Chapter 4. Three-Dimensional Path Independent Integral for Coupled Magneto-Thermo-Elastic Fracture Domain

4.1. Introduction

The analysis of the magneto-elastically coupled fracture problem for giant magnetostrictive material such as Terfenol-D is essentially a magnetization and stress dependent elasticity problem requiring a meticulous evaluation of the state of stress. Additionally, the nature of the failure in such bi-nonlinear material is influenced by the state of tension and compression regions in the flexure loading. When the stress-strain behaviour is different in tension and compression, it becomes necessary to formulate a bi-nonlinear fracture theory to account the singularity at the crack front. This happen to be a classical stress-dependent elasticity problem and additionally, for magnetostriction fracture studies, the multi-physics of magnetic field domain must be coupled to the elasticity solution. It would be unwise to ignore this real field phenomena which might otherwise have caused many unwarranted failures in design. So cause, the three-dimensional treatment is necessary to analyses this nonlinear fracture phenomenon [55,61,62].

Hence, a coupled multiphysics three-dimensional numerical approach was required to evaluate the path independent fracture parameter referred to as the 3D J-integral. In chapter, a three-dimensional J-integral definition for characterizing the fracture behaviour has been formulated employing the Eshelby [63] four-dimensional dynamic energy-momentum tensor. The formulated 3-D path independent integral is limited to mode I crack cases. For maintaining the generality, the influence of magneto-thermo-elasticity as discussed in previous chapter is taken care of with the thermal strain, magnetic strain, and the magnetic body force terms. However, the experimental investigation in this thesis is primarily focused on the magneto-elastic physics. But the generalised expression has the application in the coupled magneto-thermoelastic domain also.

4.2. Formulation of the path independent 3D J-Integral

In elastodynamics, considering the rectangular cartesian coordinates X_1, X_2, X_3 along with the time variable as $X_4 = t$ and taking for the Lagrangian density $L = \mathcal{T} - \mathcal{W}$ where $\mathcal{T} = \frac{1}{2}\rho\dot{\boldsymbol{u}}^2$ is the kinetic energy density, and $\mathcal{W}$ is the strain energy density; Eshelby [63] defined the components of a four-dimensional dynamic energy-momentum tensor with components

$$\mathcal{T}_{lj} = \left(\mathcal{W} - \frac{1}{2}\rho\dot{\boldsymbol{u}}^2\right)\delta_{lj} - \sigma_{ij}u_{i,l} \qquad i,j,l = 1,2,3 \tag{4.1}$$

$$s_j = \mathcal{T}_{j4} = -\sigma_{ij}\dot{u}_i \tag{4.2}$$

$$g_l = \mathcal{T}_{4l} = \rho\dot{u}_i u_{i,l} \tag{4.3}$$

$$\mathcal{T}_{44} = \frac{1}{2}\rho\dot{\boldsymbol{u}}^2 + \mathcal{W} \tag{4.4}$$

where σ_{ij} is stress tensor, u_i is the displacement, δ_{lj} is Kronecker symbol and $(\dot{a})$ represents the time derivative.

Considering a homogeneous medium and external body forces f_i in the account, Eshelby gives the conservation law

$$\frac{\partial \mathcal{T}_{lj}}{\partial X_j} + \frac{\partial g_l}{\partial t} = f_i u_{i,l} \tag{4.5}$$

The conservation law, with the help of Eq. (4.1) and (4.3), is transformed as

$$\frac{\partial}{\partial X_j}\left[\left(\mathcal{W} - \frac{1}{2}\rho\dot{\boldsymbol{u}}^2\right)\delta_{lj} - \sigma_{ij}u_{i,l}\right] + \frac{\partial}{\partial t}\left[\rho\dot{u}_i u_{i,l}\right] - f_i u_{i,l} = 0 \tag{4.6}$$

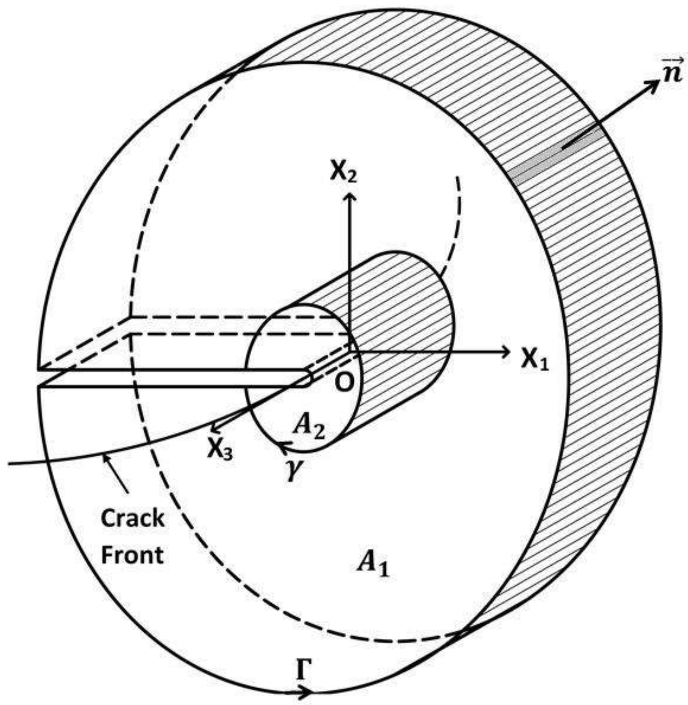

Figure 4.1 Configuration of three-dimensional crack front in solid; A_2: fracture process region

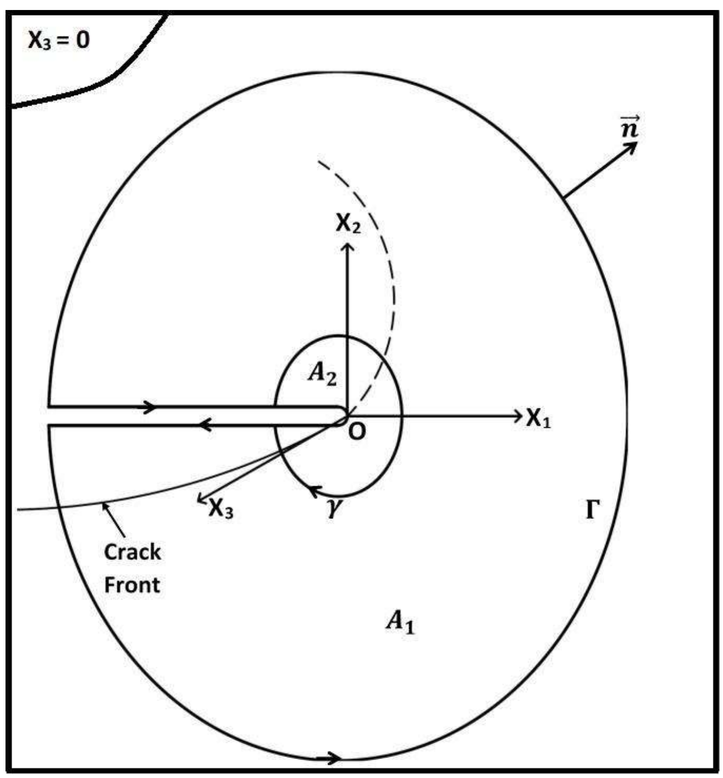

Figure 4.2 Arbitrary integration contour perpendicular to crack front at O, in the plane $X_3 = 0$.

Consider a three-dimensional solid material with traction-free crack and the local axes $(O - X_1, X_2, X_3)$ defined at a point O along the crack front, as shown in **Figure 4.1**. The directions of X_1, X_2 are taken as coincidental with the normal of the crack front contour and the direction of X_3 is set as tangential to the crack front contour. **Figure 4.2** shows an arbitrary integration area A_1 containing a crack tip in the plane $\mathrm{X}_3 = 0$. Let A_2 is the similar surface area which accommodated the crack tip singularity. Here, Γ and γ represent the contours of area A_1 and A_2 respectively. Now, taking conservation law for Eshelby dynamic energy-momentum tensor [63] into account, as shown in Eq.s (4.1)-(4.4), we can integrate Eq. (4.6) over any area in the plane $\mathrm{X}_3 = 0$, which does not contain the singularity. Then

$$\int_{A_1 - A_2} \left\{ \frac{\partial}{\partial X_j} \left[\left(\mathcal{W} - \frac{1}{2} \rho \dot{\boldsymbol{u}}^2 \right) \delta_{lj} - \sigma_{ij} u_{i,l} \right] + \frac{\partial}{\partial t} \left[\rho \dot{u}_i u_{i,l} \right] - f_i u_{i,l} \right\} dA = 0 \tag{4.7}$$

Considering the superposition of integration domain Eq. (4.7) can be rearranged as

$$\int_{A_2} \left\{ \frac{\partial}{\partial X_j} \left[\left(\mathcal{W} - \frac{1}{2} \rho \dot{\boldsymbol{u}}^2 \right) \delta_{lj} - \sigma_{ij} u_{i,l} \right] + \frac{\partial}{\partial t} \left[\rho \dot{u}_i u_{i,l} \right] - f_i u_{i,l} \right\} dA \tag{4.8}$$

$$= \int_{A_1} \left\{ \frac{\partial}{\partial X_j} \left[\left(\mathcal{W} - \frac{1}{2} \rho \dot{\boldsymbol{u}}^2 \right) \delta_{lj} - \sigma_{ij} u_{i,l} \right] + \frac{\partial}{\partial t} \left[\rho \dot{u}_i u_{i,l} \right] - f_i u_{i,l} \right\} dA$$

With the area A_2 containing the crack front singularity tends to zero, the path independent J-integral can be defined as

$$J_l^n(s) = \int_{A_1} \left\{ \frac{\partial}{\partial X_j} \left[\left(\mathcal{W} - \frac{1}{2} \rho \dot{\boldsymbol{u}}^2 \right) \delta_{lj} - \sigma_{ij} u_{i,l} \right] + \frac{\partial}{\partial t} \left[\rho \dot{u}_i u_{i,l} \right] - f_i u_{i,l} \right\} dA \tag{4.9}$$

where $J_l^n(s)$ denote the rate of energy dissipated in the fracture process region with respect to crack extension per unit area and is dependent on the position of crack front s. The superscript n is defined to make clear that the contour area A_1 should be perpendicular to the crack front.

$$J_l^n(s) = \int_{A_1} \left\{ \frac{\partial}{\partial X_j} \left[\mathcal{W} \delta_{lj} - \sigma_{ij} u_{i,l} \right] \right\} dA - \int_{A_1} \rho \dot{u}_i \frac{\partial \dot{u}_i}{\partial X_j} \delta_{lj} dA + \int_{A_1} \rho \dot{u}_i \dot{u}_{i,l} dA + \int_{A_1} (\rho \ddot{u}_i - f_i) u_{i,l} dA \tag{4.10}$$

The strain energy density $\mathcal{W}$ for a magnetostrictive material subjected to the coupled magneto-thermo-elastic field is defined by [63]

$$\sigma_{ij} = \frac{\partial \mathcal{W}(\varepsilon_{ij})}{\partial \varepsilon_{ij}} \tag{4.11}$$

4.2.1. Coupled Magneto-Thermo-Elastic field contribution in J-integral

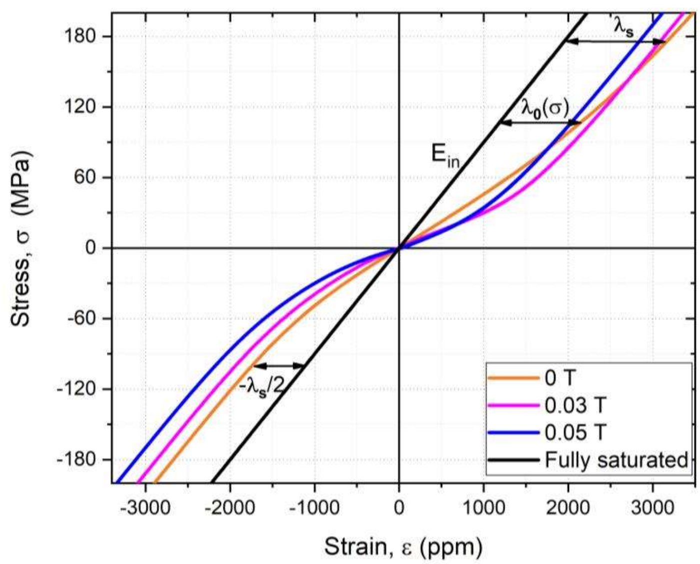

Figure 4.3 Stress-strain curves predicted by the constitutive model at different magnetic field levels when $N_{xx} = 0.096$

Giant magnetostrictive materials exhibit distinct nonlinear stress-strain behaviour under tension and compression as plotted in **Figure 4.3** employing numerical evaluation (Ref: Eqn. (3.29) and (3.30)). Depending on the stress and magnetization distribution, this nonlinear elastic nature of material must be considered in Eq. (4.10) for the J-integral computation. In the present investigation, the novel hysteretic nonlinear magneto-thermo-elastic constitutive model proposed in Chapter 2. (ref. [127]) was implemented. To better understand the nonlinear magneto-elastic behaviour, stress-strain curves for the uncracked Terfenol-D specimen have been plotted for the unidirectional case (i.e., in its longitudinal direction or x-direction) in **Figure 4.3**, as predicted by the constitutive model at different magnetic field levels. Considering that the Terfenol-D specimen have no internal defects, the geometric demagnetizing factor acts as a sole contributor to the demagnetizing factor (N). For a cuboid shape Terfenol-D specimen, the value of shape demagnetizing factor is calculated from the approximate expression given in the existing study [60]. The demagnetization factor in x-direction is evaluated as $N_{xx} = 0.096$ and in remaining directions as $N_{yy} = N_{zz} = 0.452$. The plot shows that the rotation or movement of magnetic domains generates nonlinearity in the stress-strain curve before the material is magnetized (i.e., at 0 T). The linear stress-strain curve independent of domain rotation is sketched as a black line with the slope of E_{in}. As previously discussed, E_{in} is the intrinsic Young's modulus value when the magnetization of the magnetostrictive material approaches a saturation. Here, the nonlinear contribution caused by the magnetic domain rotation is expressed by the $\lambda_0(\sigma)$. For an isotropic positive magnetostrictive material (e.g., Terfenol-D), the $\lambda_0(\sigma)$ adheres the value $-\frac{\lambda_s}{2}$ when stresses approaches to $-\infty$ and λ_s when stress approaches to ∞ [4]. Further, the contribution of magnetization increases the stress-strain curve nonlinearity in the presence of a low magnetic field (i.e., at 0.03 T and 0.05 T).

Thus, for the 3-D magneto-elastic field, we can broadly decompose the strain tensor ε_{ij} into four components [127] as given in as Eq. (3.29), that are linear elastic strain ε_{ij}, nonlinear elastic strain ε_{ij}^{n}, thermal strain $\varepsilon_{ij}^{\mathrm{T}}$ and magnetostrictive strain ε_{ij}^{m}. Now ε_{ij} can be expressed as

$$\varepsilon_{ij} = \varepsilon_{ij}^{e} + \varepsilon_{ij}^{n} + \varepsilon_{ij}^{T} + \varepsilon_{ij}^{m} \tag{4.12}$$

The linear strain ε_{ij}^{e} caused by the applied stresses alone, which is independent of domain rotation and external field variables, can be expressed as

$$\varepsilon_{ij}^{e} = \frac{1}{E_{in}}\left[(1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}\right] \tag{4.13}$$

The applied stresses can rotate the magnetic domains and generate the nonlinear strain $\varepsilon_{ij}^{n}\ or\ \lambda_{0ij}$ even in the absence of magnetic field. ε_{ij}^{n} exhibits a saturation phenomenon once the rotation of all domains is completed. After the saturation level, ε_{ij}^{n} gives a constant strain value as the stress increases. ε_{ij}^{n} can be written as

$$\varepsilon_{ij}^{n} = \lambda_{0ij} = \begin{cases} \lambda_s\delta_{ij}\left(\frac{\tilde{\sigma}_{ij}}{|\tilde{\sigma}|}\right) tanh\left(\frac{|\tilde{\sigma}|}{\sigma_s}\right) & \tilde{\sigma} \geq 0 \\ \frac{\lambda_s\delta_{ij}}{2}\left(\frac{\tilde{\sigma}_{ij}}{|\tilde{\sigma}|}\right) tanh\left(\frac{2|\tilde{\sigma}|}{\sigma_s}\right) & \tilde{\sigma} < 0 \end{cases} \tag{4.14}$$

The magnetostrictive strain, ε_{ij}^{m}, is induced on the application of a magnetic field and has a functional dependency on the coupled interaction of the magneto-thermo-elastic variables, can be derived as

$$\varepsilon_{ij}^{m} = \frac{\lambda_s}{\left(M_s(T)\right)^2}\left[\frac{3}{2}M_iM_j - \frac{1}{2}M_kM_k\delta_{ij}\right] - \frac{1}{\left(M_s(T)\right)^2}M_kM_k\begin{cases} \lambda_s\delta_{ij}\left(\frac{\tilde{\sigma}_{ij}}{|\tilde{\sigma}|}\right) tanh\left(\frac{|\tilde{\sigma}|}{\sigma_s}\right) & \tilde{\sigma} \geq 0 \\ \frac{\lambda_s\delta_{ij}}{2}\left(\frac{\tilde{\sigma}_{ij}}{|\tilde{\sigma}|}\right) tanh\left(\frac{2|\tilde{\sigma}|}{\sigma_s}\right) & \tilde{\sigma} < 0 \end{cases} \tag{4.15}$$

The thermal strain component ε_{ij}^{T}, shows the effect of temperature variation on the linear thermal expansion and the magnetostriction of the material, can be derived as

$$\varepsilon_{ij}^{T} = \alpha\delta_{ij}\Delta T + \frac{\beta\delta_{ij}}{\left(M_s(T)\right)^2}\Delta T M_k M_l \delta_{kl} \tag{4.16}$$

The elastic strain energy density function, $\mathcal{W}^e(\varepsilon_{ij}^e)$, using the Eq. (4.11) obtained as

$$\sigma_{ij} = \frac{\partial\mathcal{W}^e(\varepsilon_{ij}^e)}{\partial\epsilon_{ij}^e} \tag{4.17}$$

and $W^e(\varepsilon_{ij}^e)$ does not explicitly rely on X_j. As well as, in the presence of an external magnetic field B_{0i} $(= \mu_0 H_{0i})$, the external force f_i acting on a body is [4]

$$f_i = M_j\frac{\partial B_{0i}}{\partial X_j} \tag{4.18}$$

Here, introducing equations (4.11)-(4.18) into equation (4.10) leads to

$$J_l^n(s) = \int_{A_1}\left\{\frac{\partial}{\partial X_j}\left[\mathcal{W}^e\delta_{lj} - \sigma_{ij}u_{i,l}\right]\right\}dA - \int_{A_1}\rho\dot{u}_i\frac{\partial\dot{u}_i}{\partial X_j}\delta_{lj}dA + \int_{A_1}\rho\dot{u}_i\dot{u}_{i,l}dA$$
$$+ \int_{A_1}\left[\sigma_{ij}\left(\frac{\partial\varepsilon_{ij}^n}{\partial X_j} + \frac{\partial\varepsilon_{ij}^m}{\partial X_j} + \frac{\partial\varepsilon_{ij}^T}{\partial X_j}\right)\delta_{lj} + \left(\rho\ddot{u}_i - M_j\frac{\partial B_{0i}}{\partial X_j}\right)u_{i,l}\right]dA \tag{4.19}$$

Taking $l = 1$ for a traction free crack, the path independent *J*-integral can be simplified as

$$J_1^n(s) = \int_{A_1}\left[\frac{\partial}{\partial X_1}(\mathcal{W}^e) - \frac{\partial}{\partial X_j}\left(\sigma_{ij}u_{i,1}\right)\right]dA$$
$$+ \int_{A_1}\left[\sigma_{ij}\left(\frac{\partial\varepsilon_{ij}^n}{\partial X_1} + \frac{\partial\varepsilon_{ij}^m}{\partial X_1} + \frac{\partial\varepsilon_{ij}^T}{\partial X_1}\right) + \left(\rho\ddot{u}_i - M_j\frac{\partial B_{0i}}{\partial X_j}\right)u_{i,1}\right]dA \tag{4.20}$$

On the application of Gauss's divergence theorem to the first integral of equation (4.20), the terms related to $j = 1, 2$ are coplanar with area A_1 in the plane $X_3 = 0$ and can be transformed to a line integral. The terms related to $j = 3$ is not coplanar with area A_1 and cannot be expressed as a line integral. Thus, the final form of the 3-D *J*-integral is obtained as

$$J_1^n(s) = \int_\Gamma \left[\mathcal{W}^e n_1 - \sigma_{ij} n_j u_{i,1}\right] d\Gamma - \int_{A_1} \frac{\partial}{\partial X_3}\left(\sigma_{i3} u_{i,1}\right) dA + \int_{A_1} \left[\sigma_{ij}\left(\frac{\partial \varepsilon_{ij}^n}{\partial X_1} + \frac{\partial \varepsilon_{ij}^m}{\partial X_1} + \frac{\partial \varepsilon_{ij}^T}{\partial X_1}\right) + \left(\rho \ddot{u}_i - M_j \frac{\partial B_{0i}}{\partial X_j}\right) u_{i,1}\right] dA \tag{4.21}$$

where n_j represents the components of the outward normal vector $\vec{\boldsymbol{n}}$ to the contour Γ of the area A_1 in the plane $X_3 = 0$. The right-hand side expression of equation (4.21) is path-area independent and constant for any contour.

For a two-dimensional problem, upon ignoring the variation in the X_3 direction, the J-integral can be derived as

$$J_1(s) = \int_\Gamma \left[\mathcal{W}^e n_1 - \sigma_{ij} n_j u_{i,1}\right] d\Gamma + \int_{A_1} \left[\sigma_{ij}\left(\frac{\partial \varepsilon_{ij}^n}{\partial X_1} + \frac{\partial \varepsilon_{ij}^m}{\partial X_1} + \frac{\partial \varepsilon_{ij}^T}{\partial X_1}\right) + \left(\rho \ddot{u}_i - M_j \frac{\partial B_{0i}}{\partial X_j}\right) u_{i,1}\right] dA \tag{4.22}$$

Since the fracture parameter characterization analysis in this thesis is only limited to magneto-elastic field, thus Eqns (4.21) and (4.22) can be updated by removing thermal stress terms as

$$J_1^n(s) = \int_\Gamma \left[\mathcal{W}^e n_1 - \sigma_{ij} n_j u_{i,1}\right] d\Gamma - \int_{A_1} \frac{\partial}{\partial X_3}\left(\sigma_{i3} u_{i,1}\right) dA + \int_{A_1} \left[\sigma_{ij}\left(\frac{\partial \varepsilon_{ij}^n}{\partial X_1} + \frac{\partial \varepsilon_{ij}^m}{\partial X_1}\right) + \left(\rho \ddot{u}_i - M_j \frac{\partial B_{0i}}{\partial X_j}\right) u_{i,1}\right] dA \tag{4.23}$$

For a two-dimensional problem,

$$J_1(s) = \int_\Gamma \left[\mathcal{W}^e n_1 - \sigma_{ij} n_j u_{i,1}\right] d\Gamma + \int_{A_1} \left[\sigma_{ij}\left(\frac{\partial \varepsilon_{ij}^n}{\partial X_1} + \frac{\partial \varepsilon_{ij}^m}{\partial X_1}\right) + \left(\rho \ddot{u}_i - M_j \frac{\partial B_{0i}}{\partial X_j}\right) u_{i,1}\right] dA \tag{4.24}$$

4.3. Summary

The analysis of the magneto-elastically coupled fracture problem for giant magnetostrictive material is mainly a magnetization and stress-dependent elasticity problem. Thus, a new expression of conservation integral $J_1(s)$ for a mode I crack has been proposed to have the physical meaning of energy release rate (both in two dimensional and three dimensional cases) for a homogeneous, isotropic giant magnetostrictive material considering combined effects of inertia, thermal and magnetostriction effect.