x,y,z)dxdy.\end{aligned}$$

Therefore, we get

$$\begin{aligned}(f\otimes g)(z) &= \int_0^\infty\int_0^\infty f(x)g(y)W_a(x,y,z)dxdy \\ &= \int_0^\infty (\tau_{z,a}f)(y)g(y)dy. \qquad \square\end{aligned}$$

Theorem 2. *Let $f,g\in L^1(\mathbb{R}^+)\cap L^2(\mathbb{R}^+)$. Then the Watson convolution product can be written in* *the following form*:

$$E_\psi[W(f\otimes g)](\omega) = \int_0^\infty\int_0^\infty (Wf)(\eta)(Wg)(\xi)w(\omega,\eta,\xi) \times\left(\int_0^\infty \overline{(W\psi)}(a\eta)\overline{(W\psi)}(a\xi)\frac{da}{a}\right)d\eta d\xi, \tag{23}$$

where

$$E_\psi = \int_0^\infty \frac{\overline{(W\psi)}(a\omega)}{a}da.$$

Proof. From (16), we have

$$W[(W_\psi\phi)(b,a)](\omega) = \overline{(W\psi)}(a\omega)(W\phi)(\omega). \tag{24}$$

Putting $\phi = f\otimes g$, in the above expression

$$W[W_\psi(f\otimes g)(b,a)](\omega) = \overline{(W\psi)}(a\omega)(W(f\otimes g))(\omega). \tag{25}$$

Thus,

$$\overline{(W\psi)}(a\omega)(W(f\otimes g))(\omega) = W[W_\psi(f\otimes g)(b,a)](\omega). \tag{26}$$

Using (19), we have

$$\overline{(W\psi)}(a\omega)(W(f\otimes g))(\omega) = W[(W_\psi f)(b,a)(W_\psi g)(b,a)](\omega). \tag{27}$$

In view of (16), we get

$$\overline{(W\psi)}(a\omega)W(f\otimes g)(\omega) = W\{W^{-1}[(Wf)(\omega)\overline{(W\psi)}(a\omega)](b) \times W^{-1}[(Wg)(\omega)\overline{(W\psi)}(a\omega)](b)\}. \tag{28}$$

By the definition of Watson convolution, we get

$$\begin{aligned}\overline{(W\psi)}(a\omega)W(f\otimes g)(\omega) &= W(W^{-1}\{(Wf)(\omega)(\overline{W\psi}(a\omega))\#(Wg)(\omega) \times(\overline{W\psi}(a\omega))\}(b))\\ &= (Wf)(\omega)(\overline{W\psi})(a\omega)\#(Wg)(\omega)(\overline{W\psi})(a\omega).\end{aligned}$$

Let $F_a = (Wf)(\omega)\overline{(W\psi)}(a\omega)$ and $G_a = (Wg)(\omega)\overline{(W\psi)}(a\omega)$.

Then we have

$$\overline{(W\psi)}(a\omega)W(f\otimes g)(\omega) = (F_a\#G_a)(\omega) \tag{29}$$

$$= \int_0^\infty\int_0^\infty F_a(\eta)G_a(\xi)w(\omega,\xi,\eta)d\xi d\eta. \tag{30}$$

Putting the value of F_a and G_a in (30), the right-hand side of the above expression will be

$$(F_a\#G_a)(\omega) = \int_0^\infty\int_0^\infty (Wf)(\omega)\overline{(W\psi)}(a\eta) \times(Wg)(\omega)\overline{(W\psi)}(a\xi)w(\omega,\xi,\eta)d\xi d\eta. \tag{31}$$

Therefore, using (29) and (31)

$$\int_0^\infty W(f\otimes g)(\omega)\frac{\overline{(W\psi)}(a\omega)}{a}da = \int_0^\infty\left(\int_0^\infty\int_0^\infty (Wf)(\omega)\overline{(W\psi)}(a\eta)(Wg)(\omega) \times\overline{(W\psi)}(a\xi)w(\omega,\xi,\eta)d\xi d\eta\right)\frac{da}{a}.$$

Thus, we can write

$$\int_0^\infty W(f\otimes g)(\omega)\frac{\overline{(W\psi)}(a\omega)}{a}da = \int_0^\infty\left(\int_0^\infty\int_0^\infty (Wf)(\omega)(Wg)(\omega) \times w(\omega,\xi,\eta)(\overline{W\psi})(a\eta)(\overline{W\psi})(a\xi)d\xi d\eta\right) \times\frac{da}{a}.$$

This implies

$$E_\psi W(f\otimes g)(\omega) = \int_0^\infty\left(\int_0^\infty\int_0^\infty (Wf)(\omega)(Wg)(\omega) \times w(\omega,\xi,\eta)\overline{(W\psi)}(a\eta)(\overline{(W\psi)}(a\xi) \times d\xi d\eta\right)\frac{da}{a},$$

where

$$E_\psi = \int_0^\infty \frac{\overline{(W\psi)}(a\omega)}{a}da. \tag{32}$$

□

Theorem 3. *Let $\psi \in L^2(\mathbb{R}^+)$ be the basis wavelet and satisfies the admissibility condition*

$$C_\psi = \int_0^\infty \frac{|(W\psi)(a\omega)|^2}{a} da. \tag{33}$$

Then

$$\int_0^\infty \frac{|(W\psi)(a\omega)(W\psi)(a\eta)|}{a} da \leq C_\psi. \tag{34}$$

Proof. Let

$$\int_0^\infty \frac{|W\psi(a\omega)W\psi(a\eta)|}{a} da = \int_0^\infty \frac{|W\psi(a\omega)W\psi(a\eta)|}{a^{1/2}a^{1/2}} da.$$

By applying Holder inequality, we get

$$\int_0^\infty \frac{|(W\psi)(a\omega)(W\psi)(a\eta)|}{a} da \leq \int_0^\infty \left(\frac{|(W\psi)(a\omega)|}{a^{1/2}} da\right)^{1/2} \times \int_0^\infty \left(\frac{|(W\psi)(a\eta)|}{a^{1/2}} da\right)^{1/2}.$$

$$\int_0^\infty \frac{|(W\psi)(a\omega)(W\psi)(a\eta)|}{a} da \leq C_\psi^{1/2} C_\psi^{1/2} = C_\psi.$$

□

Theorem 4. *Let $f, g \in L^2(\mathbb{R}^+)$, then we have*

$$W(f \otimes g)(\omega) = C'_\psi (Wf \# Wg)(\omega). \tag{35}$$

Proof.

$$\int_0^\infty \overline{(W\psi)}(a\eta)\overline{(W\psi)}(a\xi)\frac{da}{a} = \int_0^\infty \left(\frac{\overline{(W\psi)}(a\eta)}{a^{1/2}} da^{1/2}\right) \times \left(\frac{\overline{(W\psi)}(a\xi)}{a^{1/2}} da^{1/2}\right). \tag{36}$$

If we put $a\eta = u$ and $a\xi = v$, then we get

$$\int_0^\infty \overline{(W\psi)}(a\eta)\overline{(W\psi)}(a\xi)\frac{da}{a} = \int_0^\infty \left(\frac{\overline{(W\psi)}(u)}{(u/\eta)^{1/2}} \frac{du^{1/2}}{\eta^{1/2}}\right)\left(\frac{\overline{(W\psi)}(v)}{(v/\xi)^{1/2}} \frac{dv^{1/2}}{\xi^{1/2}}\right). \tag{37}$$

Taking $u = v$, we have

$$\int_0^\infty \overline{(W\psi)}(a\eta)\overline{(W\psi)}(a\xi)\frac{da}{a} = \int_0^\infty \left(\frac{\overline{(W\psi)}(u)}{(u)^{1/2}} du^{1/2}\right)\left(\frac{\overline{(W\psi)}(u)}{(u)^{1/2}} du^{1/2}\right) \tag{38}$$

$$= \int_0^\infty \frac{[\overline{(W\psi)}(u)]^2}{u} du = D_\psi. \tag{39}$$

Using (39) in (23), we obtained

$$\begin{aligned} E_\psi[W(f \otimes g)](\omega) &= \int_0^\infty \int_0^\infty (Wf)(\eta)(Wg)(\xi) w(\omega,\eta,\xi) \times \left(\int_0^\infty \overline{(W\psi)}(a\eta)\overline{(W\psi)}(a\xi)\frac{da}{a}\right) d\eta d\xi \\ &= D_\psi \int_0^\infty \int_0^\infty (Wf)(\eta)(Wg)(\xi) w(\omega,\eta,\xi) d\eta d\xi \\ &= D_\psi((Wf)\#(Wg))(\omega). \end{aligned}$$

Thus,

$$W(f \otimes g)(\omega) = \frac{D_\psi}{E_\psi}(Wf\#Wg)(\omega) = C'_\psi(Wf\#Wg)(\omega),$$

where $C'_\psi = \frac{D_\psi}{E_\psi}$. □

Lemma 5. *Let $f, g \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$ and $Wf, Wg \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$, then we have the following relation*:

$$(f \otimes g)(w) = C'_\psi(fg)(w) \quad a.e.$$

Proof. From Theorem 4, we have

$$W(f \otimes g)(w) = C'_\psi(Wf\#Wg)(w). \tag{40}$$

Taking inverse Watson transform on both sides in (40), we get

$$\begin{aligned} W^{-1}(W(f \otimes g)(w)) &= W^{-1}(C'_\psi(Wf\#Wg)(w)) \\ &= C'_\psi W^{-1}((Wf\#Wg)(w)) \\ &= C'_\psi W^{-1}(Wf)(w') W^{-1}(Wg)(w'). \end{aligned}$$

Therefore,

$$(f \otimes g)(w) = C'_\psi(fg)(w) \quad \text{a.e.} \tag{41}$$

□

Theorem 6. *Let $f \in L^p(\mathbb{R}^+)$, $g \in L^{p'}(\mathbb{R}^+)$ then for $1 \leq p, p' < \infty$ and $\psi \in L^q(\mathbb{R}^+) \cap L^{q'}(\mathbb{R}^+)$ such that $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$*

$$|W_\psi(f \otimes g)(b,a)| \leq \|f\|_p \|\psi\|_{p'} \|g\|_q \|\psi\|_{q'}. \tag{42}$$

Proof. Using (19), we have

$$W_\psi(f \otimes g)(b,a) = (W_\psi f)(b,a)(W_\psi g)(b,a).$$

Applying convolution formula (10) in (16), we get

$$\begin{aligned}|W_\psi(f \otimes g)(b,a)| &= |(f\#\psi)(b)||(g\#\psi)(b)| \\ &\leq \|f\|_p \|\psi\|_{p'} \|g\|_q \|\psi\|_{q'}.\end{aligned}$$ □

Theorem 7. *Let $f \in L^2(\mathbb{R}^+)$, $g \in L^2(\mathbb{R}^+)$ then*

$$\int_0^\infty \int_0^\infty W_\psi(f \otimes g)(b,a)\frac{dadb}{a} \leq \|f\|_2 \|g\|_2. \tag{43}$$

Proof. Using (19), we have

$$\begin{aligned}&\int_0^\infty \int_0^\infty W_\psi(f \otimes g)(b,a)\frac{dadb}{a} \\ &\quad = \int_0^\infty \int_0^\infty (W_\psi f)(b,a)(W_\psi g)(b,a)\frac{dadb}{a}.\end{aligned}$$

Using (17), we obtained

$$\int_0^\infty \int_0^\infty W_\psi(f \otimes g)(b,a)\frac{dadb}{a} = C_\psi |\langle f, g\rangle|.$$

By using Cauchy–Schwartz inequality, we get

$$\int_0^\infty \int_0^\infty W_\psi(f \otimes g)(b,a)\frac{dadb}{a} \leq C_\psi \|f\|_2 \|g\|_2.$$ □

Theorem 8. *Let $f, g \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$, then we have*

$$\|W(f \otimes g)\|_2 \leq C'_\psi \|f\|_1 \|g\|_2. \tag{44}$$

Proof. Using Theorem (4), we have

$$\begin{aligned}\|W(f \otimes g)\|_2 &= C'_\psi ||Wf\#Wg||_2 \\ &\leq C'_\psi \|Wf\|_1 \|Wg\|_2 \\ &\leq C'_\psi \|f\|_1 \|g\|_2.\end{aligned}$$ □

Theorem 9. *Let $k_n(\omega) = (Wg_n)(\omega)$ for $n \in \mathbb{N}$ and $\phi(\omega) = (Wf)(\omega)$ satisfy the following conditions*:

(1) $k_n(\omega) \geq 0,\ 0 < \omega < \infty$.
(2) $\int_0^\infty k_n(\omega)d\omega = 1, \omega = 0, 1, 2, \ldots.$
(3) $\lim_{n\to\infty} \int_\delta^\infty k_n(\omega)d\omega = 0$, *for each* $\delta > 0$.
(4) $\phi(\omega) \in L^\infty(\mathbb{R}^+)$.
(5) ϕ *is continuous at* ω_0.

Then $\lim_{n\to\infty} W(f \otimes g_n)(\omega_0) = C'_\psi (Wf)(\omega_0)$, *where* C'_ψ *defined in theorem* (4).

Proof. From (35), we write

$$\begin{aligned}W(f \otimes g_n)(\omega_0) &= C'_\psi (Wf\#Wg_n)(\omega_0) \\ &= C'_\psi (\phi\#k_n)(\omega_0).\end{aligned} \tag{45}$$

Let

$$\begin{aligned}I &= (\phi\#k_n)(\omega_0) - \phi(\omega_0) \\ &= \int_0^\infty \int_0^\infty [\phi(\omega) - \phi(\omega_0)]k_n(x)w(\omega_0, \omega, x)d\omega dx.\end{aligned} \tag{46}$$

Since ϕ is continuous at ω_0, then for given $\epsilon > 0$, we choose $\delta > 0$ so small that $|\phi(\omega) - \phi(\omega_0)| < \epsilon$ for $|\omega - \omega_0| < \delta$.

Let

$$I_1 = \int_0^\delta \int_0^\infty [\phi(\omega) - \phi(\omega_0)]k_n(x)w(\omega_0, \omega, x)d\omega dx \tag{47}$$

and

$$I_2 = \int_\delta^\infty \int_0^\infty [\phi(\omega) - \phi(\omega_0)]k_n(x)w(\omega_0, \omega, x)d\omega dx. \tag{48}$$

Now

$$\begin{aligned}|I_2| &\leq \int_\delta^\infty \int_0^\infty |\phi(\omega) - \phi(\omega_0)|k_n(x)w(\omega_0, \omega, x)d\omega dx \\ &\leq 2\|\phi\|_\infty \int_\delta^\infty \left(\int_0^\infty w(\omega_0, \omega, x)d\omega\right) k_n(x)dx \\ &= 2\|\phi\|_\infty \int_\delta^\infty k_n(x)dx\end{aligned}$$

since $(\int_0^\infty w(\omega_0, \omega, x)d\omega = 1)$.

Taking limit $n \to \infty$ in the last expression and using (iii), we get $\lim_{n\to\infty} I_2 = 0$.

Now we have

$$\begin{aligned}|I_1| &\leq \int_0^\delta \int_0^\infty |\phi(\omega) - \phi(\omega_0)|k_n(x)w(\omega_0, \omega, x)d\omega dx \\ &\leq \epsilon \int_0^\delta \int_0^\infty k_n(x)w(\omega_0, \omega, x)d\omega dx \\ &\leq \epsilon \int_0^\delta \left(\int_0^\infty w(\omega_0, \omega, x)d\omega\right) k_n(x)dx \\ &\leq \epsilon \int_0^\delta k_n(x)dx \leq \epsilon.\end{aligned}$$

Therefore, $\lim_{n\to\infty} ||I|| \leq \epsilon$. Since ϵ is arbitrary, we have $\lim_{n\to\infty} I = 0$.

From (45), we have

$$\lim_{n\to\infty} W(f\otimes g_n)(\omega_0) = \lim_{n\to\infty} C'_\psi(\phi\# k_n)(\omega_0) = C'_\psi\phi(\omega_0) = C'_\psi(Wf)(\omega_0). \qquad \square$$

4. HEURISTIC TREATMENT OF THE WATSON WAVELET TRANSFORM

In this section, authors discussed the heuristic treatment of the Watson wavelet transform and investigated inversion formula of Watson wavelet transform.

Theorem 10. *Let $(W_\psi f)(b,a)$ be the Watson wavelet transform and $(W^*_\psi f)(b,a)$ be the adjoint Watson wavelet transform on a function $f \in L^2(\mathbb{R}^+)$ with respect to the wavelet $\psi \in L^2(\mathbb{R}^+)$. Then*

$$f = \int_0^\infty W^*_\psi W_\psi f \frac{da}{a}, \tag{49}$$

where $f(t) = k(t\xi)$.

Proof. Watson wavelet transform is given by

$$(W_\psi\phi)(b,a) = \int_0^\infty f(t)\overline{\psi_{b,a}}(t)dt.$$

From (13), we find

$$(W_\psi\phi)(b,a) = \int_0^\infty f(t)\left(\frac{1}{a}\int_0^\infty w\left(\frac{b}{a},\frac{t}{a},z\right)\overline{\psi}(z)dz\right)dt. \tag{50}$$

Putting $f(t) = k(t\xi)$ in (50), we get

$$(W_\psi\phi)(b,a) = \int_0^\infty k(t\xi)\left(\frac{1}{a}\int_0^\infty w\left(\frac{b}{a},\frac{t}{a},z\right)\overline{\psi}(z)dz\right)dt. \tag{51}$$

On choosing $\frac{t}{a} = u$, we obtained

$$\begin{aligned}(W_\psi\phi)(b,a) &= \frac{1}{a}\int_0^\infty\left(\int_0^\infty k(\xi ua)w\left(\frac{b}{a},u,z\right)\overline{\psi}(z)dz\right)du \times a \\ &= \int_0^\infty\left(\int_0^\infty k(\xi ua)w\left(\frac{b}{a},u,z\right)du\right)\overline{\psi}(z)dz \\ &= \int_0^\infty k(b\xi)k(\xi za)\overline{\psi}(z)dz \\ &= k(b\xi)\overline{(W\psi)}(a\xi).\end{aligned} \tag{52}$$

Now taking adjoint operator of Watson wavelet transform and using (52), we get

$$\begin{aligned}(W^*_\psi W_\psi\phi)(t) &= \int_0^\infty (W_\psi\phi)(b,a)\psi_{b,a}(t)db \\ &= \overline{(W\psi)}(a\xi)\int_0^\infty k(b\xi)\left(\frac{1}{a}\int_0^\infty w\left(\frac{b}{a},\frac{t}{a},z\right) \times\overline{\psi}(z)dz\right)db \\ &= \overline{(W\psi)}(a\xi)\int_0^\infty\left(\frac{1}{a}\int_0^\infty w\left(\frac{b}{a},\frac{t}{a},z\right) \times k(b\xi)db\right)\overline{\psi}(z)dz.\end{aligned}$$

If we substitute $\frac{b}{a} = v$, then

$$\begin{aligned}(W^*_\psi W_\psi\phi)(t) &= \overline{(W\psi)}(a\xi)\int_0^\infty\left(\frac{1}{a}\int_0^\infty w\left(v,\frac{t}{a},z\right) \times k(av\xi)adv\right)\overline{\psi}(z)dz \\ &= \overline{(W\psi)}(a\xi)\int_0^\infty k(t\xi)k(za\xi)\overline{\psi}(z)dz \\ &= k(t\xi)\overline{(W\psi)}(a\xi)\int_0^\infty k(za\xi)\overline{\psi}(z)dz \\ &= k(t\xi)\overline{(W\psi)}(a\xi)(W\psi)(a\xi) \\ &= k(t\xi)|(W\psi)(a\xi)|^2.\end{aligned}$$

Therefore,

$$\int_0^\infty (W^*_\psi W_\psi\phi)(t)\frac{da}{a} = k(t\xi)\int_0^\infty |(W\psi)(a\xi)|^2\frac{da}{a},$$

$$f(t) = k(t\xi) = \frac{\int_0^\infty (W^*_\psi W_\psi\phi)(t)\frac{da}{a}}{\int_0^\infty |(W\psi)(a\xi)|^2\frac{da}{a}}.$$

By imposing the normalization

$$\int_0^\infty |(W\psi)(a\xi)|^2 \frac{da}{a} = 1,$$

we obtain the Watson wavelet representation

$$f(t) = \int_0^\infty (W_\psi^* W_\psi \phi)(t) \frac{da}{a}. \qquad \square$$

Theorem 11. *Suppose that ψ is a continuum Watson wavelet with*

$$A_\psi = \int_0^\infty \frac{|\psi(w)|^2}{w} = 1.$$

Then for any $f \in L^2(\mathbb{R}^+)$, we have the $L^2(\mathbb{R}^+)$ inversion formula

$$\begin{aligned} f(x) &= \int_{\mathbb{R}_+^2} (W_\psi f)(b,a)\psi_{b,a}(x) \frac{dbda}{a} \\ &= \lim_{\epsilon\to 0, A,B\to\infty} \int_{\epsilon<a<A, b<B} (W_\psi f)(b,a) \\ &\quad \times \psi_{b,a}(x) \frac{dbda}{a}, \end{aligned}$$

where $S(\epsilon, A, B)f = \int_{\epsilon<a<A,b<B} (W_\psi f)(b,a)\psi_{b,a}(x) \frac{dbda}{a}$.

Proof. Let

$$\begin{aligned} &\|f - S(\epsilon, A, B)f\|_2 \\ &\quad = \sup_{\|g\|_2=1} |\langle f - S(\epsilon, A, B)f, g\rangle|. \end{aligned} \tag{53}$$

Applying Fubini's theorem, we have

$$\begin{aligned} &\langle S(\epsilon, A, B)f, g\rangle \\ &\quad = \int_{\mathbb{R}^+} \bar{g}(x) \left(\int_{\epsilon<a<A,b<B} (W_\psi f)(b,a) \right. \\ &\qquad \left. \times \psi_{b,a}(x) \frac{dbda}{a} \right) dx \\ &\quad = \int_{\mathbb{R}^+} (W_\psi f)(b,a) \\ &\qquad \times \left(\int_{\epsilon<a<A,b<B} \bar{g}(x)\psi_{b,a}(x)dx \right) \frac{dbda}{a} \\ &\quad = \int_{\epsilon<a<A,b<B} (W_\psi f)(b,a)\overline{(W_\psi g)}(b,a)(x) \\ &\qquad \times \frac{dbda}{a}. \end{aligned}$$

Using Cauchy–Schwartz inequality, we have

$$\begin{aligned} &|\langle f - S(\epsilon, A, B)f, g\rangle| \\ &\quad = \left| \int_{(\epsilon<a<A,b<B)^c} (W_\psi f)(b,a) \right. \\ &\qquad \left. \overline{(W_\psi g)}(b,a)(x) \frac{dbda}{a} \right| \\ &\quad = \left(\int_{(\epsilon<a<A,b<B)^c} |(W_\psi f)(b,a)|^2 \frac{dbda}{a} \right)^{1/2} \\ &\qquad \times \left(\int_{\mathbb{R}_+^2} |(W_\psi g)(b,a)|^2 \frac{dbda}{a} \right)^{1/2}. \end{aligned}$$

Then by using (17), we get

$$\begin{aligned} &|\langle f - S(\epsilon, A, B)f, g\rangle| \\ &\quad = \left(\int_{(\epsilon<a<A,b<B)^c} |(W_\psi f)(b,a)|^2 \frac{dbda}{a} \right)^{1/2} \\ &\qquad \times A_\psi \|g\|_2, \end{aligned} \tag{54}$$

where $\epsilon \to 0$ and $A, B \to \infty$, the region of integration decreases to empty set. Hence, the last integral tends to zero by the dominated convergence theorem. This gives that

$$\|S(\epsilon, A, B)f - f\|_2 \to 0. \qquad \square$$

Theorem 12. *Suppose that ψ is a continuum Watson wavelet with $\langle \psi, \psi\rangle_w = 1$ and*

$$C_{\psi,s} = \int_0^\infty \frac{|(W\psi)(\xi)|}{\xi^{2s}} d\xi < \infty. \tag{55}$$

Then

$$\int_0^\infty \int_0^\infty \frac{|(W_\psi f)(b,a)|^2}{a^{2s}} dadb = C_{\psi,s} ||f||_{2,s}^2.$$

Proof. From (14), we have

$$\begin{aligned} &(W_\psi f)(b,a)\overline{(W_\psi f)}(b,a)db \\ &\quad = \int_0^\infty W^{-1}((Wf)(u)\overline{(W\psi)}(au))(b) \\ &\qquad \times \overline{W^{-1}((Wg)(u)\overline{(W\psi)}(au))(b)} db. \end{aligned} \tag{56}$$

Using Parseval formula of Watson wavelet transform (17)

$$\int_0^\infty\int_0^\infty \frac{(W_\psi f)(b,a)\overline{(W_\psi f)(b,a)}}{a^{2s}}dadb$$

$$= \int_0^\infty\int_0^\infty \frac{(Wf)(u)\overline{(W\psi)}(au) \times \overline{(Wg)(u)\overline{(W\psi)}(au)}}{a^{2s}}dadb$$

$$= \int_0^\infty\int_0^\infty (Wf)(u)\overline{(Wg)}(u)\frac{|(W\psi)(au)|^2}{a^{2s}}dadb.$$

Take $f = g$, we get

$$\int_0^\infty\int_0^\infty \frac{|(W_\psi f)(b,a)|^2}{a^{2s}}dadb = \int_0^\infty\int_0^\infty |(Wf)(u)|^2\frac{|(W\psi)(au)|^2}{a^{2s}}dadb.$$

Putting $au = \xi$ in the second term of the above expression, we get

$$\int_0^\infty\int_0^\infty \frac{|(W_\psi f)(b,a)|^2}{a^{2s}}dadb$$

$$= \int_0^\infty\int_0^\infty |(Wf)(u)|^2\left(\frac{|(W\psi)(\xi)|^2}{\xi^{2s}}d\xi\right) u^{2s}du$$

$$= C_{\psi,s}\int_0^\infty |(Wf)(u)|^2u^{2s}du$$

$$= C_{\psi,s}||f||^2_{2,s}. \qquad \square$$

5. TWO-WAVELET MULTIPLIERS

With the help of Refs. 7 and 24, two-wavelet multipliers are introduced and expressed in terms of Watson wavelet convolution product by exploiting the theory of the Watson transform.

From Ref. 24, let $\pi : \mathbb{R}^+ \to U(L^2(\mathbb{R}^+))$ be the unitary representation of the multiplicative group $\mathbb{R}^+$ on $L^2(\mathbb{R}^+)$ defined by

$$(\pi_\xi u)(x) = k(x\xi)u(x), \quad x,\xi \in \mathbb{R}^+ \tag{57}$$

for all functions u in $L^2(\mathbb{R}^+)$, where $U(L^2(\mathbb{R}^+))$ is the group of all unitary operators on $L^2(\mathbb{R}^+)$.

Then the Watson two-wavelet multipliers associated with the unitary representation are given as

$$(P_{\sigma,\phi,\psi}u)(x) = \int_{\mathbb{R}^+} \sigma(\xi) < u, \pi_\xi\phi > (\pi_\xi\psi)(x)d\xi. \tag{58}$$

Theorem 13. *If $\sigma(\xi) \in L^1(\mathbb{R}^+)$ and $\phi,\psi \in L^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)$. Then the Watson two-wavelet multipliers can be expressed in the following form*:

$$(P_{\sigma,\phi,\psi}u)(x) = \frac{1}{C'_\psi}\psi(x)W^{-1}[\sigma(\xi)W(u\otimes\phi)(\xi)](x), \tag{59}$$

$u \in L^1(\mathbb{R}^+)$.

Proof. Using (58), we have

$$(P_{\sigma,\phi,\psi}u)(x) = \int_{\mathbb{R}^+} \sigma(\xi)\langle u,\pi_\xi\phi\rangle(\pi_\xi\psi)(x)d\xi,$$

$$= \int_{\mathbb{R}^+} \sigma(\xi)\left(\int_{\mathbb{R}^+} u(\eta)(\pi_\xi\phi)(\eta)d\eta\right) \times (\pi_\xi\psi)(x)d\xi.$$

From (57), we get

$$(P_{\sigma,\phi,\psi}u)(x)$$

$$= \int_{\mathbb{R}^+} \sigma(\xi)\left(\int_{\mathbb{R}^+} k(\eta\xi)\phi(\eta)u(\eta)d\eta\right) \times k(x\xi)\psi(x)d\xi$$

$$= \int_{\mathbb{R}^+} \sigma(\xi)W(u\phi)(\xi)k(x\xi)\psi(x)d\xi$$

$$= \psi(x)\int_{\mathbb{R}^+} k(x\xi)\sigma(\xi)W(u\phi)(\xi)d\xi$$

$$= \psi(x)W^{-1}[\sigma(\xi)W(u\phi)(\xi)](x). \tag{60}$$

With help of Lemma 5, we have

$$(P_{\sigma,\phi,\psi}u)(x) = \frac{1}{C'_\psi}(\psi(x)W^{-1}[\sigma(\xi)W(u\otimes\phi)(\xi)])(x). \qquad \square$$

Theorem 14. *Let $\sigma \in L^1(\mathbb{R}^+)$ and $\phi,\psi \in L^1(\mathbb{R}^+)\cap L^2(\mathbb{R}^+)$. Then the two-wavelet multipliers $P_{\sigma,\phi,\psi} : L^1(\mathbb{R}^+) \to L^1(\mathbb{R}^+)$ are*

$$\|P_{\sigma,\phi,\psi}\|_1 \le \frac{1}{C'_\psi}\|\sigma\|_\infty\|\phi\|_2\|\bar{\psi}\|_2.$$

Proof. From (59), we have

$$\|P_{\sigma,\phi,\psi}u\|_1 = \|\psi(x)W^{-1}[\sigma(\xi)W(\bar{\phi}u)(\xi)](x)\|_1.$$

Using Lemma 5, we get

$$\|P_{\sigma,\phi,\psi}u\|_1 = \frac{1}{C'_\psi}\|\psi(x)W^{-1}[\sigma(\xi)W(\bar{\phi}\otimes u)(\xi)](x)\|_1.$$

Applying Holder inequality, we obtained

$$\|P_{\sigma,\phi,\psi}u\|_1 \le \frac{1}{C'_\psi}\|\psi\|_2\|W^{-1}[\sigma(\xi)W(\bar{\phi}\otimes u)(\xi)]\|_2.$$

By Parseval relation (15), we have

$$\|P_{\sigma,\phi,\psi}u\|_1 \leq \frac{1}{C'_\psi}\|\psi\|_2\|\sigma(\xi)W(\bar{\phi}\otimes u)(\xi)\|_2$$

$$\leq \frac{1}{C'_\psi}\|\psi\|_2\|\sigma\|_\infty\|W(\bar{\phi}\otimes u)\|_2.$$

Again applying Parseval formula (15), we get

$$\|P_{\sigma,\phi,\psi}u\|_1 \leq \frac{1}{C'_\psi}\|\psi\|_2\|\sigma\|_\infty\|\bar{\phi}\otimes u\|_2.$$

Using Theorem 8, we get the required result

$$\|P_{\sigma,\phi,\psi}u\|_1 \leq \frac{1}{C'_\psi}\|\sigma\|_\infty\|\psi\|_2\|\bar{\phi}\|_2\|u\|_1. \qquad \square$$

Theorem 15. *Let $\sigma \in L^1(\mathbb{R}^+)$ and $\phi,\psi \in L^1(\mathbb{R}^+)\cap L^\infty(\mathbb{R}^+)$. Then the two-wavelet multipliers $P_{\sigma,\phi,\psi}: L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$ are estimated by*

$$\|P_{\sigma,\phi,\psi}\|_2 \leq \frac{1}{C'_\psi}\|\phi\|_\infty\|\sigma\|_\infty\|\bar{\psi}\|_1.$$

Proof. From (59), we have

$$\|P_{\sigma,\phi,\psi}u\|_2 = \|\psi(x)W^{-1}[\sigma(\xi)W(\bar{\phi}u)(\xi)](x)\|_2.$$

Using Lemma 5, we get

$$\|P_{\sigma,\phi,\psi}u\|_2 = \frac{1}{C'_\psi}\|\psi(x)W^{-1}[\sigma(\xi)W(\bar{\phi}\otimes u)(\xi)](x)\|_2$$

$$\leq \frac{1}{C'_\psi}\|\psi\|_\infty\|W^{-1}[\sigma(\xi)W(\bar{\phi}\otimes u)(\xi)](x)\|_2.$$

By Parseval relation (15), we have

$$\|P_{\sigma,\phi,\psi}u\|_2 \leq \frac{1}{C'_\psi}\|\psi\|_\infty\|\sigma(\xi)W(\bar{\phi}\otimes u)(\xi)\|_2$$

$$\leq \frac{1}{C'_\psi}\|\psi\|_\infty\|\sigma\|_\infty\|W(\bar{\phi}\otimes u)\|_2.$$

Again applying Parseval formula (15), we get

$$\|P_{\sigma,\phi,\psi}u\|_2 \leq \frac{1}{C'_\psi}\|\psi\|_\infty\|\sigma\|_\infty\|\bar{\phi}\otimes u\|_2.$$

Using Theorem 8, we get the required results

$$\|P_{\sigma,\phi,\psi}u\|_2 \leq \frac{1}{C'_\psi}\|\psi\|_\infty\|\sigma\|_\infty\|u\|_2\|\bar{\phi}\|_1. \qquad \square$$

From Wong[7] in p. 499, we recall the following fact which is very useful to find the trace class of Watson two-wavelet multipliers.

Let X be a complex and separable Hilbert space of infinite dimension in which the inner product is denoted by $\langle , \rangle$. Let $A : X \to X$ be a compact operator. If we denote $A^* : X \to X$ the adjoint of $A : X \to X$, then the linear operator $(A^*A)^{1/2} : X \to X$ is positive and compact. Let $\{\phi_k : k = 1, 2, \ldots\}$ be an orthonormal basis for X consisting of eigenvectors of $(A^*A)^{1/2} : X \to X$, and let $s_k(A)$ be the eigenvalue corresponding to the eigenvector $\phi_k, k = 1, 2, \ldots$. We say that the compact operator $A : X \to X$ is in the trace class S_1 if $\sum_{k=1}^\infty s_k(A) < \infty$. It can be shown that S_1 is a Banach space in which the norm $\| \|_{S_1}$ is given by

$$\|A\|_{S_1} = \sum_{k=1}^\infty s_k(A), \quad A \in S_1. \tag{61}$$

Proposition 16. *Let $A : X \to X$ be a bounded linear operator on a Hilbert space X and let $\{\phi_k : k = 1, 2, 3, \ldots\}$ be any orthonormal basis for X. Then the series $\sum_{k=1}^\infty \langle A\phi_k, \phi_k\rangle$ is absolutely convergent and the sum is independent of the choice of the orthonormal basis $\{\phi_k : k = 1, 2, 3, \ldots\}$.*

In view of Proposition 16, we can define the trace class S_1 of any linear operator $A : X \to X$ by

$$\mathrm{tr}(A) = \sum_{k=1}^\infty \langle A\phi_k, \phi_k\rangle. \tag{62}$$

Theorem 17. *Let $\sigma \in L^1(\mathbb{R}^+)$ and ϕ, ψ be any functions in $L^2(\mathbb{R}^+)$ such that $\|\phi\|_2 = 1 = \|\psi\|_2$. Then the two Watson wavelet multipliers $P_{\sigma,\phi,\psi}$: $L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$ are in trace class.*

$$|\mathrm{tr}(P_{\sigma,\phi,\psi})| \leq K \int_{\mathbb{R}^+} |\sigma(\xi)|d\xi. \tag{63}$$

Proof. Let $\{\phi_k\}$ be a sequence of orthonormal basis for $L^2(\mathbb{R}^+)$. Then, using (62) we have

$$|\mathrm{tr}(P_{\sigma,\phi,\psi}\phi_k)| = \left|\sum_{k=1}^\infty \langle P_{\sigma,\phi,\psi}\phi_k, \phi_k\rangle\right|.$$

From (58), we get

$$|\mathrm{tr}(P_{\sigma,\phi,\psi}\phi_k)| = \left|\sum_{k=1}^\infty \int_{\mathbb{R}^+} \sigma(\xi)\langle\phi_k, \pi_\xi\phi\rangle\langle\pi_\xi\psi, \phi_k\rangle d\xi\right|$$

$$\leq \sum_{k=1}^\infty \left|\int_{\mathbb{R}^+} \sigma(\xi)\langle\phi_k, \pi_\xi\phi\rangle\langle\pi_\xi\psi, \phi_k\rangle d\xi\right|.$$

By using the definition of inner product, we have

$$\begin{aligned}|\mathrm{tr}(P_{\sigma,\phi,\psi}\phi_k)| &\leq \sum_{k=1}^{\infty}\int_{\mathbb{R}^+}|\sigma(\xi)\langle\phi_k,\pi_\xi\phi\rangle\overline{\langle\phi_k,\pi_\xi\psi\rangle}|d\xi\\ &\leq \sum_{k=1}^{\infty}\int_{\mathbb{R}^+}|\sigma(\xi)\langle\phi_k,\pi_\xi\phi\rangle\overline{\langle\phi_k,\pi_\xi\psi\rangle}|d\xi.\end{aligned}$$

Since $\{\phi_k\}$ is an orthonormal basis and applying Cauchy–Schwartz inequality

$$\begin{aligned}|\mathrm{tr}(P_{\sigma,\phi,\psi})| &\leq \int_{\mathbb{R}^+}|\sigma(\xi)|\sum_{k=1}^{\infty}\|\phi_k\|_2^2\|\pi_\xi\phi\|_2\|\pi_\xi\psi\|_2 d\xi\\ &\leq K\|\phi\|_2\|\psi\|_2\int_{\mathbb{R}^+}|\sigma(\xi)|d\xi\\ &< \infty.\end{aligned}$$

□

Theorem 18. *Let $\sigma\in L^1(\mathbb{R}^+)$ and ϕ and ψ be any functions in $L^2(\mathbb{R}^+)\cap L^\infty(\mathbb{R}^+)$ such that $\|\phi\|_2 = 1 = \|\psi\|_2$. Then the trace class of Watson two-wavelet multiplier can be expressed in term of Watson wavelet convolution product.*

$$\begin{aligned}\mathrm{tr}(P_{\sigma,\phi,\psi}\phi_k) = \sum_{k=1}^{\infty}\frac{1}{C'_\psi}\int_{\mathbb{R}^+}\sigma(\xi)W(\phi_k\otimes\phi)(\xi)\\ \times W(\psi\otimes\phi_k)(\xi)d\xi. \qquad (64)\end{aligned}$$

Proof. Let $\{\phi_k\}$ be a sequence of orthonormal basis for X. Then, using formula (62), (57) and Fubini theorem, we have

$$\begin{aligned}\mathrm{tr}(P_{\sigma,\phi,\psi}\phi_k) &= \sum_{k=1}^{\infty}\langle P_{\sigma,\phi,\psi}\phi_k,\phi_k\rangle\\ &= \sum_{k=1}^{\infty}\int_{\mathbb{R}^+}\sigma(\xi)\langle\phi_k,\pi_\xi\phi\rangle\langle\pi_\xi\psi,\phi_k\rangle d\xi\\ &= \sum_{k=1}^{\infty}\int_{\mathbb{R}^+}\sigma(\xi)\langle\phi_k,\pi_\xi\phi\rangle\langle\pi_\xi\psi,\phi_k\rangle d\xi.\end{aligned}$$

Using the definition of inner product, we have

$$\mathrm{tr}(P_{\sigma,\phi,\psi}) = \sum_{k=1}^{\infty}\int_{\mathbb{R}^+}\sigma(\xi)\langle\phi_k,\pi_\xi\phi\rangle\overline{\langle\phi_k,\pi_\xi\psi\rangle}d\xi.$$

From (57), we find that

$$\mathrm{tr}(P_{\sigma,\phi,\psi}) = \sum_{k=1}^{\infty}\int_{\mathbb{R}^+}\sigma(\xi)W(\phi_k\phi)(\xi)\overline{W(\phi_k\psi)}(\xi)d\xi.$$

Using Lemma 5, we express the trace class in term of Watson wavelet convolution product

$$\begin{aligned}\mathrm{tr}(P_{\sigma,\phi,\psi}\phi_k)\\ = \sum_{k=1}^{\infty}\frac{1}{C'_\psi}\int_{\mathbb{R}^+}\sigma(\xi)W(\phi_k\otimes\phi)\overline{W(\phi_k\otimes\psi)}d\xi.\end{aligned}$$

□

Theorem 19. *Let $\sigma\in L^1(\mathbb{R}^+)$ and ϕ and ψ be any functions in $L^2(\mathbb{R}^+)\cap L^\infty(\mathbb{R}^+)$ such that $\|\phi\|_2 = 1 = \|\psi\|_2$. Then the trace class of Watson two-wavelet multiplier is in S_1.*

$$\begin{aligned}|\mathrm{tr}(P_{\sigma,\phi,\psi}\phi_k)|\\ \leq \sum_{k=1}^{\infty}\frac{1}{C'_\psi}\int_{\mathbb{R}^+}|\sigma(\xi)W(\phi_k\otimes\phi)W(\psi\otimes\phi_k)|d\xi\\ < \infty.\end{aligned}$$

Proof. With the help of (64), we find that

$$\begin{aligned}|\mathrm{tr}(P_{\sigma,\phi,\psi}\phi_k)|\\ &\leq \sum_{k=1}^{\infty}\frac{1}{C'_\psi}\mathrm{sup}|W(\phi_k\otimes\phi)(\xi)|\\ &\quad\times\int_{\mathbb{R}^+}|\sigma(\xi)\overline{W(\phi_k\otimes\psi)}|d\xi\\ &\leq \sum_{k=1}^{\infty}\frac{1}{C'_\psi}||\phi_k\otimes\phi||_1\int_{\mathbb{R}^+}|\sigma(\xi)\overline{W(\phi_k\otimes\psi)}|d\xi\\ &\leq \sum_{k=1}^{\infty}\frac{1}{C'_\psi}||\phi_k\otimes\phi||_1\mathrm{sup}|\overline{W(\phi_k\otimes\psi)}|\\ &\quad\times\int_{\mathbb{R}^+}|\sigma(\xi)|d\xi\\ &\leq \sum_{k=1}^{\infty}\frac{1}{C'_\psi}||\phi_k\otimes\phi||_1||\overline{\phi_k\otimes\psi}||_1\int_{\mathbb{R}^+}|\sigma(\xi)|d\xi.\end{aligned}$$

Using Lemma 5 and applying Holder inequality, we get

$$\begin{aligned}|\mathrm{tr}(P_{\sigma,\phi,\psi}\phi_k)|\\ &\leq \sum_{k=1}^{\infty}\frac{1}{C'_\psi}||\phi_k\phi||_1||\overline{\phi_k\psi}||_1\int_{\mathbb{R}^+}|\sigma(\xi)|d\xi\\ &\leq \sum_{k=1}^{\infty}||\phi_k||_2\phi||_2||\overline{\phi_k}||_2\overline{\psi}||_2\int_{\mathbb{R}^+}|\sigma(\xi)|d\xi\end{aligned}$$

$$\leq \sum_{k=1}^{\infty} ||\phi_k||_2^2 ||\phi||_2 \overline{\psi}||_2 \int_{\mathbb{R}^+} |\sigma(\xi)| d\xi$$

$$\leq \int_{\mathbb{R}^+} |\sigma(\xi)| d\xi < \infty. \qquad \square$$

6. CONCLUSIONS

From the works of Pathak,[15] Titchmarsh[18] and Schuitman,[22] the authors reached the conclusion that the Watson transform contains strong mathematical background and rich calculus. The results of the Watson transform are interesting and from Refs. 22 and 25 it is observed that Fourier transform, Laplace transform and Hankel transform are examples of the Watson transform. Various mathematical relations of Watson transform with Laplace transform, Hankel transform and Fourier transform are given in Ref. 25. This work provides an integral representation of the Watson wavelet convolution product and shows the relationship between the Watson wavelet convolution product and Watson convolution. A heuristic treatment of the inversion formula of the Watson wavelet transform is developed, and its estimation is expressed in terms of the Sobolev type space. This theory is used to derive estimations of two-wavelet multipliers associated with the Watson transform. Later on, authors were able to find the connection between Watson wavelet convolution product and trace class of two-wavelet multipliers. The results are very useful for finding the many problems of the convolution product.

In the entire work, the authors tried to take all the research work in the form of references. This work is helpful to those researchers who are interested in the area of Watson wavelet transform, Watson convolution and Watson wavelet multipliers. The results of this paper are the generalization of the work of Ref. 17.

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