A Study of Graphs of Finite Groups

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by

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Chapter 5

The Superpower Graphs of Finite Groups

This chapter is presented in a two-fold, namely we first discuss the structural characterizations of superpower graph $S(G)$ defined on any finite group G in Section 5.1. We discuss in particular about the properties such as perfectness, Eulerian, comparability graph, separating sets in $S(G)$ and their connections to the underlying quotient graph. Secondly, in Section 5.2, we discuss a generalization of results presented in the previous two chapters. That is we consider the superpower graphs defined on the class of finite groups, namely the groups having an element of exponent order. Here we discuss the properties such as the dominant set, vertex connectivity, Hamiltonian-like properties of these graphs.

5.1 Structure of Superpower Graph of any Group

In this section, we will present few results on the superpower graphs defined on any finite group G. We will discuss the properties that are common between the underlying quotient graph, namely *the Order graph*, of the superpower graph.

5.1.1 The order graph

Recall that $\pi(G)$ denotes the set of orders of elements of G. Recall that we refer to the *order graph* of G whose vertex set is $\pi(G)$, and there is an edge joining m and n if one of them divides the other. We denote this graph by $\mathcal{O}(G)$. We define, the *weighted order graph* is obtained from $\mathcal{O}(G)$ by labelling each vertex m with the number of elements of order m in G.

Note that the graph $S(G)$ can be reconstructed uniquely from the weighted order graph of G; simply blow up each vertex of $\mathcal{O}(G)$ to a complete graph whose number of vertices is equal to the weight of that vertex, and lift each edge to all edges between the corresponding sets. Note also that $\mathcal{O}(G)$ is isomorphic to an induced subgraph of $S(G)$, obtained by choosing one element of G of each possible order.

Let Γ and Δ be graphs. We say that Γ is Δ-*free* if it has no induced subgraph isomorphic to Δ. Two vertices v, w in a graph Γ are *twins* if they have the same neighbours, possibly excepting each other: that is, either v and w are not joined and have equal neighbourhoods, or they are joined and $N(v) \setminus \{w\} = N(w) \setminus \{v\}$, where $N(v)$ is the neighbourhood of v.

Theorem 5.1. *Let Δ be a graph containing no pair of twins, and G a finite group, then $S(G)$ is Δ-free if and only if $\mathcal{O}(G)$ is Δ-free.*

Proof. In an embedding of Δ into $S(G)$ as induced subgraph, two vertices of Δ mapping to elements of G with the same order must be twins. So the mapping taking an element of Δ to the order of its image must be an injection, and is an embedding in $\mathcal{O}(G)$. The converse is clear. □

Theorem 5.2. *For any finite group G, $S(G)$ is perfect.*

Proof. The graph $\mathcal{O}(G)$ is the comparability graph of the partial order on $\omega(G)$ defined by divisibility, so by Dilworth's theorem [46] it is perfect. Now by the Strong Perfect Graph Theorem [47], a graph is perfect if and only if it contains neither an odd cycle of length at least 5 nor the complement of one. The odd cycles and their complements have no pairs of twins, so it follows from Theorem 5.1 that $S(G)$ is also perfect. □

First we observe the following connection between the $\mathcal{O}(G)$ and $S(G)$.

Theorem 5.3. *For any finite group G, if $\mathcal{O}(G)$ is Hamiltonian, then $S(G)$ is Hamiltonian.*

Proof. Suppose $\mathcal{O}(G)$ is Hamiltonian. Consider a Hamiltonian cycle in $\mathcal{O}(G)$ and lift it to a Hamiltonian cycle of G by choosing a Hamiltonian path within the set of vertices of each possible order and joining their ends according to the edges in the cycle in $\mathcal{O}(G)$. □

The above results suggests a general question:

Problem 5.4. Which graph-theoretic properties can be lifted from $\mathcal{O}(G)$ to $S(G)$?

5.1.2 Comparability graph

In this section we show that $S(G)$ is the comparability graph of a partial order, and hence is perfect, and that there is no further restriction on the induced subgraphs. Unlike the proof in the preceding section, this does not depend on the Strong Perfect Graph Theorem.

A *partial preorder* on a set X is a reflexive and transitive relation R on X. Its *comparability graph* is the graph on the vertex set X in which x and y are joined if $x \mathrel{\mathrm{R}} y$ or $y \mathrel{\mathrm{R}} x$. A partial preorder is a *partial order* if $x \mathrel{\mathrm{R}} y$ and $y \mathrel{\mathrm{R}} x$ imply $x = y$. A partial order is a *total order* if, for any $x, y \in X$, either $x \mathrel{\mathrm{R}} y$ or $y \mathrel{\mathrm{R}} x$.

Proposition 5.5. *The classes of comparability graphs of partial preorders and of partial orders are isomorphic.*

Proof. Every partial order is a partial preorder. Conversely, let R be a partial preorder. Define a relation $\equiv$ by the rule that $x \equiv y$ if and only if $x \mathrel{\mathrm{R}} y$ and $y \mathrel{\mathrm{R}} x$. It is easily seen that $\equiv$ is an equivalence relation. Now enlarge the relation R to a new relation $\leq$ by imposing a total order on each equivalence class of $\equiv$. The result is a partial order with the same comparability graph as R. □

Proposition 5.6. *The superpower graph of a group G is the comparability graph of a partial order.*

Proof. Define a relation R on G by the rule that $x \mathrel{\mathrm{R}} y$ if $o(x) \mid o(y)$. This relation is reflexive and transitive, thus is a partial preorder; and the superpower graph is its comparability graph. Now the result follows using the preceding Proposition. □

Proposition 5.7. *Let X be the comparability graph of a partial order. Then there is a group G such that X is an induced subgraph of the superpower graph of G.*

Proof. Let $\leq$ be a partial order on $V(X)$ whose comparability graph is X. Put $\downarrow(x) = \{y \in V(X) : y \leq x\}$. Then we have

- $\downarrow(x) = \downarrow(y)$ if and only if $x = y$;
- $\downarrow(x) \subseteq \downarrow(y)$ if and only if $x \leq y$.

To see this, observe that in each case the reverse implication is true, since $\leq$ is a transitive relation. In the other direction, suppose that $\downarrow(x) \subseteq \downarrow(y)$; then $x \in \downarrow(y)$, so $x \leq y$. If $\downarrow(x) = \downarrow(y)$, then $x \leq y$ and $y \leq x$, so $x = y$, since $\leq$ is a partial order.

Now choose distinct primes p_x for each $x \in V(X)$, and let

$$N = \prod_{x \in V(X)} p_x.$$

Let G be the cyclic group of order N, and for each $x \in V(X)$ let g_x be an element of G whose order is

$$o(g_x) = \prod_{y \in \downarrow(x)} p_y.$$

If $x \sim y$ then, without loss, $x \leq y$, so $\downarrow(x) \subseteq \downarrow(y)$, whence $o(g_x) \mid o(g_y)$. The argument reverses. So the map $x \mapsto g_x$ is an embedding of X as an induced subgraph of the superpower graph of G. $\square$

5.1.3 Eulerian property

In this section, we study when the graph $S(G)$ becomes Eulerian. In [38], it was proved that $S(G)$ is Eulerian if and only if G is a group of odd order. Now what will be the effect on order of G if we removed all dominant vertices from $S(G)$. We answer this question with the following theorem.

Theorem 5.8. *Let G be a finite non p-group and let $H_{\overline{G}}$ be the induced subgraph obtained by removing all the dominant vertices of $S(G)$. Then $H_{\overline{G}}$ is Eulerian if and only if $o(G)$ is an even integer.*

Proof. Suppose $H_{\overline{G}}$ is Eulerian. Let $\pi(\overline{G}) = \{\overline{a_1}, \overline{a_2}, \cdots, \overline{a_\ell}\}$ be the set of all element orders in $\overline{G}$. Then for any $x \in \overline{G}$ with $o(x) = \overline{a_i}$, for some i, $1 \leq i \leq \ell$, we have

$$deg_{H_{\overline{G}}}(x) = w_{\overline{a_i}}(\overline{G}) + \sum_{\overline{a_i}|\overline{a_j} \text{ or } \overline{a_j}|\overline{a_i} \quad \overline{a_i} \neq \overline{a_j}} w_{\overline{a_j}}(\overline{G}) - 1. \tag{5.1}$$

Note that for each $j, 1 \leq j \leq \ell$, number of elements of order $\overline{a_j}$, $|w_{\overline{a_j}}(\overline{G})| = t_j \phi(\overline{a_j})$, where t_j is the number of cyclic subgroups of order $\overline{a_j}$ in $\overline{G}$. Also, it is well known fact that $\phi(k)$ is odd if and only if $k \in \{1, 2\}$. Clearly, $deg_{H_{\overline{G}}}(x)$ is even if and only if Equation (5.1) is even. Since, the identity element does not belong to $\overline{G}$, Equation (5.1) will be even if and only if there are odd number of order 2 elements (namely involution elements) in $\overline{G}$ and hence in G. Thus implying $o(G)$ is even. Conversely, if n is even then by Cauchy theorem, G and hence $\overline{G}$ must have an involution and these are odd in numbers. This implies that the degree of any element in $H_{\overline{G}}$, given in Equation (5.1) is even. Thus, $H_{\overline{G}}$ is Eulerian. □

5.1.4 Separating sets

From Lemma 3.6, for any group G of order n, $\kappa(S(G)) = n - 1$ if and only if $n = p^k$, $k \in \mathbb{N}$. Moreover, for any arbitrary group G with $o(G) \neq p^k$, finding the exact formula of vertex connectivity of $S(G)$ is a difficult task. The following theorem contributes much in this direction.

Theorem 5.9. *Let G be a group of order n which is not a prime power and let T be a minimal separating set of $S(G)$. Then, for any $a \in G$, either $[a] \subseteq T$ or $[a] \cap T = \emptyset$, that is, T is a union of some equivalence classes in G.*

Proof. We know that $S(G) \setminus T$ is disconnected. Without loss of generality, one can assume that $S(G) \setminus T$ has two connected components. Let $a \in G$ and $[a] \nsubseteq T$. We show that $[a] \cap T = \emptyset$.

Suppose there exists $w \in [a] \cap T$. As $[a] \nsubseteq T$, there exists $b \in (G \setminus T) \cap [a]$. Observe that $o(b) = o(w) = o(a)$. Since $S(G) \setminus T$ is disconnected, there exist $x, y \in S(G) \setminus T$ such that there is no path between x and y in $S(G) \setminus T$. As T is a minimal separating set, $S(G) \setminus (T \setminus \{w\})$ is connected. Therefore, there exists a path P between x and y in $S(G) \setminus (T \setminus \{w\})$ such that w is an internal vertex of P. Let $P =: \langle x, \cdots, w_1, w, w_2, \cdots, y \rangle$. Let P_1 be the $x - w_1$ path in P and P_2 be the $w_2 - y$ path in P. Here w_1 and w_2 are adjacent to w and $o(w) = o(b)$ and hence b is also adjacent to w_1 and w_2. Thus $\langle P_1, b, P_2 \rangle$ is a path between x and y in $S(G) \setminus T$, which contradiction to the choice of x and y. Hence $[a] \cap T = \emptyset$. □

Corollary 5.10. *Let G be a group of odd order n where n is odd and not a prime power. Then any minimal separating set T in $S(G)$ contains odd number of vertices.*

Proof. Note that every element in G is of odd order. Since the identity element $e \in G$ is adjacent to all other vertices in $S(G)$, it follows that $e \in T$. By Theorem 5.9, we have $T = [e] \cup \bigcup_{i=1}^{s} [b_i]$ where b_i are elements of order $d > 2$ and for some $s \geq 0$. Since order of an element and its inverse are same, each equivalence class $[b_i]$ contains even number of elements and so $|T|$ is odd. □

Corollary 5.11. *Let G be a group of even order n and T be a minimal separating set in $S(G)$. If n is not prime power, then*

$$|T| = \begin{cases} \textit{even}, & \textit{if } T \textit{ contains an element of order 2;} \\ \textit{odd}, & \textit{otherwise.} \end{cases}$$

Proof. Note that the number of elements of order 2 in any group G of even order is always odd and $e \in T$. Now, the result follows from Theorem 5.9. □

5.2 Groups Having Elements of Exponent Order

In this section, we concentrate on finite groups having an element of exponent order. Since any Abelian group always contain an element of its exponent order, we will focus mainly on non-Abelian groups. The results presented in this section are in a way generalizes the results of previous two chapters.

Recall that, the order of an element x of G is denoted by $o(x)$. The *exponent* $\exp(G)$ of G is the least common multiple of the orders of the elements of G, in other words, the smallest positive integer m such that $x^m = e$ for all $x \in G$. We say that G *has an element of exponent order* if there exists $x \in G$ with $o(x) = \exp(G)$. Let $\pi(G)$ be the set of orders of elements of G. Thus, G has an element of exponent order if and only if $\pi(G)$ is the set of all divisors of m, for some integer m. Note that any finite Abelian group, or any group of prime power order, contains an element of exponent order, so this class of groups generalizes both Abelian groups and p-groups. Moreover, for any finite group G, there is an integer m such that G^m contains an element of exponent order.

5.2.1 Dominant set

As we have observed earlier that dominant vertices play an important role in characterization of graphs. In the following theorem, we find the number of dominant vertices in $S(G)$ for any finite non p-group G having an element of order $\exp(G)$.

Proposition 5.12. *Suppose that G is not of prime power order. The element x is a dominant vertex in $S(G)$ if and only if either $x = e$ or $o(x) = \exp(G)$.*

Proof. Clearly e is a dominant vertex, so suppose that x is a dominant vertex with $x \neq e$. If $o(x) \neq \exp(G)$, then there is an element y whose order does not divide $o(x)$; so $o(x)$ divides $o(y)$. Then there is a prime power p^m which divides $o(y)$ but not $o(x)$, and an element of order p^m in the subgroup generated by y; so x is not dominant. Conversely, if $o(x) = \exp(G)$, then x is dominant, by definition. □

Theorem 5.13. *Let G be a finite non p-group and $\mathrm{dom}(S(G))$ be the set of all dominant vertices in the superpower graph $S(G)$ of G. Then*

$$|\mathrm{dom}(S(G))| = \begin{cases} t\phi(\exp(G)) + 1, & \textit{if } G \textit{ has an element of exponent order;} \\ 1, & \textit{otherwise.} \end{cases}$$

where t is the number of distinct cyclic subgroups of order $\exp(G)$ in G.

Proof. If there is no element of exponent order, then e is the only dominant vertex by Proposition 5.12. Otherwise, if $o(x) = \exp(G)$, then $\langle x \rangle$ contains $\phi(\exp(G))$ elements of order $\exp(G)$, all of which are dominant and generate the same cyclic group. So, $w_{\exp(G)}(G)$ contains union of all cyclic group of order $\exp(G)$ contains $\phi(\exp(G))$ elements, and there is no overlap between these sets of size $\phi(\exp(G))$. If t is the number of distinct cyclic subgroups of order $\exp(G)$ in G. Therefore, by Lemma 3.4 and Proposition 5.12, we have $|\mathrm{dom}(S(G))| = t\phi(\exp(G)) + 1$. □

Remark 5.14. For any non p, non-Abelian finite simple group G, $|\text{dom}(S(G))| = 1$. The proof follows from Theorem 5.13 and the fact that G does not contain an element of order $\exp(G)$, [48].

As mentioned in the proof of Theorem 5.13, if G contains an element of order $\exp(G)$ then $S(G)$ always contains a dominant vertex other than identity and hence we have the following corollary.

Corollary 5.15. *For any finite group G having an element order $\exp(G)$, the superpower graph $S(G)$ is dominatable.*

5.2.2 Vertex connectivity

In this section we first give a tight lower bound for the vertex connectivity of $S(G)$ for any finite group G having an element of order $\exp(G)$, which extend the results [38, Theorem 2.7] and [15, Theorems 2.11].

Theorem 5.16. *Let G be a finite group having an element of order $\exp(G)$. Then $\kappa(S(G)) \geq t\phi(\exp(G)) + 1$, where t is the number of distinct cyclic subgroups of order a_k in G. Further, $\kappa(S(G)) = t\phi(\exp(G)) + 1$ if and only if $\exp(G) = pq$, where p, q are different primes.*

Proof. Let G be a finite group having an element of order $a_k = \exp(G)$. Then to disconnect $S(G)$, we need to remove at least all vertices of $\text{dom}(S(G))$, since these vertices are adjacent to all other vertices of $S(G)$. This implies that $\kappa(S(G)) \geq t\phi(\exp(G)) + 1$.

Next we prove the second part of the statement. Let us suppose $\exp(G)$ is product of two distinct primes, that is $a_k = pq$ and $\overline{G} = G \setminus (\{e\} \cup w_{a_k}(G))$. Since there is no path between elements of order p and q, the induced subgraph $H_{\overline{G}}$ is disconnected. Thus $\kappa(S(G)) = t\phi(a_k) + 1$.

Conversely, assume that $\kappa(S(G)) = t\phi(a_k) + 1$. Suppose G has an element of order $\exp(G) = a_k$ which satisfies $a_k = p^{\alpha_1}q^{\beta_1}$ and α_1, β_1 are integers, $\alpha_1, \beta_1 \geq 1$ and either $\alpha_1 > 1$ or $\beta_1 > 1$. We will prove that $\kappa(S(G)) > t\phi(a_k) + 1$ by showing that $H_{\overline{G}}$ is connected. For $u, v \in V(H_{\overline{G}})$, take $w, x, z \in V(H_{\overline{G}})$ with $o(w)$ is the least prime divisor of $o(u)$ and $o(z)$ is the least prime divisor of $o(v)$ and $o(x)$ is the product of least prime divisors of $o(u)$ and $o(v)$. Then $P := \langle u, w, x, z, v\rangle$ is a path between u and v. Also, by the same way it can be shown that $H_{\overline{G}}$ is connected if $\exp(G)$ has at least three prime divisors. Otherwise, $H_{\overline{G}}$ is connected. Thus, $\exp(G)$ has at most two prime divisors with $a_k = pq$ in G. □

Let G be a finite group of order $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_m^{\alpha_m}, m \geq 2$ having an element of order $a_k = \exp(G)$. Let the prime decomposition of $a_k = p_1^{\beta_1}p_2^{\beta_2}\cdots p_s^{\beta_s}, 1 \leq s \leq m, \beta_i \geq 0, 1 \leq i \leq s$. Let $a_0 = p_1^{\beta_1}, a_1 = \dfrac{a_k}{a_0}$ and $\pi(G) = \{a_0, a_1, a_2, \cdots, a_k\}$ be the set of all orders of elements in G.

Next we find the tight upper bound for the vertex connectivity of $S(G)$ using the notations defined above.

Theorem 5.17. *Let G be a finite group of order $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_m^{\alpha_m}, m \geq 2$ and having an element of order $\exp(G)$. Then there exists a minimal separating set T of $S(G)$ with*

$$\kappa(S(G)) \leq |T| = \sum_{(a_i|a_1 \quad or \quad a_1|a_i, a_i\neq a_1)} t_i\phi(a_i), \tag{5.2}$$

where t_i is the number of distinct cyclic subgroups of order a_i in G. Also, this bound is tight.

Proof. Consider the graph $S(G)$ and define a set T in $S(G)$ as follows:

$$T = \{w_{a_i}(G)|\ a_i|a_1\ or\ a_1|a_i,\ 2 \leq i \leq k\},$$

Clearly, T is a separating set of $S(G)$, since there is no path between any vertices of the cliques $w_{a_0}(G)$ and $w_{a_1}(G)$. Let A and B are two connected component of $S(G) \setminus T$ such that $w_{a_0}(G) \in A$ and $w_{a_1}(G) \in B$. Now, we prove that this set is a minimal separating set by showing that for any non empty subset $T^\dagger$ of T, there is a path, connecting $u \in w_{a_0}(G)$ and $v \in w_{a_1}(G)$ in $S(G)\setminus T^\dagger$. Without loss of generality assume that $T \setminus T^\dagger = \{x\}$ with $x \in w_{a_r}(G)$. Since either $a_r|a_1$ or $a_1|a_r$, there exists a path $P_1(u, x)$, connecting u and x in $A \cup w_r(G)$. Similarly, let $y \in B$ such that $o(y) = a_r p_s^{\beta_s}$, then there exists a path $P_2(y, v)$, connecting y to v in B. Now, consider the path $P = \langle P_1(u, x), x, y, P_2(y, v)\rangle$ which connects u to v in $S(G) \setminus T^\dagger$. Thus, T is a minimal separating set of $S(G)$. If t_i, number of cyclic subgroups of order a_i in G, $1 \leq i \leq r$. Then

$$|T| = \sum_{(a_i|a_1\quad or\quad a_1|a_i), a_i \neq a_1} t_i \phi(a_i).$$

Thus, $\kappa(S(G)) \leq |T|$.

Now, we show that the obtained bound is in fact tight. That is, there exists a minimum separating set T, with $|T|$ given above serve the $\kappa(S(G))$ for the group $G \simeq D_{2n}$ when $n = 2^\alpha p^\beta$. Clearly, in line with the above proof, the separating set given by $T = \{w_{p^i}(G) : 0 \leq i \leq \beta - 1\} \cup \{w_{2^j p^\beta}(G) : 1 \leq j \leq \alpha\}$ with $|T| = p^{\beta-1} + (2^\alpha - 1)(p^\beta - p^{\beta-1})$ is minimal. Next we show that T is, in fact, minimum. Let $T^\dagger$ be any other minimal separating set of $S(D_{2n})$.
<u>Claim $|T^\dagger| \geq |T|$</u>: Without loss of generality assume that $S(G) \setminus T^\dagger$ has two components say A and B. Let $x \in A$ and $y \in B$ be such that there is no path joining x and y in $S(G) \setminus T^\dagger$. Clearly, both x and y are not of odd order, otherwise they will

be adjacent. Also, both x are y are not of even order, otherwise they can be joined through an element of $w_2(G)$. Note that if $w_2(G) \subset T^\dagger$, then $|T^\dagger| \geq |T|$. Finally, let us assume that $o(x) = p^\ell$ and $o(y) = 2^s p^t$ for some $1 \leq s \leq \alpha, 1 \leq t < \ell \leq \beta$. Therefore, the minimal separating set $T^\dagger$, that separates x and y, is then given by

$$T^\dagger = \left\{w_{p^i}(G) : 0 \leq i < \ell\right\} \bigcup \left\{w_{2^j p^\ell}(G) : 1 \leq j \leq \alpha\right\} \bigcup \left\{w_{2^j p^i(G)} : 1 \leq j \leq \alpha, \ell + 1 \leq i \leq \beta\right\}$$

and with

$$|T^\dagger| = \sum_{i=0}^{\ell-1} \phi(p^i) + \sum_{j=1}^{\alpha} \phi(2^j p^\ell) + \sum_{j=1}^{\alpha} \sum_{i=\ell+1}^{\beta} \phi(2^j p^i) = p^{\ell-1} + (2^\alpha - 1)(p^\beta - p^{\ell-1}).$$

Thus, we get that $|T^\dagger| - |T| = (2^\alpha - 2)(p^{\beta-1} - p^{\ell-1}) \geq 0$. Hence, T is a minimum separating set of $S(G)$. □

5.2.3 Hamiltonian-like properties

In [18], it was proved that power graph $P(G)$ of any cyclic group of order at least three is Hamiltonian, [see Theorem 4.13]. In [15], it was proved that $S(G) = P(G)$ if and only if G is a finite cyclic group. Thus, $S(G)$ is Hamiltonian for any cyclic group of order at least three. Can we extend this result for finite groups? Unfortunately, the result may not hold in general. For instance, in the previous chapter, we have shown that the superpower graph $S(D_{2n})$ of the dihedral group D_{2n} is Hamiltonian if and only if n is an even integer whereas $S(T_{4n})$ of dicyclic group is Hamiltonian for any integer n. Next, we show that $S(G)$ is Hamiltonian under certain conditions.

Theorem 5.18. *Let G be a finite group G of order $n \geq 3$, having an element of order* $\exp(G)$. *Then $S(G)$ is Hamiltonian.*

Proof. Let G be a finite group with $a_k = \exp(G)$ and $1 = a_1 < a_2 < a_3 < \cdots < a_{k-1} < a_k$ are all orders of elements in G. Clearly, $a_i | a_k \ \forall\ i, 1 \leq i \leq k-1$ which gives that every vertex of the set $w_{a_k}(G)$ is adjacent to all other vertices of the graph $S(G)$. Also, the identity element e is adjacent to all other vertices of $S(G)$. For each element of order a, induced subgraph say H_a on $w_a(G)$ forms a clique in $S(G)$. We get Hamiltonian cycle in $S(G)$ as follows: Start from the vertex $v_1 \in \text{dom}(S(G))$. From v_1, go to any vertex of the clique H_{a_2} and traverse all vertices of H_{a_2}. Now we have a Hamiltonian path containing all the vertices in $H_{a_2} \cup \{v_1\}$. Note that the terminal vertex of this Hamiltonian path is adjacent to a vertex $v_2 \in \text{dom}(S(G))$ and $v_2 \neq v_1$. From v_2, go to any vertex of the clique H_{a_3} and traverse all vertices of H_{a_3}. Now the terminating vertex of the resulting Hamiltonian path is adjacent to a vertex $v_3 \in w_{a_k(G)}$ and $v_3 \notin \{v_1, v_2\}$. Repeat this until all the cliques $H_{a_i}(G), 2 \leq i \leq k-1$ are covered. Since $k - 2 < |\text{dom}(S(G))|$, there are sufficient number of vertices in $\text{dom}(S(G))$ to connect all disjoint cliques. Finally, complete the cycle by joining all the uncovered vertices of $\text{dom}(S(G))$ by path to v_1. The entire process of identifying a Hamiltonian cycle is given in Figure 5.1. □

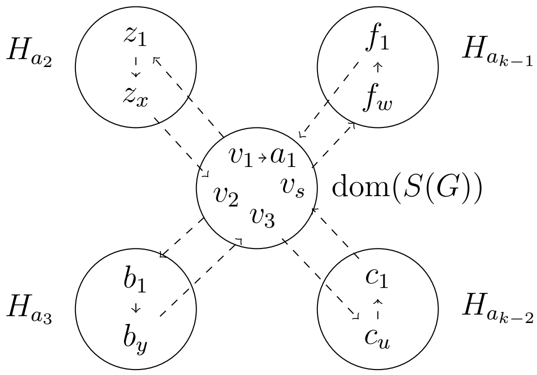

FIGURE 5.1: Hamiltonian cycle in $S(G)$

Corollary 5.19. *For any finite group G with $o(G) \geq 4$ having an element of order $\exp(G)$, $S(G)$ is 1-Hamiltonian.*

Proof. Let $a_k = \exp(G)$ be the largest order of an element in the group G and let $g \in G$. If $o(g) = a_i, 2 \leq i \leq k-1$, then g is a vertex in the clique induced by $w_{a_i}(G)$ for the divisor a_i of $o(G)$. Further $H_{a_i} \setminus \{g\}$ remains as a clique and so it has a spanning path whose initial and terminal vertices can be joined by two different vertices of $\mathrm{dom}(S(G))$. Now, the proof can be completed as in the case of Theorem 5.18.

If $o(g) = a_k$, then $g \in \mathrm{dom}(S(G))$. As seen in the proof of Theorem 5.18, there are sufficient number of vertices in $\mathrm{dom}(S(G)) \setminus \{g\}$ to connect all the disjoint cliques corresponding to all proper divisors of $o(G)$. Hence the required Hamiltonian cycle can be obtained as in Theorem 5.18. Thus, $S(G) \setminus \{g\}$ contains a Hamiltonian cycle implying that $S(G)$ is 1-Hamiltonian. □

Corollary 5.20. *For any finite group G of order at least three and having an element of order* $\exp(G)$*, $S(G)$ is pancyclic.*

Proof. Let $a_k = \exp(G)$. Clearly, $\mathrm{dom}(S(G))$ forms a clique in $S(G)$ and by Theorem 5.13, we have $|\mathrm{dom}(S(G))| = t\phi(a_k)+1$, where t is the number of cyclic subgroups of order a_k in the group G. So we have cycles of length 3 to $t\phi(a_k)+1$. Also, Theorem 5.18 implies that $S(G)$ contains a cycle of length n. For any $g_1 \in V(S(G))$, by Corollary 5.19, $S(G) \setminus \{g_1\}$ is Hamiltonian and thus $S(G)$ contains a cycle of length $n-1$. Note that, in the proof of Corollary 5.19, we see that as long as we keep choosing a vertex $g \in w_{a_i} \subset V(G) \setminus \{\mathrm{dom}(S(G))\}$, obtaining a cycle containing remaining vertices is immediate. Choose $g_2 \in w_{a_i}(G)$ (if exists), otherwise choose $\{g_2\} \in w_{a_j}(G)$ for some $2 \leq i, j \leq \ell$ and we immediately get that $S(G) \setminus \{g_1, g_2\}$ is Hamiltonian. So $S(G)$ contains a cycle of length $n-2$. Recursively deleting the vertices of w_{a_i} for each i, $2 \leq i \leq l$, we can get cycles of length $n-2$ to $t\phi(a_k)+2$. Thus $S(G)$ contains cycles of all length ℓ, for $3 \leq \ell \leq n$ and hence $S(G)$ is pancyclic. □

It is not always true that there exists a Hamiltonian path between any pair of vertices in a graph even if it is Hamiltonian. However, this happens in the case of the superpower graph $S(G)$ of any finite group G which contains an element of $\exp(G)$ and hence we have the following result.

Corollary 5.21. *For any finite group G having an element of order $\exp(G)$, $S(G)$ is Hamiltonian-connected.*

Proof. Let $u, v \in V(S(G))$ be two distinct vertices in $S(G)$. Without loss of generality, one can take $u = v_1 \in w_{a_2}(G)$ and $v = v_2 \in w_{a_3}(G)$, where a_2 and a_3 are two non-trivial distinct divisors of a_k. Start from the vertex v_1 and traverse along the spanning path in H_{a_2} and join it with a vertex v_3 of $\text{dom}(S(G))$. From v_3 go to any vertex of H_{a_4} and repeat the process until all vertices of the cliques $H_{a_i}(G) \cup \text{dom}(S(G)), 5 \leq i \leq k-1$ belongs to the path such that $v_\ell \in \text{dom}(S(G))$ is the last vertex of this path. Now, join v_{k-1} to a vertex $x \neq v_2$ of H_{a_3}. Upon completing the path from x to v_2 in H_{a_3}, we obtain the required Hamiltonian path between u and v in $S(G)$. □

Corollary 5.22. *For any finite group G having an element of order $\exp(G)$, $S(G)$ is panconnected.*

Proof. Let $u, v \in V(S(G))$ be two distinct non-adjacent vertices in $S(G)$. Then there always exists a path of every length from 2 to n. Now the required path can be obtained by inserting the vertices from $H_{a_i} \cup \text{dom}(S(G)), 2 \leq i \leq k-1$ in such a way that any two vertices of cliques H_{a_i}, H_{a_j} can be joined through a vertex of $\text{dom}(S(G))$. □

What will be the effect on Hamiltonianity of the graph $S(G)$, if we remove all dominant vertices from it? The following theorem answers this question. For a finite p-group G, $S(\overline{G})$ is a null graph, so in the following theorem, we consider class of finite non p-groups.

Theorem 5.23. *Let G be a finite non p-group having an element of order* $\exp(G)$. *Let $w_{\exp(G)}(G) = \{x \in G : o(x) = \exp(G)\}$. Then the induced subgraph $H_{\overline{G}}$ of the superpower graph $S(G)$ induced by $\overline{G} = G \setminus (\{e\} \cup w_{\exp(G)}(G))$ is Hamiltonian if and only if* $\exp(G)$ *is not a product of two distinct primes.*

Proof. Let $a_k = \exp(G)$. Assume that $H_{\overline{G}}$ is Hamiltonian. If $a_k = pq$ for two distinct primes p and q, then $H_{\overline{G}} = H_p \cup H_q$ is disconnected as there is no path connecting the vertices of H_p and H_q, a contradiction.

Conversely, assume that a_k is not a product of two distinct primes. i.e., $a_k = p_1^{\beta_1} p_2^{\beta_2} \cdots p_m^{\beta_m}$ and $\beta_i \in \mathbb{N}, m \geq 2$ and $p_1 < \cdots < p_m$ are distinct primes. Since G is not a p-group, $m \geq 2$. By the assumption on a_k, we have either $\beta_1 > 1$ or $\beta_2 > 1$ or $m \geq 3$. Since G is not a p-group and by the assumption on a_k, the largest order of an element in the set $\overline{G} = G \setminus (\{e\} \cup w_{a_k}(G))$ will be $\dfrac{a_k}{p_1}(= \overline{a_k}$, say).

Let $w_{\overline{a_k}}(\overline{G}) = \{b_1, \cdots, b_s\}$ be the set of all elements of order $\overline{a_k}$. Let $\{\overline{a_1}, \ldots, \overline{a_\ell}\}$ be the set of all non trivial divisors of $\overline{a_k}$ with $1 < \overline{a_1} < \overline{a_2} < \cdots < \overline{d_\ell} < \overline{a_k}$. Let $w_{\overline{a_i}}(\overline{G})$ be the set of all elements of order $\overline{a_i}$ in $\overline{G}$ and let $H_{\overline{a_i}}$ be the subgraph induced by $w_{\overline{a_i}}(\overline{G})$. For each $i, 1 \leq i \leq \ell$, let $P_{H_{\overline{a_i}}} = \langle v_i, u_i \cdots, x_i \rangle$ be the Hamiltonian path in $H_{\overline{a_i}}$. Then the induced subgraph H of $H_{\overline{G}}$ on the vertices of $\bigcup\limits_{1 \leq i \leq \ell} w_{\overline{a_i}} \cup w_{\overline{a_k}}$ is Hamiltonian, since $C = \langle b_1, P_{H_{\overline{a_1}}}, b_2, P_{H_{\overline{a_2}}}, b_3, \cdots, b_\ell, P_{H_{\overline{a_\ell}}}, b_{\ell+1}, \cdots, b_s \rangle$ is a Hamiltonian cycle in H.

It remains for us to include remaining vertices from $H_{\overline{G}} \setminus H$ into C appropriately to get Hamiltonian cycle in $H_{\overline{G}}$. Based on the condition on $\overline{a_k}$, we observe that the only possible subsets of different orders in $\overline{G} \setminus \{ \bigcup\limits_{1 \leq i \leq \ell} w_{\overline{a_i}}(\overline{G}) \cup w_{\overline{a_k}}(\overline{G})\}$ are of the form $w_{{p_1}^{\beta_1}}(\overline{G})$ and $w_{p_1^{\beta_1} r}(\overline{G})$, where $r = \overline{a_i}$, for some $i, 1 < i \leq \ell$. If cliques $H_{p_1^{\beta_1}}$,

$H_{p_1{}^{\beta_1}\overline{a_j}}$ exists in $H_{\overline{G}} \setminus H$, then the spanning paths $P(v_1^{'}, u_1^{'})$ of $H_{p_1^{\alpha_1}}$ and $P(v_j^{'}, u_j^{'})$ for $1 < j \leq \ell$ of $H_{p_1{}^{\alpha_1}a_j}$ are inserted into the spanning path of $H_{\overline{a_i}}$ and $H_{\overline{a_j}}$ respectively, as shown in Figure 5.2. That is, the required Hamiltonian cycle $C_{H_{\overline{G}}}$ in $H_{\overline{G}}$ is given by $\langle b_1, P_1, b_2, P_2, \cdots, b_\ell, P_l, b_{\ell+1}, \cdots, b_s\rangle$, where $P_j = \langle v_j, v_j^{'}, P(v_j^{'}, u_j^{'}), u_j, P(u_j, x_j)\rangle$(if it exists). □

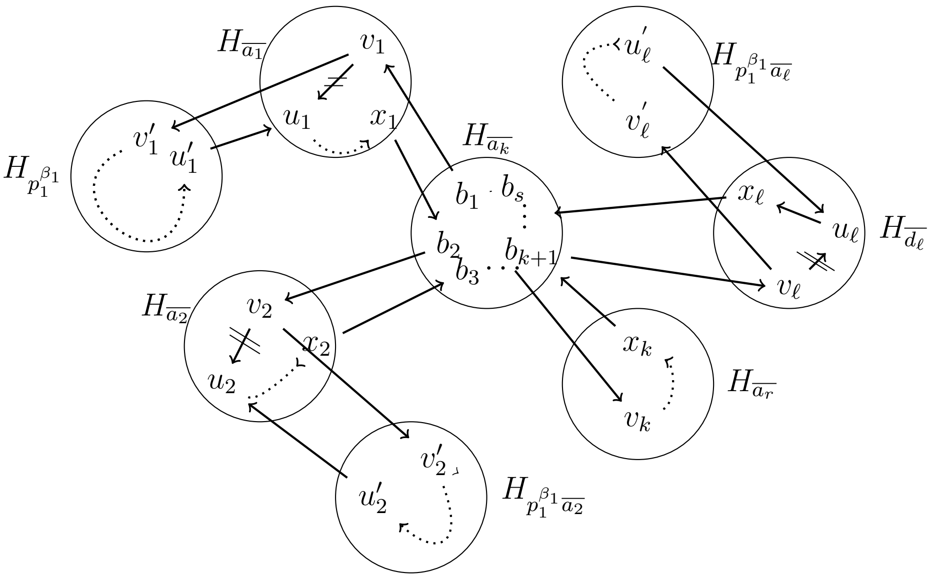

FIGURE 5.2: Hamiltonian cycle in $H_{\overline{G}}$

Corollary 5.24. *Let G be a finite non $p-$group having an element of order* $\exp(G)$*, which is not a product of two primes. Then $H_{\overline{G}}$ is 2-connected.*

Corollary 5.25. *Let G be finite non p-group which is either Abelian or nilpotent. Then the induced subgraph $H_{\overline{G}}$ of the superpower graph $S(G)$ induced by $\overline{G} = G \setminus (\{e\} \cup w_{\exp(G)}(G))$ is Hamiltonian if and only if* $\exp(G)$ *is not a product of two distinct primes.*

Proof. Result follows from the fact that both the groups mentioned above contains an element of order $\exp(G)$. □

It is well known that any graph containing Hamiltonian cycle is 2-connected and hence $S(G)$ is 2-connected for any finite group G having an element of order $\exp(G)$.

Corollary 5.26. *Let G be an finite non p-group of order n having an element of order $= a_k = \exp(G)$. If a_k is not a product of two distinct prime, then the bounds of $\kappa(S(G))$ is given by*

$t\phi(\exp(G)) + 1 \leq \kappa(S(G)) \leq |T|$, *when* $\exp(G)$ *is a product of two primes and*

$t\phi(\exp(G)) + 3 \leq \kappa(S(G)) \leq |T|$, *otherwise*

where t denotes the number of cyclic subgroups of order a_k in G and T is the minimal separating set given in Theorem 5.17 and $|T|$ is given by Equation (5.2).

Proof. Note that the required lower bound follows from Corollary 5.24 and Theorem 5.16 while the upper bound follows from Theorem 5.17. □

So far, we have discussed the Hamiltonian properties of $S(G)$ when G has an element of order $\exp(G)$. What will happen if G does not have an element of order $\exp(G)$? The following examples shows that it may or may not be Hamiltonian.

Example 5.1. *Let D_{2n} denotes the dihedral group D_{2n} of order $2n$. Then $S(D_{2n})$ does not have elements of order $\exp(G)$ if and only if n is odd. Also, $S(D_{2n})$ is not Hamiltonian if and only if n is odd.*

Example 5.2. *Consider the dicyclic group T_{4n} of order $4n$. It can be seen that $S(T_{4n})$ does not have elements of order $\exp(G)$ if and only if n is odd. Also, when n is odd, $S(T_{4n})$ is Hamiltonian.*

In [38], Hamzeh and Ashrafi proposed a conjecture which states that for any non Abelian finite simple groups G, $S(G)$ is not Hamiltonian. Consider the Mathieu groups M_{11} and M_{22}. It can be seen that corresponding graphs $S(M_{11})$ and $S(M_{22})$ are not Hamiltonian. Since these graphs contain cliques $H_{11} \subset S(M_{11})$ and $H_7 \subset$

$S(M_{22})$ such that no vertex of these cliques are adjacent to any vertex in $S(M_{11}) \setminus \{H_{11}, e\}$ and $S(M_{22}) \setminus \{H_7, e\}$, respectively.

In the following theorem we use this idea and prove that $S(A_n)$ is not Hamiltonian for $n \geq 5$, where A_n denotes non Abelian finite simple alternating group on n symbols.

Theorem 5.27. *For any alternating group $A_n, n \geq 5$ of permutation group S_n, $S(A_n)$ is not Hamiltonian.*

Proof. For given integer $n \geq 5$, there always exists a prime number $p \in (\lfloor \frac{n}{2} \rfloor, n)$ and cycle of order p in A_n. Let P_p be a spanning path in $S(A_n)$ corresponding to $w_p(A_n)$. Clearly, one of the last vertex of P_p will not adjacent to any other vertex implying that $S(A_n)$ is not Hamiltonian. □

Consider the following question posted by Hamzeh and Ashrafi in [38]:

Problem 5.28. In a finite non-Abelian simple group G, can we always find an integer r, such that elements in $w_r(G)$ of order r (as we have done in case of A_n) are not adjacent to any vertex of $S(G) \setminus H_r$ other than $e \in G$, where H_r is the induced subgraph on $w_r(G)$?

Note that, when such a subgraph H_r exists, then $S(G)$ is not Hamiltonian.

From the above remarks, can we generalize these observations to any finite non-Abelian simple groups. If so, then we can state an interesting property for any simple group as follows:

Problem 5.29. If G is a non-Abelian, non-p finite simple group then $S(G)$ is non-Hamiltonian. In other words, if $S(G)$ is Hamiltonian, then G cannot be a non-Abelian simple group.

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