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OPTIMIZATION

Generalized Hukuhara Hadamard derivative of interval-valued functions and its applications to interval optimization

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Abstract

In this article, we study the notion of gH-Hadamard derivative for interval-valued functions (IVFs) and apply it to solve interval optimization problems (IOPs). It is shown that the existence of gH-Hadamard derivative implies the existence of gH-Fréchet derivative and vise-versa. Further, it is proved that the existence of gH-Hadamard derivative implies the existence of gH-continuity of IVFs. We found that the composition of a Hadamard differentiable real-valued function and a gH-Hadamard differentiable IVF is gH-Hadamard differentiable. Further, for finite comparable IVF, we prove that the gH-Hadamard derivative of the maximum of all finite comparable IVFs is the maximum of their gH-Hadamard derivative. The proposed derivative is observed to be useful to check the convexity of an IVF and to characterize efficient points of an optimization problem with IVF. For a convex IVF, we prove that if at a point the gH-Hadamard derivative does not dominate to zero, then the point is an efficient point. Further, it is proved that at an efficient point, the gH-Hadamard derivative does not dominate zero and also contains zero. For constraint IOPs, we prove an extended Karush–Kuhn–Tucker condition using the proposed derivative. The entire study is supported by suitable examples.

Keywords Interval-valued functions · Interval optimization problems · Efficient solutions · gH-Hadamard derivative · gH-Fréchet derivative

1 Introduction

In the study of general behavior of a real-world problem, such as static or dynamic, deterministic or probabilistic, linear or nonlinear, and convex or nonconvex, several mathematical tools have been developed. In many cases, the knowledge about the underlying parameters, which influences the system's mathematical behavior, is imprecise or uncertain. Generally, one cannot measure the parameters affected by imprecision or uncertainties with exact values. In such situations, the parameters cannot be modeled by a real number. We usually overcome this deficiency using fuzzy sets, interval, or stochastic values. Interval analysis is based on representing an uncertain variable as an interval, which is a natural way of incorporating the uncertainties of parameters. As mathematical functions play a crucial role in modeling realistic problems, we analyze a special derivative of interval-valued functions (IVFs) in this article.

Three important aspects of a function are monotonicity, convexity and differentiability. In the study of monotonicity and convexity Ansari et al. (2013) of an IVF, an appropriate ordering of intervals is the prime issue. Unlike the real numbers, intervals are not linearly ordered. Due to which the whole paradigm of analyzing an IVF changes and the development of calculus for IVF is not just trivial extensions of the corresponding counterpart for conventional real-valued functions. The same reason makes the development of opti-

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mization with IVFs difficult since the very optimality notion requires an ordering of the function values.

Most often optimization with IVFs Chalco-Cano et al. (2013), Ghosh et al. (2020), Stefanini and Bede (2009) have been analyzed with respect to a partial ordering Ishibuchi and Tanaka (1990). Some researchers Bhurjee and Padhan (2016), Ghosh et al. (2018), Kumar and Bhurjee (2021) used ordering relations of intervals based on the parametric comparison of intervals. In Ref. Costa et al. (2015), an ordering relation of intervals is defined by a bijective map from the set of intervals to $\mathbb{R}^2$. However, the ordering relations of intervals Bhurjee and Padhan (2016), Ghosh et al. (2018), Costa et al. (2015); Kumar et al. (2022) can be derived from the relations described in Ref. Ishibuchi and Tanaka (1990). Recently, Ghosh et al. (2020) investigated variable ordering relations for intervals and used them in interval optimization problems (IOPs).

To observe the properties of an IVF, calculus plays an essential role. Initially, to develop the calculus of IVFs, Hukuhara Hukuhara (1967) introduced the concept of differentiability of IVFs with the help of H-difference of intervals. However, the definition of Hukuhara differentiability (H-differentiability) is found to be restrictive (see Ref. Chalco-Cano et al. (2013)). To remove the deficiencies of H-differentiability, Bede and Gal (2005) defined strongly generalized derivative (G-derivative) for IVFs and derived a Newton–Leibnitz-type formula. In order to formulate the mean-value theorem for IVFs, Markov (1979) introduced a new concept of difference of intervals and defined differentiability of IVFs using this difference. In Ref. Stefanini and Bede (2009), Stefanini and Bede defined the generalized Hukuhara differentiability (gH-differentiability) of IVFs using the concept of generalized Hukuhara difference. In defining the calculus of IVFs, the concepts of gH-derivative, gH-partial derivative, gH-subderivative, gH-gradient, and gH-differentiability for IVFs have been developed in Refs. Ghosh (2017), Stefanini and Bede (2009), Stefanini and Arana-Jiménez (2019), Ghosh et al. (2022).

To derive a Karush–Kuhn–Tucker (KKT) condition for IOPs, Guo et al. (2019) defined gH-symmetric derivative for IVFs. Ghosh (2016) analyzed the notion of gH-differentiability of multi-variable IVFs to propose the Newton method for IOPs. The concept of second-order differentiability of IVFs is introduced by Van Hoa (2015) to study the existence of a unique solution of interval differential equations. Lupulescu (2013) defined delta generalized Hukuhara differentiability on time scales using gH-difference. Chalco-Cano et al. (2011) introduced the concept of π-derivative for IVFs that generalizes Hukuhara derivative and G-derivative, and proved that this derivative is equivalent to gH-derivative. In Ref. Stefanini and Bede (2014), Stefanini and Bede defined level-wise gH-differentiability and generalized fuzzy differentiability by LU-parametric representation for fuzzy-valued functions. Kalani et al. (2016) analyzed the concept of interval-valued fuzzy derivative for perfect and semi-perfect interval-valued fuzzy mappings to derive a method for solving interval-valued fuzzy differential equations using the extension principle. Recently, Ghosh et al. (2020) and Chauhan et al. (2021) have provided the idea of gH-directional derivative, gH-Gâteaux derivative, gH-Fréchet derivative, and gH-Clarke derivative of IVFs to derive the optimality conditions for IOPs.

Despite many attempts to develop calculus for IVFs, the existing ideas are not adequate to retain two most important features of classical differential calculus—linearity of the derivative with respect to the direction and the chain rule. Although Ghosh et al. (2020) proposed some optimality conditions for IOPs using gH-directional and gH-Gâteaux derivatives, but these derivatives are not sufficient to preserve the continuity of IVFs (see Example 5.1 of Ghosh et al. (2020)) and chain rule for the composition of IVFs (see Example 2 of this article). Even though gH-Fréchet derivative in Ghosh et al. (2020) preserves linearity and continuity but it does not hold the chain rule for the composition of IVFs whose lower and upper functions are equal at each points (see example for Proposition 3.5 Shapiro (1990)). With the help of the derivative of lower and upper functions, some articles Wu (2009), Zhang et al. (2014); Ren and Wang (2017) reported KKT condition to characterize efficient solutions of constraint IOPs. However, the derivative used in Refs. Wu (2009), Zhang et al. (2014), Ren and Wang (2017) are very restrictive because this derivative is very difficult to calculate even for very simple IVF (see Example 1 of Ref. Chalco-Cano et al. (2015)). However, in this article, we derive KKT condition of constraint IOPs by gH-Hadamard derivative which do not depend on the existence of the Hadamard derivative of lower and upper functions. In addition, proposed derivative retains the linearity of the derivative with respect to direction, the existence of continuity as well as the chain rule of derivative.

1.1 Motivation and contribution

In conventional nonsmooth optimization theory, one of the mostly used idea of derivative is Hadamard derivative which is applied to characterize optimal solutions. An explicit expression of the derivative of an extremum with respect to parameters can be obtained with the help of Hadamard derivative. Therefore, it works well for most differentiable optimization problems including convex or concave problems. Correspondingly, in interval analysis and interval optimization, we expect to have a notion of the Hadamard derivative for interval-valued functions. In addition, from the literature on the analysis of IVFs, one can notice that the study of Hadamard derivative for IVFs have not been developed so far. However, the basic properties of this derivative might be beneficial for characterizing and capturing the optimal solutions of IOPs.

In this article, we define gH-Hadamard derivative of IVFs. It is proved that if an IVF is gH-Hadamard differentiable, then IVF is gH-continuous. Using the proposed concept of gH-Hadamard derivative, we prove that a gH-Fréchet differentiable IVF is gH-Hadamard differentiable and vise-versa. Further, we characterize the convexity of IVFs with the help of gH-Hadamard derivative. Besides, with the help of gH-Hadamard derivative, we provide a necessary and sufficient condition for characterizing the efficient solutions to IOPs. Further, for constraint IOPs, we derive the extended KKT necessary and sufficient condition to characterize the efficient solutions.

1.2 Delineation

The rest of the article is demonstrated in the following sequence. The next section covers some basic terminologies and notions of convex analysis and interval analysis, followed by the convexity and calculus of IVFs. In addition, a few properties of intervals, gH-directional and gH-Fréchet derivatives of an IVF are discussed in Sect. 2. In Sect. 3, we define the gH-Hadamard derivative of IVF and observe that the existence of gH-Hadamard derivative implies the existence of gH-Fréchet derivative and vise-versa. Further, it is found that the existence of gH-Hadamard derivative implies gH-continuity. In the same section, it is shown that proposed derivatives are useful to check the convexity of an IVF. In Sect. 4, we prove that at a point, in which gH-Hadamard derivative does not dominate zero is an efficient point of an IVF. Further, it is observed that at an efficient point of IVF, gH-Hadamard derivative must contain zero. In Sect. 5, a few properties of the cone of descent direction and cone of feasible direction are given. In addition, we prove the extended necessary and sufficient optimality condition for constraint IOPs in the same section. Finally, the last section concludes and draws future scopes of the study.

2 Preliminaries and terminologies

This section is devoted to some basic notions on intervals. In addition, we present basic convexity and calculus of IVFs which will be used throughout the paper. We also use the following notations.

- $\mathscr{X}$ is a real normed linear space with the norm $\|\cdot\|$
- $\mathscr{S}$ is a nonempty subset of $\mathscr{X}$
- $\mathbb{R}$ denotes the set of real numbers
- $\mathbb{R}_+$ denotes the set of nonnegative real numbers
- $I(\mathbb{R})$ is the set of all compact intervals (that is, closed and bounded intervals)
- $I(\mathbb{R})^n$ is the set of vectors whose components are compact intervals

2.1 Arithmetic of intervals and their dominance relation

Throughout the article, we denote the elements of $I(\mathbb{R})$ by bold capital letters: $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots$. We represent an element $\mathbf{P}$ of $I(\mathbb{R})$ in its interval form with the help of the corresponding small letter in the following way:

$$\mathbf{P} = [\underline{p}, \overline{p}], \text{ where } \underline{p} \text{ and } \overline{p} \text{ are real numbers such that } \underline{p} \leq \overline{p}.$$

Let $\mathbf{P}, \mathbf{Q} \in I(\mathbb{R})$ and $\lambda \in \mathbb{R}$. Moore's (1966; 1979) interval addition, subtraction, product, division and scalar multiplication are denoted by $\mathbf{P} \oplus \mathbf{Q}$, $\mathbf{P} \ominus \mathbf{Q}$, $\mathbf{P} \odot \mathbf{Q}$, $\mathbf{P} \oslash \mathbf{Q}$, and $\lambda \odot \mathbf{P}$, respectively. In defining $\mathbf{P} \oslash \mathbf{Q}$, it is assumed that $0 \notin \mathbf{Q}$.

Since $\mathbf{P} \ominus \mathbf{P} \neq \mathbf{0}$ for any nondegenerate interval (whose lower and upper limits are not equal) $\mathbf{P}$, we use the following concept of difference of intervals in this article.

Definition 1 Stefanini (2008) Let $\mathbf{P}$ and $\mathbf{Q}$ be two elements of $I(\mathbb{R})$. The *gH-difference* between $\mathbf{P}$ and $\mathbf{Q}$, denoted by $\mathbf{P} \ominus_{gH} \mathbf{Q}$, is defined by the interval $\mathbf{C}$ such that

$$\mathbf{P} = \mathbf{Q} \oplus \mathbf{C} \quad \text{or} \quad \mathbf{Q} = \mathbf{P} \ominus \mathbf{C}.$$

It is to be noted that for $\mathbf{P} = \left[\underline{p},\ \overline{p}\right]$ and $\mathbf{Q} = \left[\underline{q},\ \overline{q}\right]$,

$$\mathbf{P} \ominus_{gH} \mathbf{Q} = \Big[\min\{\underline{p} - \underline{q}, \overline{p} - \overline{q}\},\ \max\{\underline{p} - \underline{q}, \overline{p} - \overline{q}\}\Big] \text{ and } \mathbf{P} \ominus_{gH} \mathbf{P} = \mathbf{0}.$$

In the following, we provide a domination relation for intervals based on a *minimization* type optimization problems: a *smaller value* is *better*.

Definition 2 Ishibuchi and Tanaka (1990) Let $\mathbf{P} = [\underline{p}, \overline{p}]$ and $\mathbf{Q} = [\underline{q}, \overline{q}]$ be two intervals in $I(\mathbb{R})$.

(i) $\mathbf{Q}$ is said to be *dominated by* $\mathbf{P}$ if $\underline{p} \leq \underline{q}$ and $\overline{p} \leq \overline{q}$, and then we write $\mathbf{P} \preceq \mathbf{Q}$.
(ii) $\mathbf{Q}$ is said to be *strictly dominated by* $\mathbf{P}$ if either '$\underline{p} \leq \underline{q}$ and $\overline{p} < \overline{q}$' or '$\underline{p} < \underline{q}$ and $\overline{p} \leq \overline{q}$', and then we write $\mathbf{P} \prec \mathbf{Q}$.
(iii) If $\mathbf{Q}$ is not dominated by $\mathbf{P}$, then we write $\mathbf{P} \npreceq \mathbf{Q}$; if $\mathbf{Q}$ is not strictly dominated by $\mathbf{P}$, then we write $\mathbf{P} \nprec \mathbf{Q}$.
(iv) If $\mathbf{P}$ is dominated by $\mathbf{Q}$ or $\mathbf{Q}$ is dominated by $\mathbf{P}$, then $\mathbf{P}$ and $\mathbf{Q}$ are said to be *comparable*.
(v) If $\mathbf{P} \npreceq \mathbf{Q}$ and $\mathbf{Q} \npreceq \mathbf{P}$, then we say that *none of* $\mathbf{P}$ *and* $\mathbf{Q}$ *dominates the other*, or $\mathbf{P}$ and $\mathbf{Q}$ are *not comparable*.

Notice that if $\mathbf{Q}$ is strictly dominated by $\mathbf{P}$, then $\mathbf{Q}$ is dominated by $\mathbf{P}$. Moreover, if $\mathbf{Q}$ is not dominated by $\mathbf{P}$, then $\mathbf{Q}$ is not strictly dominated by $\mathbf{P}$.

Lemma 1 *For $\boldsymbol{P}$ and $\boldsymbol{Q}$ in $I(\mathbb{R})$,*

(i) *If $\boldsymbol{Q} \not\prec \boldsymbol{0}$ and $\boldsymbol{Q} \preceq \boldsymbol{P}$, then $\boldsymbol{P} \not\prec \boldsymbol{0}$,*
(ii) *If $\boldsymbol{P} \oplus \boldsymbol{Q} \not\prec \boldsymbol{0}$ and $\boldsymbol{Q} \preceq \boldsymbol{0}$, then $\boldsymbol{P} \not\prec \boldsymbol{0}$.*

Proof See Appendix A □

Definition 3 Chauhan and Ghosh (2021) Let $\mathbf{P} = [\underline{p}, \bar{a}] = \{a(t) : a(t) = \underline{p} + t(\bar{a} - \underline{p}), 0 \le t \le 1\}$ and $\mathbf{Q} = [\underline{q}, \bar{b}] = \{b(t) : b(t) = \underline{q} + t(\bar{b} - \underline{q}), 0 \le t \le 1\}$ be two elements of $I(\mathbb{R})$. Then, $\mathbf{Q}$ is said to be *better strictly dominated by* by $\mathbf{P}$ if $a(t) < b(t)$ for all $t \in [0, 1]$, and then we write $\mathbf{P} < \mathbf{Q}$.

Lemma 2 Chauhan and Ghosh (2021) *Let $\boldsymbol{P} = [\underline{p}, \overline{p}]$ and $\boldsymbol{Q} = [\underline{q}, \overline{q}]$ be in $I(\mathbb{R})$. Then, $\boldsymbol{P} < \boldsymbol{Q}$ if and only if $\underline{p} < \underline{q}$ and $\overline{p} < \overline{q}$.*

Definition 4 Moore (1966) A function $\|\cdot\|_{I(\mathbb{R})} : I(\mathbb{R}) \to \mathbb{R}_+$ defined by

$$\|\mathbf{P}\|_{I(\mathbb{R})} = \max\{|\underline{p}|, |\bar{p}|\}, \quad \text{for all } \mathbf{P} = [\underline{p}, \overline{p}] \in I(\mathbb{R}),$$

is called a *norm on* $I(\mathbb{R})$.

Definition 5 For two comparable intervals $\mathbf{P}$ and $\mathbf{Q}$ of $I(\mathbb{R})$ with $\mathbf{P} \preceq \mathbf{Q}$, their *maximum* is $\max\{\mathbf{P}, \mathbf{Q}\} = \mathbf{Q}$.

2.2 Convexity and calculus of IVFs

A function $\mathbf{\Phi} : \mathscr{S} \to I(\mathbb{R})$ is known as an IVF. For each $x \in \mathscr{S}$, $\mathbf{\Phi}$ can be presented by the following interval:

$$\mathbf{\Phi}(t) = \left[\underline{\Phi}(t),\ \overline{\Phi}(t)\right],$$

where $\underline{\Phi}$ and $\overline{\Phi}$ are real-valued functions on $\mathscr{S}$ such that $\underline{\Phi}(t) \le \overline{\Phi}(t)$ for all $t \in \mathscr{S}$. In addition, $\mathbf{\Phi}$ is said to be degenerate IVF if $\underline{\Phi}(t) = \overline{\Phi}(t)$ for all $t \in \mathscr{S}$.

If $\mathscr{S}$ is convex, then the IVF $\mathbf{\Phi}$ is said to be convex Wu (2007) on $\mathscr{S}$ if for any $t_1, t_2 \in \mathscr{S}$,

$$\mathbf{\Phi}(\lambda_1 t_1 + \lambda_2 t_2) \preceq \lambda_1 \odot \mathbf{\Phi}(t_1)$$
$$\oplus \lambda_2 \odot \mathbf{\Phi}(t_2) \text{ for all } \lambda_1, \lambda_2 \in [0, 1] \text{ with } \lambda_1 + \lambda_2 = 1.$$

The IVF $\mathbf{\Phi} : \mathscr{S} \to I(R)$ is said to be *gH-continuous* Ghosh (2017) at a point $\bar{t}$ of $\mathscr{S}$ if

$$\lim_{\substack{\|d\| \to 0 \\ \bar{t}+d \in \mathscr{S}}} \left(\mathbf{\Phi}(\bar{t} + d) \ominus_{gH} \mathbf{\Phi}(\bar{t})\right) = \mathbf{0}.$$

If $\mathbf{\Phi}$ is gH-continuous at each point t in $\mathscr{S}$, then $\mathbf{\Phi}$ is said to be gH-continuous on $\mathscr{S}$.

Definition 6 Ghosh et al. (2020) An IVF $\mathbf{\Phi} : \mathscr{S} \to I(\mathbb{R})$ is said to be gH-Lipschitz continuous on $\mathscr{S} \subseteq \mathbb{R}^n$ if there exists $M > 0$ such that

$$\|\mathbf{\Phi}(t_1) \ominus_{gH} \mathbf{\Phi}(t_2)\|_{I(\mathbb{R})} \le M\|t_1 - t_2\|_{\mathscr{S}} \quad \text{for all } t_1, t_2 \in \mathscr{S}.$$

The constant M is called a Lipschitz constant.

Lemma 3 Wu (2007) *$\mathbf{\Phi}$ is a convex IVF on a convex set $\mathscr{S} \subseteq \mathscr{X}$ if and only if $\underline{\Phi}$ and $\overline{\Phi}$ are convex on $\mathscr{S}$.*

Definition 7 Ghosh et al. (2020) Let $\mathscr{S}$ be a linear subspace of $\mathscr{X}$. The function $\mathbf{\Phi} : \mathscr{S} \to I(\mathbb{R})$ is said to be *linear* if

(i) $\mathbf{\Phi}(\lambda t) = \lambda \odot \mathbf{\Phi}(t)$ for all $t \in \mathscr{S}$ and all $\lambda \in \mathbb{R}$, and
(ii) for all $t,\ y \in \mathscr{S}$,

'either $\mathbf{\Phi}(t) \oplus \mathbf{\Phi}(y) = \mathbf{\Phi}(t + y)$' or 'none of $\mathbf{\Phi}(t) \oplus \mathbf{\Phi}(y)$ and $\mathbf{\Phi}(t + y)$ dominates the other '.

Lemma 4 *Let $\mathscr{S}$ be a linear subspace of $\mathscr{X}$ and $\mathbf{\Phi} : \mathscr{S} \to I(\mathbb{R})$ be a linear IVF. Then, the following results hold.*

(i) *If $\mathbf{\Phi}(t) \not\prec \boldsymbol{0}$ for all $t \in \mathscr{S}$, then $\boldsymbol{0}$ and $\mathbf{\Phi}(t)$ are not comparable.*
(ii) *If $\mathbf{\Phi}(t) \preceq \boldsymbol{0}$ for all $t \in \mathscr{S}$, then $\mathbf{\Phi}(t) = \boldsymbol{0}$.*

Proof See Appendix B □

Definition 8 Ghosh et al. (2020) Let $\mathscr{S}$ be a nonempty subset of $\mathbb{R}^n$ and $\mathbf{\Phi} : \mathbb{R}^n \to I(\mathbb{R})$ be an IVF. A point $\bar{t} \in \mathscr{S}$ is said to be an *efficient point* of the IOP

$$\min_{t \in \mathscr{S} \subset \mathbb{R}^n} \mathbf{\Phi}(t) \tag{2.1}$$

if $\mathbf{\Phi}(t) \not\prec \mathbf{\Phi}(\bar{t})$ for all $t \in \mathscr{S}$.

Definition 9 Ghosh et al. (2020) Let $\mathbf{\Phi}$ be an IVF on a nonempty subset $\mathscr{S}$ of $\mathscr{X}$. Let $\bar{t} \in \mathscr{S}$ and $h \in \mathscr{X}$. If the limit

$$\lim_{\lambda \to 0+} \frac{1}{\lambda} \odot \left(\mathbf{\Phi}(\bar{t} + \lambda h) \ominus_{gH} \mathbf{\Phi}(\bar{t})\right)$$

exists finitely, then the limit is said to be *gH-directional derivative* of $\mathbf{\Phi}$ at $\bar{t}$ in the direction h, and it is denoted by $\mathbf{\Phi}_{\mathscr{D}}(\bar{t})(h)$.

Definition 10 Chauhan et al. (2021) Let $\mathbf{\Phi}$ be an IVF defined on a nonempty subset $\mathscr{S}$ of $\mathscr{X}$. For $\bar{t} \in \mathscr{S}$ and $h \in \mathscr{X}$, if the limit superior

$$\limsup_{\substack{t \to \bar{t} \\ \lambda \to 0+}} \frac{1}{\lambda} \odot \left(\mathbf{\Phi}(t + \lambda h) \ominus_{gH} \mathbf{\Phi}(t)\right)$$

$$= \lim_{\delta \to 0} \left(\sup_{t \in \overline{\mathscr{B}}(\bar{t},\delta) \cap \mathscr{S},\ \lambda \in (0,\delta)} \frac{1}{\lambda} \odot \left(\mathbf{\Phi}(t + \lambda h) \ominus_{gH} \mathbf{\Phi}(t) \right) \right)$$

exists finitely, then the limit superior value is called *upper* gH*-Clarke derivative* of $\mathbf{\Phi}$ at $\bar{t}$ in the direction h, and it is denoted by $\mathbf{\Phi}_{\mathscr{C}}(\bar{t})(h)$. If this limit superior exists for all $h \in \mathscr{X}$, then $\mathbf{\Phi}$ is said to be *upper* gH*-Clarke differentiable* at $\bar{t}$.

Definition 11 Ghosh et al. (2020) Let $\mathscr{S}$ be a nonempty open subset of $\mathscr{X}$, $\mathbf{\Phi} : \mathscr{S} \to I(\mathbb{R})$ be an IVF and $\bar{t} \in \mathscr{S}$. Suppose that there exists a gH-continuous and linear mapping $\mathbf{G} : \mathscr{X} \to I(\mathbb{R})$ with the following property:

$$\lim_{\|h\| \to 0} \frac{\|\mathbf{\Phi}(\bar{t} + h) \ominus_{gH} \mathbf{\Phi}(\bar{t}) \ominus_{gH} \mathbf{G}(h)\|_{I(\mathbb{R})}}{\|h\|} = 0,$$

then $\mathbf{G}$ is said to be gH*-Fréchet derivative* of $\mathbf{\Phi}$ at $\bar{t}$, and we write $\mathbf{G} = \mathbf{\Phi}_{\mathscr{F}}(\bar{t})$.

3 *gH*-Hadamard derivative of interval-valued functions

In this section, we present the concept of gH-Hadamard derivative for IVFs. It is worth noting that if an IVF $\mathbf{\Phi}$ has a gH-Hadamard derivative at $\bar{t}$, then $\mathbf{\Phi}$ must be gH-continuous at $\bar{t}$.

Definition 12 Let $\mathbf{\Phi}$ be an IVF on a nonempty subset $\mathscr{S}$ of $\mathscr{X}$. For $\bar{t} \in \mathscr{S}$ and $v \in \mathscr{X}$, if the limit

$$\mathbf{\Phi}_{\mathscr{H}}(\bar{t})(v) := \lim_{\substack{\lambda \to 0+ \\ h \to v}} \frac{1}{\lambda} \odot \left(\mathbf{\Phi}(\bar{t} + \lambda h) \ominus_{gH} \mathbf{\Phi}(\bar{t}) \right)$$

exists and $\mathbf{\Phi}_{\mathscr{H}}(\bar{t})$ is a linear IVF from $\mathscr{X}$ to $I(\mathbb{R})$, then $\mathbf{\Phi}_{\mathscr{H}}(\bar{t})(v)$ is called gH*-Hadamard derivative* of $\mathbf{\Phi}$ at $\bar{t}$ in the direction v. If this limit exists for all $v \in \mathscr{X}$, then $\mathbf{\Phi}$ is said to be gH*-Hadamard differentiable* at $\bar{t}$.

Remark 1 The limit $\mathbf{\Phi}_{\mathscr{H}}(\bar{t})(v)$ exists if for all sequences $\{\lambda_n\}$ and $\{h_n\}$ with $\lambda_n > 0$ for all n such that $\lim_{n\to\infty} \lambda_n = 0$, $\lim_{n\to\infty} h_n = v$,

$$\lim_{n \to \infty} \frac{1}{\lambda_n} \odot \left(\mathbf{\Phi}(\bar{t} + \lambda_n h_n) \ominus_{gH} \mathbf{\Phi}(\bar{t}) \right)$$

exists and the limit value is a linear IVF on $\mathscr{S}$.

Example 1 Let $\mathscr{S} = \mathscr{X} = \mathbb{R}^n$ and consider the IVF $\mathbf{\Phi}(t) = \|t\|^2 \odot \mathbf{C},\ t \in \mathbb{R}^n$, where $\mathbf{C} = [\underline{c}, \overline{c}] \in I(\mathbb{R})$. We calculate the gH-Hadamard derivative of $\mathbf{\Phi}$ at $\bar{t} = 0$.

For any $\bar{t} \in \mathscr{S}$ and $v \in \mathscr{X}$, we see that

$$\lim_{\substack{\lambda \to 0+ \\ h \to v}} \frac{1}{\lambda} \odot \left(\mathbf{\Phi}(\bar{t} + \lambda h) \ominus_{gH} \mathbf{\Phi}(\bar{t}) \right)$$

$$= \lim_{\substack{\lambda \to 0+ \\ h \to v}} \frac{1}{\lambda} \odot \left(\|\bar{t} + \lambda h\|^2 \odot \mathbf{C} \ominus_{gH} \|\bar{t}\|^2 \odot \mathbf{C} \right)$$

$$= \lim_{\substack{\lambda \to 0+ \\ h \to v}} \frac{1}{\lambda} \odot \left(\left[\|\bar{t} + \lambda h\|^2 \underline{c},\ \|\bar{t} + \lambda h\|^2 \overline{c} \right] \ominus_{gH} \left[\|\bar{t}\|^2 \underline{c},\ \|\bar{t}\|^2 \overline{c} \right] \right)$$

$$= \lim_{\substack{\lambda \to 0+ \\ h \to v}} \frac{1}{\lambda} \odot \left(\left[\min\{\|\bar{t} + \lambda h\|^2 \underline{c} - \|\bar{t}\|^2 \underline{c},\ \|\bar{t} + \lambda h\|^2 \overline{c} - \|\bar{t}\|^2 \overline{c}\}, \max\{\|\bar{t} + \lambda h\|^2 \underline{c} - \|\bar{t}\|^2 \underline{c},\ \|\bar{t} + \lambda h\|^2 \overline{c} - \|\bar{t}\|^2 \overline{c}\} \right] \right)$$

$$= \lim_{\substack{\lambda \to 0+ \\ h \to v}} \frac{1}{\lambda} \odot \left(\left[\min\left\{ \left(2\bar{t}^\top (\lambda h) + \|\lambda h\|^2 \right) \underline{c},\ \left(2\bar{t}^\top (\lambda h) + \|\lambda h\|^2 \right) \overline{c} \right\}, \max\left\{ \left(2\bar{t}^\top (\lambda h) + \|\lambda h\|^2 \right) \underline{c},\ \left(2\bar{t}^\top (\lambda h) + \|\lambda h\|^2 \right) \overline{c} \right\} \right] \right)$$

$$= \lim_{\substack{\lambda \to 0+ \\ h \to v}} \frac{1}{\lambda} \odot \left(\left(2\bar{t}^\top (\lambda h) + \|\lambda h\|^2 \right) \odot \mathbf{C} \right)$$

$$= 2\bar{t}^\top v \odot \mathbf{C},\ \text{by } gH\text{-continuity of } \bar{t}^\top h \odot \mathbf{C}.$$

Hence, $\mathbf{\Phi}_{\mathscr{H}}(\bar{t})(v) = 2\bar{t}^\top v \odot \mathbf{C}$ and $\mathbf{\Phi}_{\mathscr{H}}(\bar{t})$ is a linear IVF from $\mathscr{X}$ to $I(\mathbb{R})$ by Definition 7. Therefore, $\mathbf{\Phi}$ is gH-Hadamard differentiable at $\bar{t}$ with $\mathbf{\Phi}_{\mathscr{H}}(\bar{t})(v) = 2\bar{t}^\top v \odot \mathbf{C}$.

Remark 2 By Definition 12 of gH-Hadamard differentiable and Definition 9 of gH-directional differentiable IVF, we can observe that a gH-Hadamard differentiable IVF is gH-directional differentiable IVF. However, converse is not true. For instance, consider the IVF $\mathbf{\Phi}(t) = |t| \odot \mathbf{C}$, $\mathbf{C} \geq \mathbf{0}$ for all $t \in \mathbb{R}$. We see that for all $h \in \mathbb{R}$ and at $\bar{t} = 0$,

$$\lim_{\lambda \to 0+} \frac{1}{\lambda} \odot \left(\mathbf{\Phi}(\bar{t} + \lambda h) \ominus_{gH} \mathbf{\Phi}(\bar{t}) \right) = |h| \odot \mathbf{C}.$$

Again, for any $v \in \mathbb{R}$ and $\bar{t} = 0$, we see that

$$\mathbf{\Phi}_{\mathscr{H}}(\bar{t})(v) = \lim_{\substack{\lambda \to 0+ \\ h \to v}} \frac{1}{\lambda} \odot \left(\mathbf{\Phi}(\bar{t} + \lambda h) \ominus_{gH} \mathbf{\Phi}(\bar{t}) \right) = \lim_{\substack{\lambda \to 0+ \\ h \to v}} \left(\left(\frac{1}{\lambda} \odot \lambda \right) \odot (|h| \odot \mathbf{C}) \right) = |v| \odot \mathbf{C}.$$

Hence, for $h = 2$, $v = -3$ and $\mathbf{C} = [5, 9]$,

$$\mathbf{\Phi}_{\mathscr{H}}(\bar{t})(h + v) = |h + v| \odot \mathbf{C} = 1 \odot [5, 9] = [5, 9].$$

In addition,

$$\mathbf{\Phi}_{\mathscr{H}}(\bar{t})(h) \oplus \mathbf{\Phi}_{\mathscr{H}}(\bar{t})(v) = |2| \odot [5, 9] \oplus |-3| \odot [5, 9]$$

$= [10, 18] \oplus [15, 27] = [35, 45].$

Since $\mathbf{\Phi}_{\mathscr{H}}(\bar{t})(h+v) \neq \mathbf{\Phi}_{\mathscr{H}}(\bar{t})(h) \oplus \mathbf{\Phi}_{\mathscr{H}}(\bar{t})(v)$ and $\mathbf{\Phi}_{\mathscr{H}}(\bar{t})(h+v) < \mathbf{\Phi}_{\mathscr{H}}(\bar{t})(h) \oplus \mathbf{\Phi}_{\mathscr{H}}(\bar{t})(v)$, property (7) of Definition 7 does not hold. Hence, $\mathbf{\Phi}_{\mathscr{H}}(\bar{t})$ is not linear. Therefore, $\mathbf{\Phi}_{\mathscr{H}}(\bar{t})(h)$ does not exist.

Next, remark shows the relation between upper gH-Clarke derivative and gH-Hadamard derivative for gH-Lipschitz continuous IVFs.

Remark 3 By Theorem 2 of Chauhan et al. (2021) and Remark 2, it is clear that the gH-Hadamard derivative at any point of a convex gH-Lipschitz continuous IVF coincides with the upper gH-Clarke derivative at that point. However, there are some convex gH-Lipschitz continuous IVF whose upper gH-Clarke derivative does not coincide with gH-Hadamard derivative. For instance, consider the IVF $\mathbf{\Phi}(t) = |t| \odot \mathbf{C}$, $\mathbf{C} = [\underline{c}1, \overline{c}2] \succeq \mathbf{0}$ for all $t \in \mathbb{R}$.

Since $\underline{\mathbf{\Phi}}(t) = \underline{c}|t|$ and $\overline{\mathbf{\Phi}}(t) = \overline{c}|t|$, where $\mathbf{\Phi}(t) = [\underline{\mathbf{\Phi}}(t), \overline{\mathbf{\Phi}}(t)]$ are Lipschitz continuous, by Lemma 4 of Chauhan et al. (2021), $\mathbf{\Phi}$ is gH-Lipschitz continuous.

In addition, $\mathbf{\Phi}$ is convex by Lemma 3 of Chauhan et al. (2021) as $\underline{\mathbf{\Phi}}(t)$ and $\overline{\mathbf{\Phi}}(t)$ are convex.

We see that for $h \in \mathbb{R}$,

$$\begin{aligned}
&\limsup_{\substack{t\to 0\\ \lambda\to 0+}} \frac{1}{\lambda} \odot \left(\mathbf{\Phi}(t+\lambda h) \ominus_{gH} \mathbf{\Phi}(t)\right)\\
&= \limsup_{\substack{t\to 0\\ \lambda\to 0+}} \frac{1}{\lambda} \odot \left(|t+\lambda h| \odot \mathbf{C} \ominus_{gH} |t| \odot \mathbf{C}\right)\\
&\preceq \limsup_{\substack{t\to 0\\ \lambda\to 0+}} \frac{1}{\lambda} \odot \left(|t| \odot \mathbf{C} \oplus \lambda|h| \odot \mathbf{C} \ominus_{gH} |t| \odot \mathbf{C}\right),\\
&\text{(by Lemma 1 of [8] as } |a+b| \odot \mathbf{C}\\
&\preceq |a| \odot \mathbf{C} \oplus |b| \odot \mathbf{C})\\
&= \limsup_{\substack{t\to 0\\ \lambda\to 0+}} \frac{1}{\lambda} \odot (\lambda|h| \odot \mathbf{C})\\
&= |h| \odot \mathbf{C}. \qquad (3.1)
\end{aligned}$$

Further,

$$\begin{aligned}
&\limsup_{\substack{t\to 0\\ \lambda\to 0+}} \frac{1}{\lambda} \odot \left(|t+\lambda h| \odot \mathbf{C} \ominus_{gH} |t| \odot \mathbf{C}\right)\\
&\succeq \limsup_{\lambda\to 0+} \frac{1}{\lambda} \odot \left(2\lambda|h| \odot \mathbf{C} \ominus_{gH} \lambda|h| \odot \mathbf{C}\right)\\
&(\text{ taking } t = \lambda h, h > 0)\\
&(\text{since } \limsup_{t\to 0}|t| \geq \limsup_{\lambda\to 0+}|\lambda h| \text{for any perticular direction } t = \lambda h)\\
&= |h| \odot \mathbf{C}. \qquad (3.2)
\end{aligned}$$

Then, from the inequalities (3.1) and (3.2), we get the gH-Clarke derivative $\mathbf{\Phi}_{\mathscr{C}}(0)(h) = |h| \odot \mathbf{C}$.

Again, for any $v \in \mathbb{R}$ and $\bar{t} = 0$, we see that

$$\begin{aligned}
\mathbf{\Phi}_{\mathscr{H}}(\bar{t})(v) &= \lim_{\substack{\lambda\to 0+\\ h\to v}} \frac{1}{\lambda} \odot \left(\mathbf{\Phi}(\bar{t}+\lambda h) \ominus_{gH} \mathbf{\Phi}(\bar{t})\right)\\
&= \lim_{\substack{\lambda\to 0+\\ h\to v}} \left(\left(\frac{1}{\lambda} \odot \lambda\right) \odot (|h| \odot \mathbf{C})\right)\\
&= |v| \odot \mathbf{C}.
\end{aligned}$$

Hence, the limit value is not a linear IVF (for explanation see Remark 2) on $\mathscr{S}$. Therefore, $\mathbf{\Phi}_{\mathscr{H}}(\bar{t})(h)$ does not exist.

Theorem 1 *Let $\mathscr{X} = \mathbb{R}^n$, $\mathscr{S}$ be a nonempty subset of $\mathscr{X}$, $\mathbf{\Phi}$ be an IVF on $\mathscr{S}$ and $\bar{t} \in \mathscr{S}$. Then, the following statements are equivalent:*

(i) *$\mathbf{\Phi}$ is gH-Fréchet differentiable at $\bar{t}$.*
(ii) *$\mathbf{\Phi}$ is gH-Hadamard differentiable at $\bar{t}$.*

Proof (i) $\implies$ (ii). Since $\mathbf{\Phi}$ is gH-Fréchet differentiable at $\bar{t} \in \mathscr{S}$, there exists a gH-continuous and linear IVF $\mathbf{G}$ such that

$$\lim_{\lambda\to 0+} \frac{\|\mathbf{\Phi}(\bar{t}+\lambda h) \ominus_{gH} \mathbf{\Phi}(\bar{t}) \ominus_{gH} \mathbf{G}(\lambda h)\|_{I(\mathbb{R})}}{\|\lambda h\|} = 0,$$
$$\text{for all } h \in \mathscr{X}\backslash\{\hat{0}\}$$

$$\text{or, } \lim_{\lambda\to 0+} \frac{1}{\lambda} \|\mathbf{\Phi}(\bar{t}+\lambda h) \ominus_{gH} \mathbf{\Phi}(\bar{t}) \ominus_{gH} \mathbf{G}(\lambda h)\|_{I(\mathbb{R})} = 0,$$
$$\text{for all } h \in \mathscr{X}\backslash\{\hat{0}\}. \qquad (3.3)$$

Since $\mathbf{G}$ is linear, and thus $\mathbf{G}(\lambda\, h) = \lambda \odot \mathbf{G}(h)$, the equation (3.3) gives

$$\lim_{\lambda\to 0+} \frac{1}{\lambda} \odot \left(\mathbf{\Phi}(\bar{t}+\lambda h) \ominus_{gH} \mathbf{\Phi}(\bar{t}) \ominus_{gH} \lambda \odot \mathbf{G}(h)\right) = \mathbf{0},$$
$$\text{for all } h \in \mathscr{X}\backslash\{\hat{0}\}$$

$$\text{or, } \lim_{\lambda\to 0+} \frac{1}{\lambda} \odot \left(\mathbf{\Phi}(\bar{t}+\lambda h)\right.$$
$$\left.\ominus_{gH} \mathbf{\Phi}(\bar{t})\right) = \mathbf{G}(h), \quad \text{for all } h \in \mathscr{X}\backslash\{\hat{0}\}.$$

Since $\mathbf{G}$ is gH-continuous, we have

$$\lim_{\substack{\lambda\to 0+\\ h\to v}} \frac{1}{\lambda} \odot \left(\mathbf{\Phi}(\bar{t}+\lambda h) \ominus_{gH} \mathbf{\Phi}(\bar{t})\right) = \mathbf{G}(v).$$

Hence, $\mathbf{\Phi}$ is gH-Hadamard differentiable at $\bar{t}$.
(ii) $\implies$ (i). As $\mathbf{\Phi}$ is gH-Hadamard differentiable at $\bar{t} \in \mathscr{S}$, $\mathbf{\Phi}_{\mathscr{H}}(\bar{t})(v)$ exists for all v and $\mathbf{\Phi}_{\mathscr{H}}(\bar{t})$ is a linear IVF. Let

$$Q(h) = \frac{1}{\|h\|} \odot \left(\mathbf{\Phi}(\bar{t}+h) \ominus_{gH} \mathbf{\Phi}(\bar{t}) \ominus_{gH} \mathbf{\Phi}_{\mathscr{H}}(\bar{t})(h)\right), \ h \neq \hat{0}.$$

Consider a sequence $\{h_n\}$ converging to 0. As $\mathscr{W} = \{h/\|h\| : h \in \mathscr{X}, h \neq \hat{0}\}$ is a compact set, there exists a subsequences $\{h_{n_k}\}$ and a point $\bar{v} \in \mathscr{W}$ such that $w_{n_k} = \frac{h_{n_k}}{\|h_{n_k}\|} \to \bar{v} \in \mathscr{W}$.

Note that the sequence $\{t_{n_k}\}$, defined by $t_{n_k} = \|h_{n_k}\|$, converges to 0. Since $\mathbf{\Phi}_{\mathscr{H}}(\bar{t})(\bar{v})$ exists and
$\mathbf{\Phi}_{\mathscr{H}}(\bar{t})(w_{n_k}) \to \mathbf{\Phi}_{\mathscr{H}}(\bar{t})(\bar{v})$ as $k \to \infty$, we have

$$Q(h_{n_k}) = \frac{1}{t_{n_k}} \odot \big(\mathbf{\Phi}(\bar{t} + t_{n_k} w_{n_k}) \ominus_{gH} \mathbf{\Phi}(\bar{t})\big) \ominus_{gH} \mathbf{\Phi}_{\mathscr{H}}(\bar{t})(w_{n_k}) \to \mathbf{0} \text{ as } k \to \infty.$$

This implies that $\lim_{k\to\infty} \|Q(h_{n_k})\|_{I(\mathbb{R})} = 0$.

As $\{h_n\}$ is an arbitrarily chosen sequence that converges to 0, $\lim_{\|h\|\to 0} \|Q(h)\|_{I(\mathbb{R})} = 0$. Hence, $\mathbf{\Phi}$ is gH-Fréchet differentiable at $\bar{t}$. □

Remark 4 If $\mathscr{X}$ is infinite dimensional, then Theorem 1 is not true. For instance, see Example 1 of Yu and Liu (2013). According to this example, there exists a degenerate IVF $\mathbf{\Phi}$ which is gH-Hadamard differentiable at $\bar{t}$ but not gH-Fréchet differentiable at $\bar{t}$.

Theorem 2 *Let $\mathscr{S}$ be a nonempty subset of $\mathscr{X} = \mathbb{R}^n$. If the function $\mathbf{\Phi} : \mathscr{S} \to I(\mathbb{R})$ has a gH-Hadamard derivative at $\bar{t} \in \mathscr{S}$, then the function $\mathbf{\Phi}$ is gH-continuous at $\bar{t}$.*

Proof Since $\mathbf{\Phi}$ is gH-Hadamard differentiable at $\bar{t} \in \mathscr{S}$, $\mathbf{\Phi}$ is gH-Fréchet differentiable at $\bar{t}$ by Theorem 1. In addition, from Theorem 5.1 in Ghosh et al. (2020), the function $\mathbf{\Phi}$ is gH-continuous at $\bar{t}$. □

Remark 5 The converse of Theorem 2 is not true. For instance, consider the gH-continuous IVF $\mathbf{\Phi}(t) = \|t\| \odot \mathbf{C}$ for all $t \in \mathbb{R}^n$. Therefore, for any $v \in \mathbb{R}^n$ and $\bar{t} = 0$, we see that

$$\lim_{\substack{\lambda\to 0+\\ h\to v}} \frac{1}{\lambda} \odot \big(\mathbf{\Phi}(\bar{t} + \lambda h) \ominus_{gH} \mathbf{\Phi}(\bar{t})\big) = \|v\| \odot \mathbf{C}.$$

Hence, the limit value is not a linear IVF on $\mathscr{S}$. Therefore, $\mathbf{\Phi}_{\mathscr{H}}(\bar{t})(h)$ does not exist.

Remark 6 By the definitions of gH-directional (Definition 3.1 in Ghosh et al. (2020)), gH-Gâteaux (Definition 4.3 in Ghosh et al. (2020)) and gH-Hadamard (Definition 12) derivatives of IVF $\mathbf{\Phi}$, it is clear that if $\mathbf{\Phi}_{\mathscr{H}}(\bar{t})(h)$ exists, then $\mathbf{\Phi}_{\mathscr{D}}(\bar{t})(h)$ and $\mathbf{\Phi}_{\mathscr{G}}(\bar{t})(h)$ exist and they are equal to $\mathbf{\Phi}_{\mathscr{H}}(\bar{t})(h)$. However, the converse is not true. For instance, consider the IVF $\mathbf{\Phi} : \mathbb{R}^2 \to I(\mathbb{R})$ defined by

$$\mathbf{\Phi}(t, y) = \begin{cases} \left(\frac{t^6}{(y-t^2)^2+t^8}\right) \odot [3,\ 9], & \text{if } (t, y) \neq (0, 0), \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

For $\bar{t} = (0, 0)$ and arbitrary $h = (h_1, h_2) \in \mathbb{R}^2$, we have

$$\lim_{\lambda\to 0+} \frac{1}{\lambda} \odot \big(\mathbf{\Phi}(\bar{t} + \lambda h) \ominus_{gH} \mathbf{\Phi}(\bar{t})\big) = \lim_{\lambda\to 0+} \frac{1}{\lambda} \odot \left(\left(\frac{\lambda^6 h_1^6}{(\lambda h_2 - \lambda^2 h_1^2)^2 + \lambda^8 h_1^8}\right) \odot [3,\ 9]\right) = \mathbf{0}.$$

Hence, $\mathbf{\Phi}$ is gH-directional and gH-Gâteaux differentiable at $\bar{t}$ with $\mathbf{\Phi}_{\mathscr{D}}(\bar{t})(h) = \mathbf{\Phi}_{\mathscr{G}}(\bar{t})(h) = \mathbf{0}$.

Let $\lambda_n = \frac{1}{n}$ and $h_n = (\frac{1}{n}, \frac{1}{n^3})$ for $n \in \mathbb{N}$. Then, for $\bar{t} = (0, 0)$, we have

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \odot \big(\mathbf{\Phi}(\bar{t} + \lambda_n h_n) \ominus_{gH} \mathbf{\Phi}(\bar{t})\big) = \lim_{n\to\infty} n^5 \odot [3, 9]. \tag{3.4}$$

Hence, $\mathbf{\Phi}_{\mathscr{H}}(\bar{t})(0)$ does not exist.

Theorem 3 *Let $\mathscr{S}$ be a nonempty convex subset of $\mathbb{R}^n$ and the IVF $\mathbf{\Phi} : \mathscr{S} \to I(\mathbb{R})$ has gH-Hadamard derivative at every $\bar{t} \in \mathscr{S}$. If the function $\mathbf{\Phi}$ is convex on $\mathscr{S}$, then*

$$\mathbf{\Phi}(v) \ominus_{gH} \mathbf{\Phi}(\bar{t}) \nprec \mathbf{\Phi}_{\mathscr{H}}(\bar{t})(v - \bar{t}), \quad \textit{for all } v \in \mathscr{S}.$$

Proof Since $\mathbf{\Phi}$ is convex on $\mathscr{S}$, for any $\bar{t},\ h \in \mathscr{S}$ and $\lambda,\ \lambda' \in (0, 1]$ with $\lambda + \lambda' = 1$, we have

$$\begin{aligned}
&\mathbf{\Phi}(\bar{t} + \lambda(h - \bar{t})) = \mathbf{\Phi}(\lambda h + \lambda' \bar{t}) \preceq \lambda \odot \mathbf{\Phi}(h) \oplus \lambda' \odot \mathbf{\Phi}(\bar{t}) \\
&\quad\Longrightarrow \mathbf{\Phi}(\bar{t} + \lambda(h - \bar{t})) \ominus_{gH} \mathbf{\Phi}(\bar{t}) \preceq (\lambda \odot \mathbf{\Phi}(h) \\
&\quad \oplus \lambda' \odot \mathbf{\Phi}(\bar{t})) \ominus_{gH} \mathbf{\Phi}(\bar{t}) \\
&\quad\Longrightarrow \mathbf{\Phi}(\bar{t} + \lambda(h - \bar{t})) \ominus_{gH} \mathbf{\Phi}(\bar{t}) \preceq \lambda \odot (\mathbf{\Phi}(h) \ominus_{gH} \mathbf{\Phi}(\bar{t})) \\
&\quad\Longrightarrow \frac{1}{\lambda} \odot (\mathbf{\Phi}(\bar{t} + \lambda(h - \bar{t})) \ominus_{gH} \mathbf{\Phi}(\bar{t})) \preceq \mathbf{\Phi}(h) \ominus_{gH} \mathbf{\Phi}(\bar{t}).
\end{aligned}$$

From Theorem 2, $\mathbf{\Phi}$ is gH-continuous. Thus, as $\lambda \to 0+$ and $h \to v$, we obtain

$$\mathbf{\Phi}_{\mathscr{H}}(\bar{t})(v - \bar{t}) \preceq \mathbf{\Phi}(v) \ominus_{gH} \mathbf{\Phi}(\bar{t}), \quad \text{for all } v \in \mathscr{S}. \tag{3.5}$$

If possible, let

$$\mathbf{\Phi}(v') \ominus_{gH} \mathbf{\Phi}(\bar{t}') \prec \mathbf{\Phi}_{\mathscr{H}}(\bar{t}')(v' - \bar{t}') \text{ for some } v' \in \mathscr{X}.$$

Then,

$$\mathbf{\Phi}(v') \ominus_{gH} \mathbf{\Phi}(\bar{t}') \prec \mathbf{\Phi}_{\mathscr{H}}(\bar{t}')(v' - \bar{t}'),$$

which contradicts (3.5). Hence,

$$\mathbf{\Phi}(v) \ominus_{gH} \mathbf{\Phi}(\bar{t}) \nprec \mathbf{\Phi}_{\mathscr{H}}(\bar{t})(v - \bar{t}), \quad \text{for all } v \in \mathscr{S}.$$

□

Remark 7 The converse of Theorem 3 is not true. For example, let us consider the IVF $\mathbf{\Phi} : \mathbb{R} \to I(\mathbb{R})$ defined by

$$\mathbf{\Phi}(t) = [-4t^2, 6t^2].$$

At $\bar{t} = 0 \in \mathbb{R}$, for arbitrary $v \in \mathbb{R}$, we have

$$\mathbf{\Phi}_{\mathscr{H}}(\bar{t})(v) = \lim_{\substack{\lambda \to 0+ \\ h \to v}} \frac{1}{\lambda} \odot \left(\mathbf{\Phi}(\bar{t} + \lambda h) \ominus_{gH} \mathbf{\Phi}(\bar{t})\right) = \mathbf{0}.$$

Hence, $\mathbf{\Phi}(v) \ominus_{gH} \mathbf{\Phi}(\bar{t}) \nprec \mathbf{\Phi}_{\mathscr{H}}(\bar{t})(v - \bar{t})$ for all $v \in \mathbb{R}$. However, $\underline{\Phi}$ is not convex on $\mathbb{R}$. Thus, from Lemma 3, $\mathbf{\Phi}$ is not convex on $\mathbb{R}$.

Remark 8 For a convex IVF $\mathbf{\Phi}$ on $\mathscr{S} \subset \mathbb{R}^n$, the inequality '$\mathbf{\Phi}_{\mathscr{H}}(\bar{t})(v-\bar{t}) \ominus_{gH} \mathbf{\Phi}_{\mathscr{H}}(v)(v-\bar{t}) \preceq \mathbf{0}$ for all $\bar{t}, v \in \mathscr{S}$' is not true. For instance, consider the convex IVF $\mathbf{\Phi} : \mathbb{R} \to I(\mathbb{R})$ defined by

$$\mathbf{\Phi}(t) = [t^2, 3t^2].$$

At $\bar{t} \in \mathbb{R}$, for arbitrary $v \in \mathbb{R}$, we have $\mathbf{\Phi}_{\mathscr{H}}(\bar{t})(v - \bar{t}) = 2\bar{t}(v-\bar{t}) \odot [1, 3]$. For $\bar{t} = 1$ and $v = 2$, we obtain $\mathbf{\Phi}_{\mathscr{H}}(\bar{t})(v-\bar{t}) \ominus \mathbf{\Phi}_{\mathscr{H}}(v)(v - \bar{t}) = [-10, 2] \npreceq \mathbf{0}$.

Notes 1 Let $\mathscr{S}$ be a nonempty convex subset of $\mathbb{R}^n$ and the IVF $\mathbf{\Phi} : \mathscr{S} \to I(\mathbb{R})$ be gH-Lipschitz continuous convex function. Then, at $\bar{t} \in \mathscr{S}$

$$\mathbf{\Phi}(h) \ominus_{gH} \mathbf{\Phi}(\bar{t}) \nprec \mathbf{\Phi}_{\mathscr{C}}(\bar{t})(h - \bar{t}), \quad \text{for all } h \in \mathscr{S}.$$

Theorem 4 *Let $\mathbf{\Phi} : \mathbb{R}^n \to I(\mathbb{R})$ be an IVF and $\bar{t} \in \mathbb{R}^n$. Then, for a given direction $v \in \mathbb{R}^n$, the following statements are equivalent:*

(i) *$\mathbf{\Phi}$ is gH-Hadamard differentiable at $\bar{t}$;*
(ii) *There exists a linear IVF $\boldsymbol{L} : \mathbb{R}^n \to I(\mathbb{R})$ such that for any path $f : \mathbb{R} \to \mathbb{R}^n$ with $f(0) = \bar{t}$ for which $f_{\mathscr{D}}(0)(1)$ exists, we have*

$$(\mathbf{\Phi} \circ f)_{\mathscr{D}}(0)(1) = \boldsymbol{L}(\bar{t})(v), \quad \textit{where } v = f_{\mathscr{D}}(0)(1).$$

Proof (i) $\Longrightarrow$ (ii). Let $\{\delta_n\}$ be a sequence of positive real numbers with $\delta_n \to 0^+$ and $h_n = \frac{1}{\delta_n}(f(\delta_n) - f(0))$ for all $n \in \mathbb{N}$. Since $f_{\mathscr{D}}(0)(1)$ exists, we have

$$\lim_{n\to\infty} h_n = \lim_{n\to\infty} \frac{1}{\delta_n} \odot (f(\delta_n) - f(0)) = f_{\mathscr{D}}(0)(1) = v. \tag{3.6}$$

If $\mathbf{\Phi}$ is gH-Hadamard differentiable at $\bar{t}$, then

$$\mathbf{\Phi}_{\mathscr{H}}(\bar{t})(v) = \lim_{n\to\infty} \frac{1}{\delta_n} \odot \left(\mathbf{\Phi}(\bar{t} + \delta_n h_n) \ominus_{gH} \mathbf{\Phi}(\bar{t})\right)$$

$$= \lim_{n\to\infty} \frac{1}{\delta_n} \odot \left(\mathbf{\Phi}(f(\delta_n)) \ominus_{gH} \mathbf{\Phi}(f(0))\right),$$

since $f(0) = \bar{t}$ and $h_n = \frac{1}{\delta_n}(f(\delta_n) - f(0))$

$$= \lim_{n\to\infty} \frac{1}{\delta_n} \odot \left((\mathbf{\Phi} \circ f)(\delta_n) \ominus_{gH} (\mathbf{\Phi} \circ f)(0)\right).$$

Hence, $(\mathbf{\Phi} \circ f)_{\mathscr{D}}(0)(1) = \mathbf{\Phi}_{\mathscr{H}}(\bar{t})(v)$. Due to the linearity of $\mathbf{\Phi}_{\mathscr{H}}(\bar{t})(v)$ on $\mathbb{R}^n$, by taking $\mathbf{L}(\bar{t})(v) = \mathbf{\Phi}_{\mathscr{H}}(\bar{t})(v)$, we get the desired result.
(ii) $\Longrightarrow$ (i). If possible, assume that $\mathbf{\Phi}$ is not gH-Hadamard differentiable at $\bar{t}$. Then, there exist sequences $h_n \to v$ and $\delta_n \to 0^+$ such that

$$\text{`either } \lim_{n\to\infty} \frac{1}{\delta_n} \odot \left(\mathbf{\Phi}(\bar{t} + \delta_n h_n) \ominus_{gH} \mathbf{\Phi}(\bar{t})\right) \text{ does not exist' or `limit value is not linear IVF on } \mathbb{R}^n\text{'}. \tag{3.7}$$

Since $h_n \to v$ and $\delta_n \to 0^+$, for every $\epsilon > 0$, there exist a natural number N and a real number a such that

$$\|h_n\| \leq a, \quad \|h_n - v\| < \epsilon, \text{ and } \quad \delta_n < \epsilon/a \text{ for all } n > N. \tag{3.8}$$

Using the sequences $\{h_n\}$ and $\{\delta_n\}$, we construct a function $f : \mathbb{R} \to \mathbb{R}^n$ as follows:

$$f(\delta) = \begin{cases} \bar{t} + \delta v, & \text{if } \delta \leq 0, \\ \bar{t} + \delta h_n, & \text{if } \delta_n \leq \delta < \delta_{n-1}, n \geq 2, \\ \bar{t} + \delta h_1, & \text{if } \delta \geq \delta_1. \end{cases}$$

Thus, the function f yields $f(0) = \bar{t}$ and $f_{\mathscr{D}}(0)(1) = v$ (for details, see p. 92 in Delfour (2012)). By hypothesis, $(\mathbf{\Phi} \circ f)_{\mathscr{D}}(0)(1)$ exists and equals to $\mathbf{L}(\bar{t})(v)$, where $v = f_{\mathscr{D}}(0)(1)$. From the construction of f, we have

$$\lim_{n\to\infty} \frac{1}{\delta_n} \odot \left((\mathbf{\Phi} \circ f)(\delta_n) \ominus_{gH} (\mathbf{\Phi} \circ f)(0)\right) = \mathbf{L}(\bar{t})(v)$$

or,

$$\lim_{n\to\infty} \frac{1}{\delta_n} \odot \left(\mathbf{\Phi}(f(\delta_n)) \ominus_{gH} \mathbf{\Phi}(f(0))\right) = \mathbf{L}(\bar{t})(v)$$

or,

$$\lim_{n\to\infty} \frac{1}{\delta_n} \odot \left(\mathbf{\Phi}(\bar{t} + \delta_n h_n) \ominus_{gH} \mathbf{\Phi}(\bar{t})\right) = \mathbf{L}(\bar{t})(v),$$

which contradicts to (3.7). Therefore, $\mathbf{\Phi}$ is gH-Hadamard differentiable at $\bar{t}$. □

Theorem 5 (Chain rule). *Let $H : \mathbb{R}^m \to \mathbb{R}^n$ be a vector-valued function and $\mathbf{\Phi} : \mathbb{R}^n \to I(\mathbb{R})$ be an IVF. Assume that for a point $\bar{t} \in \mathbb{R}^m$ and direction $v \in \mathbb{R}^m$,*

(a) $H_{\mathscr{D}}(\bar{t})(v)$ *exists for all* $v \in \mathbb{R}^m$, *and*
(b) $\mathbf{\Phi}_{\mathscr{H}}(\bar{y})(z)$ *exists, where* $\bar{y} = H(\bar{t})$ *and* $z = H_{\mathscr{D}}(\bar{t})(v)$.

Then,

(i) $(\mathbf{\Phi} \circ H)_{\mathscr{D}}(\bar{t})(v)$ *exists and* $(\mathbf{\Phi} \circ H)_{\mathscr{D}}(\bar{t})(v) = \mathbf{\Phi}_{\mathscr{H}}(\bar{y})(z)$
(ii) *if* $H_{\mathscr{H}}(\bar{t})(v)$ *exists, then* $(\mathbf{\Phi} \circ H)_{\mathscr{H}}(\bar{t})(v)$ *exists and*

$$(\mathbf{\Phi} \circ H)_{\mathscr{H}}(\bar{t})(v) = \mathbf{\Phi}_{\mathscr{H}}(\bar{y})(\bar{z}), \quad \text{where} \quad \bar{y} = H(\bar{t}),\ \bar{z} = H_{\mathscr{H}}(\bar{t})(v).$$

Proof (i) For $\delta > 0$, define

$$\mathbf{Q}(\delta) = \frac{1}{\delta} \odot \left(\mathbf{\Phi}(H(\bar{t} + \delta v)) \ominus_{gH} \mathbf{\Phi}(H(\bar{t}))\right) \quad \text{and}$$
$$\theta(\delta) = \frac{1}{\delta}\left(H(\bar{t} + \delta v) - H(\bar{t})\right). \tag{3.9}$$

Then,

$$\mathbf{Q}(\delta) = \frac{1}{\delta} \odot \left(\mathbf{\Phi}(H(\bar{t}) + \delta\theta(\delta)) \ominus_{gH} \mathbf{\Phi}(H(\bar{t}))\right). \tag{3.10}$$

Since $\theta(\delta) \to H_{\mathscr{D}}(\bar{t})(v)$ as $\delta \to 0+$, from (3.9), (3.10) and the hypothesis (b), we have

$$\begin{aligned}\mathbf{\Phi}_{\mathscr{H}}(\bar{y})(z) &= \lim_{\delta \to 0+} \frac{1}{\delta} \odot \left(\mathbf{\Phi}(H(\bar{t} + \delta v)) \ominus_{gH} \mathbf{\Phi}(H(\bar{t}))\right), \\ &\quad \text{where } \bar{y} = H(\bar{t}),\ z = H_{\mathscr{D}}(\bar{t})(v) \\ &= \lim_{\delta \to 0+} \frac{1}{\delta} \odot \left((\mathbf{\Phi} \circ H)(\bar{t} + \delta v) \ominus_{gH} (\mathbf{\Phi} \circ H)(\bar{t})\right) \\ &= (\mathbf{\Phi} \circ H)_{\mathscr{D}}(\bar{t})(v).\end{aligned}$$

(ii) For $\delta > 0$ and $h \in \mathbb{R}^m$, define

$$\mathbf{Q}'(\delta, h) = \frac{1}{\delta} \odot \left(\mathbf{\Phi}(H(\bar{t} + \delta h)) gH \mathbf{\Phi}(H(\bar{t}))\right)$$
$$\text{and } \Phi(\delta, h) = \frac{1}{\delta}\left(H(\bar{t} + \delta h) - H(\bar{t})\right). \tag{3.11}$$

Then,

$$\mathbf{Q}'(\delta, h) = \frac{1}{\delta} \odot \left(\mathbf{\Phi}(H(\bar{t}) + \delta\Phi(\delta, h)) \ominus_{gH} \mathbf{\Phi}(H(\bar{t}))\right). \tag{3.12}$$

Since $\Phi(\delta, h) \to H_{\mathscr{H}}(\bar{t})(v)$ as $\delta \to 0+$ and $h \to v$, from (3.11), (3.12) and the hypothesis (b), we have

$$\begin{aligned}\mathbf{\Phi}_{\mathscr{H}}(\bar{y})(\bar{k}) &= \lim_{\substack{\delta \to 0+ \\ h \to v}} \frac{1}{\delta} \odot \left(\mathbf{\Phi}(H(\bar{t} + \delta h)) \ominus_{gH} \mathbf{\Phi}(H(\bar{t}))\right), \text{ where } \bar{y} = H(\bar{t}),\ \bar{z} = H_{\mathscr{H}}(\bar{t})(v) \\ &= \lim_{\substack{\delta \to 0+ \\ h \to v}} \frac{1}{\delta} \odot \left((\mathbf{\Phi} \circ H)(\bar{t} + \delta h) \ominus_{gH} (\mathbf{\Phi} \circ H)(\bar{t})\right) \\ &= (\mathbf{\Phi} \circ H)_{\mathscr{H}}(\bar{t})(v).\end{aligned}$$

□

The weaker assumption—the existence of $G_{\mathscr{D}}(\bar{t})(v)$ and $\mathbf{\Phi}_{\mathscr{D}}(\bar{y})(k)$ with $\bar{y} = G(\bar{t})$, $k = G_{\mathscr{D}}(\bar{t})(v)$—is not sufficient to prove Theorem 5. For the proof of this theorem, we require a strong assumption (b) of Theorem 5. This is illustrated by the following example that the composition $\mathbf{\Phi} \circ G$, of a gH-Gâteaux differentiable IVF $\mathbf{\Phi}$ and a Gâteaux differentiable vector-valued function G, is not gH-Gâteaux differentiable and even not gH-directional differentiable in any direction $v \neq 0$.

Example 2 Consider the IVF $\mathbf{\Phi} : \mathbb{R}^2 \to I(\mathbb{R})$ defined by

$$\mathbf{\Phi}(t, y) = \begin{cases} \left(\frac{t^6}{(y - t^2)^2 + t^8}\right) \odot [2,\ 6], & \text{if } (t, y) \neq (0, 0), \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

and the vector-valued function $G : \mathbb{R} \to \mathbb{R}^2$ by $G(t) = (t, t^2)$ for all $t \in \mathbb{R}$.

It is clear that G is Gâteaux differentiable function at $\bar{t} = 0$ in every direction. Note that $\bar{y} = G(\bar{t}) = (0, 0)$ and for any $h \in \mathbb{R}^2$, we have

$$\begin{aligned}&\lim_{\lambda \to 0+} \frac{1}{\lambda} \odot \left(\mathbf{\Phi}(\bar{y} + \lambda h) \ominus_{gH} \mathbf{\Phi}(\bar{y})\right) \\ &= \lim_{\lambda \to 0+} \frac{1}{\lambda} \odot \left(\left(\frac{\lambda^6 h_1^6}{(\lambda h_2 - \lambda^2 h_1^2)^2 + \lambda^8 h_1^8}\right) \odot [2,\ 6]\right) = \mathbf{0}.\end{aligned}$$

Then, due to the linearity and gH-continuity of the limit value, $\mathbf{\Phi}$ is also gH-Gâteaux differentiable IVF at $\bar{y} = G(\bar{t})$.

The composition of $\mathbf{\Phi}$ and G is

$$\mathbf{H}(t) = (\mathbf{\Phi} \circ G)(t) = \begin{cases} \left(\frac{1}{t^2}\right) \odot [2,\ 6], & \text{if } (t, y) \neq (0, 0), \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

Since for $h \neq 0$,

$$\begin{aligned}&\lim_{\lambda \to 0+} \frac{1}{\lambda} \odot \left(\mathbf{H}(\bar{t} + \lambda h) \ominus_{gH} \mathbf{H}(\bar{t})\right) \\ &= \lim_{\lambda \to 0+} \frac{1}{\lambda^3 h} \odot [2, 6]\end{aligned}$$

does not exist, $\mathbf{H} = \mathbf{\Phi} \circ G$ is not gH-directional differentiable IVF at $G(\bar{t}) = 0$ in any direction $h \neq 0$.

Theorem 6 *Let* I *be a finite set of indices and* $\mathbf{\Phi}_i : \mathscr{X} \to I(\mathbb{R})$ *be a family of IVFs such that* $\mathbf{\Phi}_{i_{\mathscr{H}}}(\bar{t})(h)$

exists for all $h \in \mathscr{X}$. For each $t \in \mathscr{X}$, let the intervals in $\{\mathbf{\Phi}_i(t) : i \in I\}$ be comparable. If $\mathbf{\Phi}(t) = \max_{i \in I} \mathbf{\Phi}_i(t)$ for all $x \in \mathscr{X}$, then,

$$\mathbf{\Phi}_{\mathscr{H}}(\bar{t})(h) = \max_{i \in \mathscr{A}(\bar{t})} \mathbf{\Phi}_{i_{\mathscr{H}}}(\bar{t})(h), \text{ where } \mathscr{A}(\bar{t}) = \{i \in I : \mathbf{\Phi}_i(\bar{t}) = \mathbf{\Phi}(\bar{t})\}.$$

Proof Let $\bar{t} \in \mathscr{X}$ and $d \in \mathscr{X}$ be such that $\bar{t} + \lambda d \in \mathscr{X}$ for $\lambda > 0$. Then,

$$\begin{aligned}
&\mathbf{\Phi}_i(\bar{t} + \delta d) \preceq \mathbf{\Phi}(\bar{t} + \delta d), \quad \text{for all } i \in I\\
&\text{or, } \mathbf{\Phi}_i(\bar{t} + \delta d) \ominus_{gH} \mathbf{\Phi}(\bar{t}) \preceq \mathbf{\Phi}(\bar{t} + \delta d) \ominus_{gH} \mathbf{\Phi}(\bar{t}), \quad \text{for all } i \in I\\
&\text{or, } \mathbf{\Phi}_i(\bar{t} + \delta d) \ominus_{gH} \mathbf{\Phi}_i(\bar{t}) \preceq \mathbf{\Phi}(\bar{t} + \delta d) \ominus_{gH} \mathbf{\Phi}(\bar{t}), \quad \text{for each } i \in \mathscr{A}(\bar{t})\\
&\text{or, } \lim_{\substack{\delta \to 0+\\ d \to h}} \frac{1}{\delta} \odot (\mathbf{\Phi}_i(\bar{t} + \delta d) \ominus_{gH} \mathbf{\Phi}_i(\bar{t}))\\
&\preceq \lim_{\substack{\delta \to 0+\\ d \to h}} \frac{1}{\delta} \odot (\mathbf{\Phi}(\bar{t} + \delta d) \ominus_{gH} \mathbf{\Phi}(\bar{t}))\\
&\text{or, } \max_{i \in \mathscr{A}(\bar{t})} \mathbf{\Phi}_{i_{\mathscr{H}}}(\bar{t})(h) \preceq \mathbf{\Phi}_{\mathscr{H}}(\bar{t})(h).
\end{aligned} \tag{3.13}$$

To prove the reverse inequality, we claim that there exists a neighborhood $\mathscr{N}(\bar{t})$ such that $\mathscr{A}(t) \subset \mathscr{A}(\bar{t})$ for all $t \in \mathscr{N}(\bar{t})$. Assume on contrary that there exists a sequence $\{t_k\}$ in $\mathscr{X}$ with $t_k \to \bar{t}$ such that $\mathscr{A}(t_k) \not\subset \mathscr{A}(\bar{t})$. We can choose $i_k \in \mathscr{A}(t_k)$ but $i_k \notin \mathscr{A}(\bar{t})$. Since $\mathscr{A}(t_k)$ is closed, $i_k \to \bar{i} \in \mathscr{A}(t_k)$. By gH-continuity of $\mathbf{\Phi}$, we have

$$\mathbf{\Phi}_{\bar{i}}(t_k) = \mathbf{\Phi}(t_k) \implies \mathbf{\Phi}_{\bar{i}}(\bar{t}) = \mathbf{\Phi}(\bar{t}),$$

which contradicts to $i_k \notin \mathscr{A}(\bar{t})$. Thus, $\mathscr{A}(t) \subset \mathscr{A}(\bar{t})$ for all $x \in \mathscr{N}(\bar{t})$.

Let us choose a sequence $\{\delta_k\}$, $\delta_k \to 0$ such that $\bar{t} + \delta_k d \in \mathscr{N}(\bar{t})$ for all $d \in \mathscr{X}$. Then,

$$\begin{aligned}
&\mathbf{\Phi}_i(\bar{t}) \preceq \mathbf{\Phi}(\bar{t}), \quad \text{for all } i \in I\\
&\text{or, } \mathbf{\Phi}(\bar{t} + \delta_k d) \ominus_{gH} \mathbf{\Phi}(\bar{t}) \preceq \mathbf{\Phi}(\bar{t} + \delta_k d) \ominus_{gH} \mathbf{\Phi}_i(\bar{t}), \quad \text{for all } i \in \mathscr{A}(\bar{t})\\
&\text{or, } \mathbf{\Phi}(\bar{t} + \delta_k d) \ominus_{gH} \mathbf{\Phi}(\bar{t}) \preceq \mathbf{\Phi}_i(\bar{t} + \delta_k d) \ominus_{gH} \mathbf{\Phi}_i(\bar{t}), \quad \text{for all } i \in \mathscr{A}(\bar{t} + \delta_k d)\\
&\text{or, } \lim_{\substack{k \to \infty\\ d \to h}} \frac{1}{\delta_k} \odot (\mathbf{\Phi}(\bar{t} + \delta_k d) \ominus_{gH} \mathbf{\Phi}(\bar{t}))\\
&\preceq \lim_{\substack{k \to \infty\\ d \to h}} \frac{1}{\delta_k} \odot (\mathbf{\Phi}_i(\bar{t} + \delta_k d) \ominus_{gH} \mathbf{\Phi}_i(\bar{t}))\\
&\text{or, } \mathbf{\Phi}_{\mathscr{H}}(\bar{t})(h) \preceq \max_{i \in \mathscr{A}(\bar{t})} \mathbf{\Phi}_{i_{\mathscr{H}}}(\bar{t})(h).
\end{aligned} \tag{3.14}$$

From (3.13) and (3.14) , we obtain

$$\mathbf{\Phi}_{\mathscr{H}}(\bar{t})(h) = \max \mathbf{\Phi}_{i_{\mathscr{H}}}(\bar{t})(h) \text{ for all } i \in \mathscr{A}(\bar{t}).$$

□

4 Characterization of efficient solutions

In this section, we present some characterizations of efficient solutions for IOPs with the help of the properties of gH-Hadamard differentiable IVFs.

Theorem 7 (Sufficient condition for efficient points). *Let $\mathscr{S}$ be a nonempty convex subset of $\mathscr{X}$ and $\mathbf{\Phi} : \mathscr{S} \to I(\mathbb{R})$ be a convex IVF. If the function $\mathbf{\Phi}$ has a gH-Hadamard derivative at $\bar{t} \in \mathscr{S}$ in the direction $v - \bar{t}$ with*

$$\mathbf{\Phi}_{\mathscr{H}}(\bar{t})(v - \bar{t}) \nprec \mathbf{0}, \quad \textit{for all } v \in \mathscr{X}, \tag{4.1}$$

then $\bar{t}$ must be an efficient point of the IOP (2.1).

Proof Assume that $\bar{t}$ is not an efficient point of $\mathbf{\Phi}$. Then, there exists at least one $y \in \mathscr{S}$ such that for any $\lambda \in (0, 1]$, we have

$$\begin{aligned}
&\lambda \odot \mathbf{\Phi}(y) \prec \lambda \odot \mathbf{\Phi}(\bar{t}),\\
&\text{or, } \lambda \odot \mathbf{\Phi}(y) \oplus \lambda' \odot \mathbf{\Phi}(\bar{t}) \prec \lambda \odot \mathbf{\Phi}(\bar{t}) \oplus \lambda' \odot \mathbf{\Phi}(\bar{t}),\\
&\text{where } \lambda' = 1 - \lambda,\\
&\text{or, } \lambda \odot \mathbf{\Phi}(y) \oplus \lambda' \odot \mathbf{\Phi}(\bar{t}) \prec (\lambda + \lambda') \odot \mathbf{\Phi}(\bar{t}) = \mathbf{\Phi}(\bar{t}).
\end{aligned}$$

Due to the convexity of $\mathbf{\Phi}$ on $\mathscr{S}$, we have

$$\begin{aligned}
&\mathbf{\Phi}(\bar{t} + \lambda(y - \bar{t})) = \mathbf{\Phi}(\lambda y + \lambda' \bar{t}) \preceq \lambda \odot \mathbf{\Phi}(y) \oplus \lambda' \odot \mathbf{\Phi}(\bar{t}) \prec \mathbf{\Phi}(\bar{t}),\\
&\text{or, } \mathbf{\Phi}(\bar{t} + \lambda(v - \bar{t})) \ominus_{gH} \mathbf{\Phi}(\bar{t}) \prec \mathbf{0},\\
&\text{or, } \mathbf{\Phi}_{\mathscr{H}}(\bar{t})(v - \bar{t}) \preceq \mathbf{0}.
\end{aligned} \tag{4.2}$$

Now we have the following two possibilities.

- **Case** I: If $\mathbf{\Phi}_{\mathscr{H}}(\bar{t})(v - \bar{t}) = \mathbf{0}$, then $\mathbf{\Phi}_{\mathscr{D}}(\bar{t})(v - \bar{t}) = \mathbf{0}$ and

$$\underline{\Phi}_{\mathscr{D}}(\bar{t})(v - \bar{t}) = 0 \text{ and } \overline{\Phi}_{\mathscr{D}}(\bar{t})(v - \bar{t}) = 0. \tag{4.3}$$

  Due to Lemma 3, $\underline{\Phi}$ and $\overline{\Phi}$ are convex on $\mathscr{S}$. From (4.3), we observe that $\bar{t}$ is a minimum point of $\underline{\Phi}$ and $\overline{\Phi}$. Consequently, $\bar{t}$ is an efficient point of $\mathbf{\Phi}$. This contradicts to our assumption that $\bar{t}$ is not efficient point of $\mathbf{\Phi}$.
- **Case** II: If $\mathbf{\Phi}_{\mathscr{H}}(\bar{t})(v - \bar{t}) \prec \mathbf{0}$, then this contradicts the assumption that $\mathbf{\Phi}_{\mathscr{H}}(\bar{t})(v - \bar{t}) \nprec \mathbf{0}$ for all $v \in \mathscr{X}$.

Hence, $\bar{t}$ is the efficient point of the IOP (2.1). □

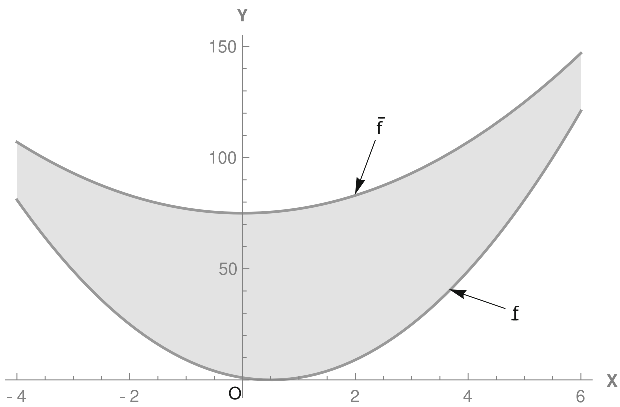

Fig. 1 IVF $\mathbf{\Phi}$ of Remark 9

Remark 9 The relation (4.1) can be seen as a variational inequality for interval-valued functions. For details as variational inequalities, we refer Ansari et al. (2013). The converse of Theorem 7 is not true. For example, consider $\mathscr{X} = \mathbb{R}$, $\mathscr{S} = [-1, 2]$, and the convex IVF $\mathbf{\Phi} : \mathscr{S} \to I(\mathbb{R})$ defined by

$$\mathbf{\Phi}(t) = [4t^2 - 4t + 1, 2t^2 + 75].$$

At $\bar{t} = 0$ and for $v \in \mathscr{X}$, $\mathbf{\Phi}_{\mathscr{H}}(\bar{t})(v) = v \odot [-4, 0]$ for all $v \in \mathscr{X}$.

From Fig. 1, it is clear that $\bar{t} = 0$ is an efficient solution of the IOP (2.1). However, for all $v > 0$ we have $\mathbf{\Phi}_{\mathscr{H}}(\bar{t})(v) \prec \mathbf{0}$.

Theorem 8 (Necessary condition for efficient points). *Let $\mathscr{S}$ be a linear subspace of $\mathscr{X}$, $\mathbf{\Phi} : \mathscr{S} \to I(\mathbb{R})$ be an IVF and $\bar{t} \in \mathscr{S}$ be an efficient point of the IOP (2.1). If the function $\mathbf{\Phi}$ has a gH-Hadamard derivative at $\bar{t}$ in every direction $v \in \mathscr{S}$, then*

$$\mathbf{\Phi}_{\mathscr{H}}(\bar{t})(v - \bar{t}) \nprec \mathbf{0}, \quad \textit{for all } v \in \mathscr{S}.$$

Proof Since the point $\bar{t}$ is an efficient point of the function $\mathbf{\Phi}$, for any $h \in \mathscr{S}$ and $\lambda > 0$, we have

$$\mathbf{\Phi}(\bar{t} + \lambda(h - \bar{t})) \ominus_{gH} \mathbf{\Phi}(\bar{t}) \nprec \mathbf{0}. \tag{4.4}$$

If $\mathbf{\Phi}_{\mathscr{H}}(\bar{t})(v - \bar{t}) \preceq \mathbf{0}$, then due to linearity of $\mathbf{\Phi}_{\mathscr{H}}(\bar{t})$ on $\mathscr{S}$, we have $\mathbf{\Phi}_{\mathscr{H}}(\bar{t})(v - \bar{t}) = \mathbf{0}$ by (ii) of Lemma 4. Therefore, $\mathbf{\Phi}_{\mathscr{H}}(\bar{t})(v - \bar{t}) \nprec \mathbf{0}$ for all $v \in \mathscr{S}$.
If $\mathbf{\Phi}_{\mathscr{H}}(\bar{t})(v - \bar{t}) \nprec \mathbf{0}$, then the result holds. □

Remark 10 One may think that in Theorem 8, instead of considering the fact that the IVF $\mathbf{\Phi}$ is defined on a linear subspace of $\mathscr{S}$, we may take $\mathbf{\Phi}$ being defined on any nonempty convex subset of $\mathscr{S}$. However, this assumption is not sufficient. For instance, consider $\mathscr{X} = \mathbb{R}$, $\mathscr{S} = [-1, 7]$, and the convex IVF $\mathbf{\Phi} : \mathscr{S} \to I(\mathbb{R})$ defined by $\mathbf{\Phi}(t) = [t^2 - 4t + 4, t^2 + 5]$. Then, at $\bar{t} \in \mathscr{S}$, $\mathbf{\Phi}_{\mathscr{H}}(\bar{t})(v) = 2v \odot [\bar{t} - 2, \bar{t}]$ for all $v \in \mathscr{X}$. Note that $\bar{t} = 0$ is an efficient point of IOP (2.1) because $\mathbf{\Phi}(y) \nprec \mathbf{\Phi}(\bar{t})$ for all $y \in \mathscr{S}$. However, $\mathbf{\Phi}_{\mathscr{H}}(\bar{t})(v) \prec \mathbf{0}$ for all $v > 0$.

Theorem 9 *Let $\mathscr{S}$ be a nonempty subset of $\mathscr{X}$, $\mathbf{\Phi} : \mathscr{S} \to I(\mathbb{R})$ be an IVF, and $\bar{t} \in \mathscr{S}$ be an efficient point of the IOP (2.1). If the IVF $\mathbf{\Phi}$ has a gH-Hadamard derivative at $\bar{t}$ in every direction $v \in \mathscr{S}$, then there exist no $v \in \mathscr{S}$ such that $\mathbf{\Phi}_{\mathscr{H}}(\bar{t})(v - \bar{t}) < \mathbf{0}$.*

Proof Since the point $\bar{t}$ is an efficient point of the function $\mathbf{\Phi}$, for any $h \in \mathscr{S}$ and $\lambda > 0$, we have

$$\mathbf{\Phi}(\bar{t} + \lambda(h - \bar{t})) \ominus_{gH} \mathbf{\Phi}(\bar{t}) \nprec \mathbf{0}.$$

This implies that

$$\lim_{\lambda \to 0+} \frac{1}{\lambda} \max\{\underline{\Phi}(\bar{t} + \lambda(h - \bar{t})) - \underline{\Phi}(\bar{t}), \overline{\Phi}(\bar{t} + \lambda(h - \bar{t})) - \overline{\Phi}(\bar{t})\} \geq 0. \tag{4.5}$$

From (4.5) and Lemma 2, there is no $v \in \mathscr{S}$ such that $\mathbf{\Phi}_{\mathscr{H}}(\bar{t})(v - \bar{t}) < \mathbf{0}$. □

Theorem 10 . *Let $\mathscr{S}$ be a linear subspace of $\mathscr{X}$, $\mathbf{\Phi} : \mathscr{S} \to I(\mathbb{R})$ be an IVF, and $\bar{t} \in \mathscr{S}$ be an efficient point of the IOP (2.1). If the IVF $\mathbf{\Phi}$ has a gH-Hadamard derivative at $\bar{t}$ in every direction $v \in \mathscr{S}$, then*

$$0 \in \mathbf{\Phi}_{\mathscr{H}}(\bar{t})(v), \quad \textit{for all } v \in \mathscr{S}.$$

The converse holds if $\mathbf{\Phi}$ is convex on $\mathscr{X}$.

Proof Let $\bar{t}$ be an efficient point of IOP (2.1). Then, by Theorem 8, we have $\mathbf{\Phi}_{\mathscr{H}}(\bar{t})(v) \nprec \mathbf{0}$ for all $v \in \mathscr{S}$. Due to linearity of $\mathbf{\Phi}_{\mathscr{H}}(\bar{t})$ and $v = -h$, we obtain $\mathbf{\Phi}_{\mathscr{H}}(\bar{t})(h) \nsucc \mathbf{0}$ for all $h \in \mathscr{S}$. Hence, $0 \in \mathbf{\Phi}_{\mathscr{H}}(\bar{t})(v)$ for all $v \in \mathscr{S}$.

Conversely, let $\mathbf{\Phi}$ be convex on $\mathscr{S}$ and assume that $\mathbf{\Phi}$ has a gH-Hadamard derivative at $\bar{t}$ in every direction $w \in \mathscr{X}$. Let $0 \in \mathbf{\Phi}_{\mathscr{H}}(\bar{t})(w)$ for all $w \in \mathscr{X}$. Then, due to linearity of $\mathbf{\Phi}_{\mathscr{H}}(\bar{t})$ on $\mathscr{S}$, we have

$$\mathbf{\Phi}_{\mathscr{H}}(\bar{t})(w) \nprec \mathbf{0} \text{ and } \mathbf{0} \nprec \mathbf{\Phi}_{\mathscr{H}}(\bar{t})(w) \text{ for all } w.$$

Hence, $\bar{t}$ is efficient point of IOP (2.1) by Theorem 7. □

5 Characterization of efficient solution using Karush–Kuhn–Tucker conditions

To characterize the efficient solutions of IOPs, we derive an extended Karush–Kuhn–Tucker necessary and sufficient conditions in this section.

Lemma 5 *Let $\mathbf{\Phi} : \mathbb{R}^n \to I(\mathbb{R})$ be a gH-Hadamard differentiable IVF at $\bar{t}$ in the direction $v \in \mathbb{R}^n$ with $\mathbf{\Phi}_{\mathscr{H}}(\bar{t})(v) \prec \mathbf{0}$. Then, there exists $\delta > 0$ such that for each $\lambda \in (0, \delta)$,*

$$\mathbf{\Phi}(\bar{t} + \lambda v) \prec \mathbf{\Phi}(\bar{t}).$$

Proof Since $\mathbf{\Phi}_{\mathscr{H}}(\bar{t})(v) \prec \mathbf{0}$, there exist $\delta, \delta' > 0$ such that for all $h \in \mathbb{R}^n$, we have

$$\frac{1}{\lambda} \odot (\mathbf{\Phi}(\bar{t} + \lambda h) \ominus_{gH} \mathbf{\Phi}(\bar{t})) \prec \mathbf{0}, \ \lambda \in (0, \delta) \text{ and } \|v - h\| < \delta'.$$

Due to gH-continuity of $\mathbf{\Phi}$ at v, we get

$$\mathbf{\Phi}(\bar{t} + \lambda v) \ominus_{gH} \mathbf{\Phi}(\bar{t}) \prec \mathbf{0}, \ \forall \lambda \in (0, \delta),$$

which implies $\mathbf{\Phi}(\bar{t} + \lambda v) \prec \mathbf{\Phi}(\bar{t}), \ \forall \lambda \in (0, \delta)$. □

Definition 13 The set of descent directions at $\bar{t} \in \mathbb{R}^n$ for a gH-Hadamard differentiable IVF $\mathbf{\Phi} : \mathbb{R}^n \to I(\mathbb{R})$, is defined by

$$\hat{\mathbf{\Phi}}(\bar{t}) = \{d \in \mathbb{R}^n : \mathbf{\Phi}_{\mathscr{H}}(\bar{t})(d) \prec \mathbf{0}\}.$$

As for any d in $\hat{\mathbf{\Phi}}(\bar{t})$, $\lambda d \in \hat{\mathbf{\Phi}}(\bar{t})$ for all $\lambda > 0$, the set $\hat{\mathbf{\Phi}}(\bar{t})$ is called the cone of descent direction.

Definition 14 Ghosh et al. (2019) Given a nonempty set $\mathscr{S} \subseteq \mathbb{R}^n$ and $\bar{t} \in \mathscr{S}$. At $\bar{t}$, the cone of feasible directions of $\mathscr{S}$ is defined by

$$\hat{\mathscr{S}}(\bar{t}) = \{d \in \mathbb{R}^n : d \neq 0, \ \bar{t} + \lambda d \in \mathscr{S}, \ \forall \lambda \in (0, \delta) \text{ and for some } \delta > 0\}.$$

Lemma 6 *Let $\bar{t} \in \mathscr{S}$ is an efficient solution of the IOP (2.1), where $\mathscr{S} \subseteq \mathbb{R}^n$ and $\mathbf{\Phi} : \mathbb{R}^n \to I(\mathbb{R})$ be a gH-Hadamard differentiable IVF at $\bar{t}$. Then, $\hat{\mathbf{\Phi}}(\bar{t}) \cap \hat{\mathscr{S}}(\bar{t}) = \emptyset$.*

Proof Assume contrary that $\hat{\mathbf{\Phi}}(\bar{t}) \cap \hat{\mathscr{S}}(\bar{t}) \neq \emptyset$ and $d \in \hat{\mathbf{\Phi}}(\bar{t}) \cap \hat{\mathscr{S}}(\bar{t})$. By Lemma 5 and Definition 14, there exist $\delta_1, \delta_2 > 0$ such that

$$\bar{t} + \lambda d \in \mathscr{S} \text{ for all } \lambda \text{ in } (0, \delta_1)$$
$$\text{and} \quad \mathbf{\Phi}(\bar{t} + \lambda d) \prec \mathbf{\Phi}(\bar{t}) \text{ for all } \lambda \text{ in } (0, \delta_2).$$

Taking $\delta = \min\{\delta_1, \delta_2\}$, we see that for all $\lambda \in (0, \delta)$,

$$\bar{t} + \lambda d \in \mathscr{S} \text{ and } \mathbf{\Phi}(\bar{t} + \lambda d) \prec \mathbf{\Phi}(\bar{t}).$$

This is contradictory to $\bar{t}$ being a local efficient point. Hence, $\hat{\mathbf{\Phi}}(\bar{t}) \cap \hat{\mathscr{S}}(\bar{t}) = \emptyset$. □

Lemma 7 *Let $\mathbf{\Psi}_i : \mathbb{R}^n \to I(\mathbb{R})$ be IVF for $i = 1, 2, \ldots, m$, and $\mathscr{S} = \{t \in X : \mathbf{\Psi}_i(t) \preceq \mathbf{0}$ for $i = 1, 2, \ldots, m\}$, where X be a nonempty open set in $\mathbb{R}^n$. Let $\mathbf{\Psi}_i$ be gH-Hadamard differentiable at $\bar{t} \in \mathscr{S}$ and gH-continuous for $i \notin I(\bar{t})$, where $I(\bar{t}) = \{i : \mathbf{\Psi}_i(\bar{t}) = \mathbf{0}\}$. For all $i \in I(\bar{t})$ define*

$$\hat{\Psi}(\bar{t}) = \{d : \mathbf{\Psi}_{i\mathscr{H}}(t)(d)_{(}\bar{t}) \prec \mathbf{0} \text{ for all } i \in I(t_0)\}.$$

Then, $\hat{\Psi}(\bar{t}) \subseteq \hat{\mathscr{S}}(\bar{t})$, where $\hat{\mathscr{S}}(\bar{t}) = \{d \in \mathbb{R}^n : d \neq 0, \ \bar{t} + \alpha d \in S \ \forall \alpha \ \in (0, \delta)$ for some $\delta > 0\}$.

Proof The proof of this Lemma is similar to Lemma 3.1 in Ghosh et al. (2019) for gH-Hadamard derivative, and therefore, we omit. □

With the help of Definition 13 and 14, and Lemma 7, we characterize an efficient solution of a constrained IOP.

Theorem 11 *Consider an IOP*

$$\left.\begin{array}{l} \min \ \mathbf{\Phi}(t) \\ \textit{such that } \ \mathbf{\Psi}_i(t) \preceq \mathbf{0}, \quad \textit{for } i = 1, 2, \ldots, m \\ t \in \mathscr{S}, \end{array}\right\} \tag{5.1}$$

where $\mathbf{\Phi} : \mathbb{R}^n \to I(\mathbb{R})$, $\mathbf{\Psi}_i : \mathbb{R}^n \to I(\mathbb{R})$ for $i = 1, 2, \ldots, m$, and $\mathscr{S}$ be a nonempty open set in $\mathbb{R}^n$. Define $I(t_0) = \{i : \mathbf{\Psi}_i(\bar{t}) = 0\}$ for a feasible point t_0. At $\bar{t}$, let $\mathbf{\Phi}$ and $\mathbf{\Psi}_i$, $i \in I(\bar{t})$, be gH-Hadamard differentiable, and for $i \notin I(\bar{t})$, $\mathbf{\Psi}_i$ be gH-continuous. If $\bar{t}$ is a local efficient solution of (5.1), then

$$\hat{\Phi}(\bar{t}) \cap \hat{\Psi}(\bar{t}) = \emptyset,$$

where $\hat{\Phi}(\bar{t}) = \{d : \mathbf{\Phi}_{\mathscr{H}}(t)(d) \prec \mathbf{0}\}$ and $\hat{\Psi}(\bar{t}) = \{d : \mathbf{G}_{i\mathscr{H}}(t)(d) \prec \mathbf{0}$ for each $i \in I(\bar{t})\}$.

Proof By Lemma 6 and Lemma 7, we obtain

$$t_0 \text{ is a local efficient solution} \implies \hat{\Phi}(\bar{t}) \cap \hat{\mathscr{S}}(\bar{t}) = \emptyset \implies \hat{\Phi}(\bar{t}) \cap \hat{\Psi}(\bar{t}) = \emptyset.$$

□

Theorem 12 *Let $\mathscr{S}$ be a nonempty open set in $\mathbb{R}^n$; $\mathbf{\Phi} : \mathbb{R}^n \to I(\mathbb{R})$ and $\mathbf{\Psi}_i : \mathbb{R}^n \to I(\mathbb{R})$ for $i = 1, 2, \ldots, m$ be IVFs. Consider the IOP:*

$$\left.\begin{array}{l} \min \ \mathbf{\Phi}(t), \\ \textit{such that } \mathbf{\Psi}_i(t) \preceq \mathbf{0}, \quad i = 1, 2, \ldots, m \\ t \in \mathscr{S}. \end{array}\right\} \tag{5.2}$$

For a feasible point $\bar{t}$, define $I(\bar{t}) = \{i : \mathbf{\Psi}_i(\bar{t}) = \mathbf{0}\}$. Let $\mathbf{\Phi}$ and $\mathbf{\Psi}_i$ be gH-Hadamard differentiable at $\bar{t}$ for $i \in I(\bar{t})$ and gH-continuous for $i \notin I(\bar{t})$. If $\bar{t}$ is a local efficient point of

(5.2), then there exist constants u_0 and u_i for $i \in I(\bar{t})$ such that

$$\begin{cases} 0 \in \left(u_0 \odot \mathbf{\Phi}_{\mathscr{H}}(\bar{t})(d) \oplus \sum\limits_{i \in I(\bar{t})} u_i \odot \mathbf{\Psi}_{i\mathscr{H}}(\bar{t})(d) \right), \\ u_0 \geq 0, u_i \geq 0 \text{ for } i \in I(\bar{t}), \\ (u_0, u_I) \neq \left(0, 0_v^{|I(\bar{t})|}\right), \end{cases}$$

where u_I is the vector whose components are u_i for $i \in I(\bar{t})$.

Further, if $\mathbf{\Psi}_i$, for all $i \notin I(\bar{t})$, are also gH-Hadamard differentiable at $\bar{t}$, then there exist constants $u_0, u_1, u_2, \ldots, u_m$ such that

$$\begin{cases} 0 \in \left(u_0 \odot \mathbf{\Phi}_{\mathscr{H}}(\bar{t})(d) \oplus \sum\limits_{i=1}^{m} u_i \odot \mathbf{G}_{i\mathscr{H}}(\bar{t})(d) \right), \\ u_i \odot \mathbf{\Psi}_i(\bar{t}) = \mathbf{0}, \ i = 1, 2, \ldots, m, \\ u_0 \geq 0, u_i \geq 0, \ i = 1, 2, \ldots, m, \\ (u_0, u) \neq \left(0, 0_v^m\right), \end{cases}$$

where u is the vector $(u_1, u_2, \ldots, u_m)$.

Proof Since $\bar{t}$ is a local efficient point of (5.2), by Theorem 11, we get

$$\begin{aligned} &\hat{\Phi}(\bar{t}) \cap \hat{\Psi}(\bar{t}) = \emptyset, \\ &\text{or, } \nexists\, d \in \mathbb{R}^n \text{ s.t. } \mathbf{\Phi}_{\mathscr{H}}(\bar{t})(d) \prec \mathbf{0} \text{ and } \mathbf{\Psi}_{i\mathscr{H}}(\bar{t})(d) \prec \mathbf{0}\ \forall\, i \in I(\bar{t}), \\ &\text{or, } \mathbf{\Phi}_{\mathscr{H}}(\bar{t})(d) \nprec \mathbf{0} \text{ and } \mathbf{\Psi}_{i\mathscr{H}}(\bar{t})(d) \nprec \mathbf{0}\ \forall\, d \in \mathbb{R}^n \text{ and } i \in I(\bar{t}), \\ &\text{or, } 0 \in \mathbf{\Phi}_{\mathscr{H}}(\bar{t})(d) \text{ and } 0 \in \mathbf{\Psi}_{i\mathscr{H}}(\bar{t})(d)\ \forall\, d \in \mathbb{R}^n \text{ and } i \in I(\bar{t}) \\ &\quad \text{by Lemma 4.} \end{aligned} \tag{5.3}$$

We can chose nonzero vector p with $p = [u_0, u_i]^\top_{i \in I(\bar{t})}$ such that

$$\begin{cases} 0 \in \left(u_0 \odot \mathbf{\Phi}_{\mathscr{H}}(\bar{t})(d) \oplus \sum\limits_{i \in I(\bar{t})} u_i \odot \mathbf{\Psi}_{i\mathscr{H}}(\bar{t})(d) \right), \\ u_0, u_i \geq 0 \text{ for } i \in I(t_0), \\ (u_0, u_I) \neq (0, 0, \cdots, 0). \end{cases}$$

This proves the first part of the theorem.

For $i \in I(\bar{t})$, $\mathbf{\Psi}_i(\bar{t}) = \mathbf{0}$. Therefore, $u_i \odot \mathbf{\Psi}_i(\bar{t}) = \mathbf{0}$. If $\mathbf{\Psi}_i$ for all $i \notin I(\bar{t})$ are also gH-differentiable at $\bar{t}$, by setting $u_i = 0$ for $i \notin I(\bar{t})$ the second part of the theorem is followed. □

Definition 15 Ghosh et al. (2019) The set of m intervals $\{\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_m\}$ is said to be linearly independent if for m real numbers $c_1, c_2, \ldots, c_m$:

$$\begin{aligned} &0 \in c_1 \odot \mathbf{X}_1 \oplus c_2 \odot \mathbf{X}_2 \oplus \ldots \oplus c_m \odot \mathbf{X}_m \quad \text{if and only if} \\ &\quad c_1 = 0, c_2 = 0, \ldots, c_m = 0. \end{aligned}$$

Theorem 13 (Extended Karush–Kuhn–Tucker necessary optimality condition). *Let $\mathscr{S}$ be a nonempty open set in $\mathbb{R}^n$ and $\mathbf{\Phi} : \mathbb{R}^n \to I(\mathbb{R})$ and $\mathbf{\Psi}_i : \mathbb{R}^n \to I(\mathbb{R})$, $i = 1, 2, \ldots, m$, be IVFs. Suppose that $\bar{t}$ is a feasible point of the following IOP:*

$$\left.\begin{aligned} &\min\ \mathbf{\Phi}(t) \\ &\textit{such that}\ \ \mathbf{\Psi}_i(t) \preceq \mathbf{0}, \quad i = 1, 2, \ldots, m \\ &t \in \mathscr{S}. \end{aligned}\right\}$$

Define $I(\bar{t}) = \{i : \mathbf{G}_i(\bar{t}) = 0\}$. Let

- (i) *$\mathbf{\Phi}$ and $\mathbf{G}_i$ be gH-Hadamard differentiable at $\bar{t}$ for all $i \in I(\bar{t})$,*
- (ii) *$\mathbf{\Psi}_i$ be gH-continuous for all $i \notin I(\bar{t})$, and*
- (iii) *the collection of intervals $\{\mathbf{\Psi}_{i\mathscr{H}}(\bar{t})(d) \ : \ i \in I(\bar{t})\}$ be linearly independent.*

If $\bar{t}$ is a local efficient solution, then there exist constants $u_i \geq 0$ for all $i \in I(\bar{t})$ such that

$$0 \in \left(u_0 \odot \mathbf{\Phi}_{\mathscr{H}}(\bar{t})(d) \oplus \sum_{i \in I(\bar{t})} u_i \odot \mathbf{\Psi}_{i\mathscr{H}}(\bar{t})(d) \right).$$

If $\mathbf{\Psi}_i$'s, for $i \notin I(\bar{t})$, are also gH-differentiable at $\bar{t}$, then there exist constants $u_1, u_2, \ldots, u_m$ such that

$$\begin{cases} 0 \in \left(u_0 \odot \mathbf{\Phi}_{\mathscr{H}}(\bar{t})(d) \oplus \sum\limits_{i=1}^{m} u_i \odot \mathbf{\Psi}_{i\mathscr{H}}(\bar{t})(d) \right), \\ u_i \odot \mathbf{\Psi}_i(\bar{t}) = \mathbf{0}, \ i = 1, 2, \ldots, m, \\ u_i \geq 0, \ i = 1, 2, \ldots, m. \end{cases}$$

Proof By Theorem 12, there exist real constants u_0 and u_i' for all $i \in I(\bar{t})$, not all zeros, such that

$$\begin{cases} 0 \in \left(u_0 \odot \mathbf{\Phi}_{\mathscr{H}}(\bar{t})(d) \oplus \sum\limits_{i \in I(\bar{t})} u_i' \odot \mathbf{\Psi}_{i\mathscr{H}}(\bar{t})(d) \right), \\ u_0 \geq 0, u_i' \geq 0 \text{ for all } i \in I(\bar{t}). \end{cases} \tag{5.4}$$

Then, we must have $u_0 > 0$. Since otherwise, the set $\{\mathbf{\Psi}_{i\mathscr{H}}(\bar{t})(d) \ : \ i \in I(\bar{t})\}$ will become linearly dependent.

Define $u_i = u_i'/u_0$. Then, $u_i \geq 0$ for all $i \in I(\bar{t})$ and

$$0 \in \left(u_0 \odot \mathbf{\Phi}_{\mathscr{H}}(\bar{t})(d) \oplus \sum_{i \in I(\bar{t})} u_i \odot \mathbf{\Psi}_{i\mathscr{H}}(\bar{t})(d) \right).$$

For $i \in I(\bar{t})$, $\mathbf{\Psi}_i(\bar{t}) = \mathbf{0}$. Therefore, $0 \in u_i \odot \mathbf{\Psi}_i(\bar{t})$. If the functions $\mathbf{\Psi}_i$ for $i \notin I(\bar{t})$ are also gH-Hadamard differentiable at t_0, then by setting $u_i = 0$ for $i \notin I(\bar{t})$, the latter part of the theorem is followed. □

Theorem 14 (Extended Karush–Kuhn–Tucker sufficient condition for efficient points). *Let $\mathscr{S}$ be a nonempty convex subset of $\mathscr{X}$; $\mathbf{\Phi} : \mathscr{S} \to I(\mathbb{R})$ and $\mathbf{\Psi}_i : \mathscr{S} \to I(\mathbb{R})$, $i = 1, 2, \cdots, m$ be interval-valued gH-Hadamard differentiable convex functions. Suppose that $\bar{t} \in \mathscr{S}$ is a feasible point of the following IOP:*

$$\left.\begin{aligned} & \min\ \ \mathbf{\Phi}(t) \\ & \text{such that}\ \ \mathbf{\Psi}_i(\bar{t}) \preceq \mathbf{0}, \quad i = 1, 2, \cdots, m \\ & t \in \mathscr{S}. \end{aligned}\right\} \tag{5.5}$$

If there exist real constants $u_1, u_2, \ldots, u_m$ for which

$$\begin{cases} \mathbf{\Phi}_{\mathscr{H}}(\bar{t})(v) \oplus \sum_{i=1}^{m} u_i \odot \mathbf{\Psi}_{i\mathscr{H}}(\bar{t})(v) \nprec \mathbf{0}, & \text{for all } v \in \mathscr{S}, \\ u_i \odot \mathbf{\Psi}_i(\bar{t}) = \mathbf{0},\ \ i = 1, 2, \cdots, m \\ u_i \geq 0,\ \ i = 1, 2, \cdots, m, \end{cases}$$

then $\bar{t}$ is an efficient point of the IOP.

Proof By the hypothesis, for every $v \in \mathscr{S}$ satisfying $\mathbf{\Psi}_i(v) \preceq \mathbf{0}$ for all $i = 1, 2, \ldots, m$, we have

$$\begin{aligned} & \mathbf{\Phi}_{\mathscr{H}}(\bar{t})(v - \bar{t}) \oplus \sum_{i=1}^{m} u_i \mathbf{\Psi}_{i\mathscr{H}}(\bar{t})(v - \bar{t}) \nprec \mathbf{0}, \\ & \implies \left(\mathbf{\Phi}(v) \ominus_{gH} \mathbf{\Phi}(\bar{t})\right) \oplus \left(\sum_{i=1}^{m} u_i \left(\mathbf{\Psi}_i(v) \ominus_{gH} \mathbf{\Psi}_i(\bar{t})\right)\right) \nprec \mathbf{0} \\ & \text{(by (3.5) of Theorem 3 and (i) of Lemma 1)}, \\ & \implies \left(\mathbf{\Phi}(v) \ominus_{gH} \mathbf{\Phi}(\bar{t})\right) \oplus \left(\sum_{i=1}^{m} u_i \left(\mathbf{\Psi}_i(v)\right)\right) \nprec \mathbf{0}, \\ & \implies \mathbf{\Phi}(v) \ominus_{gH} \mathbf{\Phi}(\bar{t}) \nprec \mathbf{0} \text{ from (ii) of Lemma 1}, \\ & \implies \mathbf{\Phi}(v) \nprec \mathbf{\Phi}(\bar{t}). \end{aligned}$$

Hence, $\bar{t}$ is an efficient point of the IOP. □

Next, remark shows that for the non gH-Hadamard differentiable gH-Lipschitz continuous IVFs, the extended Karush–Kuhn–Tucker sufficient condition for efficient points can be derived using upper gH-Clarke derivative of IVFs. This remark is also helpful to characterize the efficient solution of the Support Vector Machine (6.1) using upper gH-Clarke derivative.

Remark 11 If $\mathbf{\Phi}$ and $\mathbf{\Psi}_i$ of the system (5.5) are gH-Lipschitz continuous IVFs and there exist real constants $u_1, u_2, \ldots, u_m$ for which

$$\begin{cases} \mathbf{\Phi}_{\mathscr{C}}(\bar{t})(h) \oplus \sum_{i=1}^{m} u_i \odot \mathbf{\Psi}_{i\mathscr{C}}(\bar{t})(h) \nprec \mathbf{0}, & \text{for all } h \in \mathscr{S}, \\ u_i \odot \mathbf{\Psi}_i(\bar{t}) = \mathbf{0},\ \ i = 1, 2, \cdots, m \\ u_i \geq 0,\ \ i = 1, 2, \cdots, m, \end{cases}$$

then $\bar{t}$ is an efficient point of the IOP (5.5). Here, $\mathbf{\Phi}_{\mathscr{C}}(\bar{t})(h)$ and $\mathbf{\Psi}_{i\mathscr{C}}(\bar{t})(h)$ are upper gH-Clarke derivative of $\mathbf{\Phi}$ and $\mathbf{\Psi}_i$ at $\bar{t}$.

By the hypothesis, for every $h \in \mathscr{S}$ satisfying $\mathbf{\Psi}_i(h) \preceq \mathbf{0}$ for all $i = 1, 2, \ldots, m$, we have

$$\begin{aligned} & \mathbf{\Phi}_{\mathscr{C}}(\bar{t})(h - \bar{t}) \oplus \sum_{i=1}^{m} u_i \mathbf{\Psi}_{i\mathscr{C}}(\bar{t})(h - \bar{t}) \nprec \mathbf{0}, \\ & \implies \left(\mathbf{\Phi}(h) \ominus_{gH} \mathbf{\Phi}(\bar{t})\right) \oplus \left(\sum_{i=1}^{m} u_i \left(\mathbf{\Psi}_i(h) \ominus_{gH} \mathbf{\Psi}_i(\bar{t})\right)\right) \nprec \mathbf{0} \\ & \text{(by Note 1 and (i) of Lemma 1)}, \\ & \implies \left(\mathbf{\Phi}(h) \ominus_{gH} \mathbf{\Phi}(\bar{t})\right) \oplus \left(\sum_{i=1}^{m} u_i \left(\mathbf{\Psi}_i(h)\right)\right) \nprec \mathbf{0}, \\ & \implies \mathbf{\Phi}(h) \ominus_{gH} \mathbf{\Phi}(\bar{t}) \nprec \mathbf{0} \text{ from (ii) of Lemma 1}, \\ & \implies \mathbf{\Phi}(h) \nprec \mathbf{\Phi}(\bar{t}). \end{aligned}$$

Hence, $\bar{t}$ is an efficient point of the IOP (5.5).

6 Application to support vector machines

The standard Support Vector Machines (SVM) formulation is not applicable for imprecise or uncertain data. Thus, we formulate the hard-margin SVM problem for the interval-valued data set

$$\left\{(\mathbf{X}_i, y_i) \mid \mathbf{X}_i \in I(\mathbb{R})^n,\ \ y_i \in \{-1, 1\}, i = 1, 2, \cdots, m\right\}$$

by

$$\left.\begin{aligned} & \max_{w,b}\ \Phi_1(w, b) = \frac{1}{\|w\|}, \\ & \text{such that}\ \ \mathbf{\Psi}_i(w, b) = y_i \odot \left(w^\top \odot \mathbf{X}_i \oplus b\right) \\ & \quad \preceq [1, 1],\ \ i = 1, 2, \ldots, m. \end{aligned}\right\}$$

or,

$$\left.\begin{aligned} & \min_{w,b}\ \Phi_1'(w, b) = \|w\|, \\ & \text{such that}\ \ \mathbf{\Psi}_i(w, b) = y_i \odot \\ & \quad \left(w^\top \odot \mathbf{X}_i \oplus b\right) \preceq [1, 1],\ \ i = 1, 2, \ldots, m. \end{aligned}\right\} \tag{6.1}$$

By Definition 2.8 of Ghosh et al. (2020), we note that the functions Φ_1' is gH-Lipschitz continuous since

$$\|\Phi_1'(w_1, b) \ominus_{gH} \Phi_1'(w_2, b)\|_{I(\mathbb{R})} \leq \|w_1 - w_2\| \text{ for all } w_1, w_2 \in \mathbb{R}^n.$$

Therefore, by Theorem 1 of Chauhan et al. (2021), Φ_1' is gH-Clarke differentiable and at $\bar{t} = (\bar{w}, \bar{b})$ in the direction

$h = (h, b)$, we have

$$\Phi_{\mathscr{C}}(\bar{t})(h) = w \odot [\bar{w}, \bar{w}].$$

In addition, by Definition 10 of upper gH-Clarke derivative, $\boldsymbol{\Psi}_i$ are upper gH-Clarke differentiable at $\bar{t} = (\bar{w}, \bar{b})$ in the direction $h = (h', b')$, we have

$$\begin{aligned}
&\limsup_{\substack{x \to \bar{t} \\ \lambda \to 0+}} \frac{1}{\lambda} \odot \left(\boldsymbol{\Psi}_i(t + \lambda h) \ominus_{gH} \boldsymbol{\Psi}_i(t)\right) \\
&= \limsup_{\substack{x \to \bar{t} \\ \lambda \to 0+}} \frac{1}{\lambda} \odot \left(y_i \odot \left((w + \lambda w')^\top \odot \boldsymbol{X}_i \oplus (b + b')\right)\right. \\
&\left.\ominus_{gH} y_i \odot \left(w^\top \odot \boldsymbol{X}_i \oplus b\right)\right) \\
&= \limsup_{\substack{x \to \bar{t} \\ \lambda \to 0+}} \frac{1}{\lambda} \odot \left(y_i \odot \left((\lambda w')^\top \odot \boldsymbol{X}_i \oplus b'\right)\right) \\
&= y_i \odot \left(w'^\top \odot \boldsymbol{X}_i \oplus b'\right).
\end{aligned}$$

Hence, $\boldsymbol{\Psi}_{i\mathscr{C}}(\bar{t})(d) = (w \odot (y_i \odot \boldsymbol{X}_i) \oplus by_i)$ at $\bar{t} = (\bar{w}, \bar{b})$ in the direction $h = (h, b)$.

Here, we can only characterize the efficient solutions of the SVM (6.1) if we use the existing concepts of upper gH-Clarke derivative in Chauhan et al. (2021) and Remark 11, but we unable to identify the classifying hyperplane which separates the $+1$'s from -1's since there does not exist any article which derives the KKT necessary condition for IOPs using Clarke derivative for IVFs.

In addition, problem (6.1) cannot be solved using gH-Hadamard derivative since Φ_1' is not gH-Hadamard differentiable by Remark 3 for $w \in \mathbb{R}$. However, if we reformulate it to another SVM problem as

$$\left.\begin{aligned}
&\min_{w,b} \Phi(w, b) = \tfrac{1}{2}\|w\|^2, \\
&\text{such that } \boldsymbol{\Psi}_i(w, b) = [1, 1] \ominus_{gH} y_i \odot \left(w^\top \odot \boldsymbol{X}_i \oplus b\right) \\
&\quad \preceq \boldsymbol{0}, \quad i = 1, 2, \ldots, m,
\end{aligned}\right\} \tag{6.2}$$

then, we can solve it by gH-Hadamard derivative. We note that the functions Φ and $\boldsymbol{\Psi}_i$ are gH-Hadamard differentiable and convex. At $\bar{t} = (\bar{w}, \bar{b})$, in the direction $v = (w, b)$, we have

$$\Phi_{\mathscr{H}}(\bar{t})(v) = w \odot [\bar{w}, \bar{w}] \text{ and}$$
$$\boldsymbol{\Psi}_{i\mathscr{H}}(\bar{t})(d) = (w \odot (y_i \odot \boldsymbol{X}_i) \oplus by_i).$$

According to Theorem 13, for an efficient point $(\bar{w}, \bar{b})$ of (6.2), there exist nonnegative scalars $u_1, u_2, \ldots, u_m$ such that

$$\boldsymbol{0} \in \left(w \odot [\bar{w}, \bar{w}] \oplus \sum_{i=1}^{m} u_i \odot \oplus (w \odot (y_i \odot \boldsymbol{X}_i) \oplus by_i)\right), \tag{6.3}$$

and

$$\boldsymbol{0} = u_i \odot \boldsymbol{\Psi}_i(w^*, b^*), \quad i = 1, 2, \ldots, m. \tag{6.4}$$

The condition (6.3) can be simplified as

$$\boldsymbol{0} \in \left([w^*, w^*] \oplus \sum_{i=1}^{m} (u_i y_i) \odot \boldsymbol{X}_i\right) \text{ and } \sum_{i=1}^{m} u_i y_i = 0.$$

The data points $\boldsymbol{X}_i$ for which $u_i \neq 0$ are called support vectors. By (6.4), corresponding to any $u_i > 0$, we have $\boldsymbol{\Psi}_i(w^*, b^*) = \boldsymbol{0}$. Thus, corresponding to w^*, the value of the bias b^* is such a quantity that $\boldsymbol{\Psi}_i(w^*, b^*) = \boldsymbol{0}$ for all of those $i \in \{1, 2, \ldots, m\}$ for which $u_i > 0$.

As the functions Φ and $\boldsymbol{\Psi}_i$ are gH-Hadamard differentiable and convex, by Theorems 13 and 14, the set of conditions by which we obtain the efficient solutions of the SVM IOP (6.2) are

$$\begin{cases}
\boldsymbol{0} \in \left([w, w] \oplus \sum_{i=1}^{m} (u_i y_i) \odot \boldsymbol{X}_i\right), \\
\sum_{i=1}^{m} u_i y_i = 0, \\
\boldsymbol{0} = u_i \odot \boldsymbol{\Psi}_i(w, b), \; i = 1, 2, \ldots, m.
\end{cases} \tag{6.5}$$

Corresponding to any of the value of w that satisfies (6.5), we define the set of possible values of the bias by

$$\bigcap_{i:\, u_i > 0} \{b \mid \boldsymbol{\Psi}_i(w, b) = \boldsymbol{0}\}. \tag{6.6}$$

Using any solution $\bar{w}$ and $\bar{b}$ of (6.5) and (6.6), a classifying hyperplane and the SVM classifier function are given by

$$\bar{w}^\top \boldsymbol{X} + \bar{b} = \boldsymbol{0} \quad \text{and} \quad s^*(\boldsymbol{X}) = \text{sign}\left(\bar{w}^\top \boldsymbol{X} + \bar{b}\right),$$

where sign $(\cdot)$ denotes the sign function.

7 Conclusion and future directions

In this article, the concept of gH-hadamard derivative for IVFs has been studied (Definition 12). One can trivially notice that in the degenerate case, the Definition 12 reduces to the respective conventional definition for the real-valued functions (see Yu and Liu (2013), de Miranda and Fichmann (2005)). It has been noticed that the gH-Hadamard derivative at any point is the gH-Fréchet derivative at that point and vise-versa (Theorem 1). In addition, a gH-Hadamard differentiable IVF is found to be gH-continuous (Theorem 2). It has been shown that the gH-Hadamard derivative is helpful to characterize the convexity of an IVF (Theorem 3). It is also observed that the composition of a Hadamard differentiable real-valued function and a gH-Hadamard differentiable IVF is gH-Hadamard differentiable IVF and the chain rule is applicable (Theorem 5 and Theorem 4). Further, for a finite number of IVFs whose values are comparable at each point, it has been proven that the gH-Hadamard derivative of maximum of all finite comparable IVFs is the maximum of their gH-Hadamard derivative (Theorem 6).

In addition, it has been shown that if the objective function of an IOP is convex on the feasible set $\mathscr{S} \subseteq \mathscr{X}$ and gH-Hadamard derivative at $\bar{t} \in \mathscr{S}$ does not dominate to $\mathbf{0}$, then $\bar{t}$ is an efficient point of that IOP (Theorem 7). Further, it is proved that if the feasible set $\mathscr{S}$ is linear subspace of $\mathscr{X}$ and the objective function of IOP is gH-Hadamard differentiable at an efficient point of IOP, then gH-Hadamard derivative does not dominate to $\mathbf{0}$ (Theorem 8) and also contains 0 (Theorem 10). Moreover, for constraint IOPs, we have proved extended KKT necessary and sufficient condition to characterize the efficient solutions using gH-Hadamard derivative (Theorem 13 and Theorem 14).

As an application of the proposed gH-Hadamard derivative, we have formulated and solved SVM problem for interval-valued data.

In analogy to the current study, future research can be carried out for other generalized directional derivatives for IVFs, e.g., upper and lower Dini semiderivative, Hadamard semiderivative, upper and lower Hadamard semiderivative, Michel-Penot, etc., and their relationships with fuzzy set theory Delfour (2012), Ghosh and Chakraborty (2014, 2015).

In parallel to the proposed analysis of IVFs, another promising direction of future research can be the analysis of the fuzzy-valued functions (FVFs) as the alpha-cuts of fuzzy numbers are compact intervals Ghosh and Chakraborty (2019). Hence, we expect that some results for FVFs can be obtained in a similar way to this paper.

Appendix A: Proof of Lemma 1

Proof Let $\mathbf{P} = [\underline{p}, \overline{p}]$ and $\mathbf{Q} = [\underline{q}, \overline{q}]$.

(i) Since $\mathbf{Q} \not\prec \mathbf{0}$ and $\mathbf{Q} \preceq \mathbf{P}$, then

$$\overline{q} \geq 0 \text{ and } \overline{q} \geq \overline{p} \implies \overline{p} \geq 0 \implies \mathbf{P} \not\prec \mathbf{0}.$$

(ii) Since $\mathbf{P} \oplus \mathbf{Q} \not\prec \mathbf{0}$ and $\mathbf{Q} \preceq \mathbf{0}$, then

$$\overline{p} + \overline{q} \geq 0 \text{ and } \overline{q} \leq 0 \implies \overline{p} \geq 0 \implies \mathbf{P} \not\prec \mathbf{0}.$$

□

Appendix B: Proof of Lemma 4

Proof (i) If

$$\mathbf{\Phi}(t) \not\prec \mathbf{0} \text{ for all } t \in \mathscr{S}, \tag{7.1}$$

then due to linearity of $\mathbf{\Phi}$, we have

$$\mathbf{\Phi}(t) = (-1) \odot \mathbf{\Phi}(-t) \not\succ \mathbf{0} \text{ for all } t \in \mathscr{S} \tag{7.2}$$

since $\mathbf{\Phi}(-t) \not\prec \mathbf{0}$ by (7.1). From (7.1) and (7.2), it is clear that $\mathbf{0}$ and $\mathbf{\Phi}(t)$ are not comparable.

(ii) If $\mathbf{\Phi}(t) \preceq \mathbf{0}$ for all $x \in \mathscr{S}$, then due to linearity of $\mathbf{\Phi}$, we have $\mathbf{\Phi}(t) = (-1) \odot \mathbf{\Phi}(-t) \succeq \mathbf{0}$ for all $t \in \mathscr{S}$. Hence, $\mathbf{\Phi}(t) = \mathbf{0}$.

□

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