Chapter 4

Superpower Graphs of Some non-Abelian Finite Groups

In this chapter, we first explore the undirected superpower graphs $S(D_{2n})$ and $S(T_{4n})$ of the dihedral group D_{2n} and dicyclic group T_{4n}. Among the class of non-Abelian groups, these groups have their own importance as they hold a fundamental place and that they also appear as a subgroup of many groups. Our motive is to explore the structural properties of superpower graph of these groups and see how they differ from its corresponding power graph. In Section 4.1 we delve into the superpower graph of dihedral groups and determine the tight bounds for the vertex connectivity of $S(D_{2n})$ in Subsection 4.1.1. In addition, we also study the edge connectivity of $S(D_{2n})$ in Subsection 4.1.2, and compute their exact value with the help of minimum degree for some special values of n. In Subsection 4.1.3, we give Hamiltonian-like properties for $S(D_{2n})$. Similarly, in Section 4.2, we examine the superpower graphs of dicyclic groups and give bounds for the vertex connectivity and Hamiltonian-like properties for $S(T_{4n})$, respectively in Subsections 4.2.1 and 4.2.2.

4.1 Superpower Graphs of Dihedral Groups

Before going further, observe that finding the size and number of equivalence classes in arbitrary group G is difficult. With this in our mind, we consider the Dihedral group D_{2n} of order $2n$ (Recall that $D_{2n} = \{\langle a, b\rangle | a^n = b^2 = e, aba = b\}$) and study about the superpower graph of the same. To do this, we first identify the equivalence classes of D_{2n} with the following notation. Let $w_n(G) = [a] = \{a^i \mid gcd(i, n) = 1, 1 \leq i \leq n\}$, $w_2(G) = [b] = [a^{\frac{n}{2}}] = \bigcup_{i=1}^{n}\{a^i b\} \cup \{a^{\frac{n}{2}}\}$, $w_1(G) = [a^n] = \{e\}$ and for each non-trivial divisor $d(\neq 2)$ of n, $w_d(G) = [a^{\frac{n}{d}}] = \{a^{\mathrm{id}} \mid a^i \in [a]\}$. Note that $\mathcal{P} = \{[a^j] \mid \text{j is a divisor of n}\}$ forms a partition of D_{2n} such that the induced subgraph corresponding to each part $[a^j]$ in $\mathcal{P}$ forms a clique in $S(D_{2n})$. If n is an odd positive integer, then for each positive divisor d of n, $[a^{\frac{n}{d}}]$ is an equivalence class in D_{2n} of cardinality $\phi(d)$ and $[b]$ is an equivalence class of cardinality n. Thus,

$$S(D_{2n}) = \Delta_{D_{2n}}[K_n, K_{\phi(d_1)}, \ldots, K_{\phi(d_m)}].$$

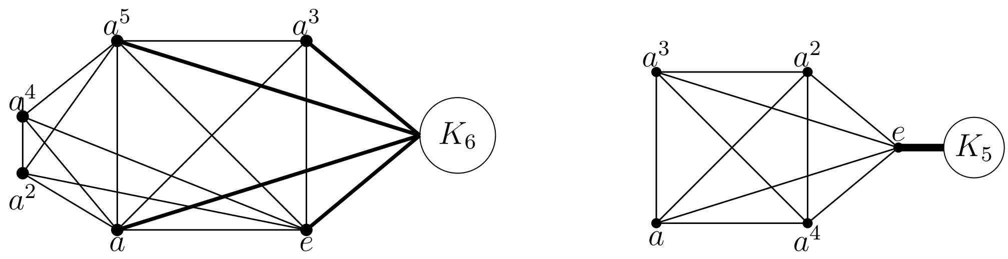
If n is an even integer, for every positive divisor $d \neq 2$ of n, $[a^{\frac{n}{d}}]$ is an equivalence class of cardinality $\phi(d)$ and $[a^{\frac{n}{2}}]$ is an equivalence class of cardinality $n+1$. Hence, for even integer n,

$$S(D_{2n}) = \Delta_{D_{2n}}[K_{n+1}, K_{\phi(d_1)}, \ldots, K_{\phi(d_m)}],$$

where $d_1, d_2, \ldots, d_m$ ($d_i \neq 2$ for each $i, 1 \leq i \leq m$) are all the positive divisors of n.

On the other hand, to understand the structure of $S(D_{2n})$, Hamzeh and Ashrafi [15] also described it as follows: Note that, in D_{2n}, $a^i b$ for every i, $1 \leq i \leq n$, all are involution of D_{2n} which forms an n-vertex clique K_n in $S(D_{2n})$. Thus, $S(D_{2n})$ contains two blocks, namely K_n and $S(\langle a\rangle)$. For an odd integer n, $S(D_{2n})$ is formed by adding edges from every vertex of K_n to the identity element of D_{2n} in the block of $S(\langle a \rangle)$ and for even integer $n \neq 2^k$, $S(D_{2n})$ can be obtained by joining every vertex of K_n to every even order element of $\langle a \rangle$ and the identity element e in the block $S(\langle a \rangle)$; see Figure 4.1. For $n = 2^k$, $S(D_{2n})$ forms a complete graph K_{2n}.

In this chapter, we follow both the structural representations and use it wherever it is convenient and/or applicable to achieve the goals. Also, we will use the notations $H_{D^*_{2n}}$ for the subgraph of $S(D_{2n})$ induced by $D^*_{2n} = D_{2n} \setminus \{e\}$ and $H_{\overline{D}_{2n}}$ for the subgraph of $S(D_{2n})$ induced by $\overline{D}_{2n} = D^*_{2n} \setminus \{x \in D_{2n} : o(x) = n\}$.



(a) $S(D_{12})$ (b) $S(D_{10})$

FIGURE 4.1: The superpower graphs of D_{12} and D_{10}[bold line between a vertex and a clique indicates that the vertex is adjacent to every vertex in the clique].

4.1.1 Vertex connectivity

Next, we state and prove a theorem on the vertex connectivity and minimum separating set of $S(D_{2n})$ where $n \neq 2^k$.

Theorem 4.1. *Let $n \geq 3$ be an integer and n is not a power of 2. Then the vertex connectivity of the superpower graph $S(D_{2n})$ of the Dihedral group D_{2n} can be determined from the following:*

(i) $\kappa(S(D_{2n})) = 1$ *if and only if n is an odd integer;*

(ii) *For even integer $n \geq 6$, $\kappa(S(D_{2n})) \geq 1 + \phi(n)$;*

(iii) *Any minimum separating set $T^\dagger$ for $S(D_{2n})$ does not contain elements of order 2.*

Proof. (i) Assume that n is an odd integer and recall the structure of $S(D_{2n})$ for an odd integer n, $S(D_{2n})$ is formed by adding edges from every vertex of K_n to the identity element of D_{2n} in the block of $S(\langle a\rangle)$. Consider the graph $H_{D^*_{2n}}$, it can be seen that there is no edge between the blocks K_n and $S(\langle a\rangle \setminus \{e\})$. Hence $T^\dagger = \{e\}$ is the minimum separating set of $S(D_{2n})$ and so $\kappa(S(D_{2n})) = 1$.

Conversely, let $\kappa(S(D_{2n})) = 1$. Suppose n is even. Let x and y be two non-identity arbitrary elements. Clearly, D_{2n} contains an element $e \neq z$ of order n. Then $P := \langle x, z, y\rangle$ is a path in the induced subgraph $\langle S(D^*_{2n})\rangle$ of $S(D_{2n})$ between x and y. Thus, $\kappa(S(D_{2n})) > 1$, which is a contradiction.

(ii) Assume that n is even. Note that e and a^k with $\gcd(n, k) = 1$ are vertices of degree $2n-1$ in $S(D_{2n})$ and so there are $\phi(n)+1$ vertices are of degree $2n-1$. Since $S(D_{2n}) = \Delta_{D_{2n}}[K_1, K_{\phi(n)}, H_{\overline{D}_{2n}}]$, we have $\kappa(S(D_{2n})) = 1+\phi(n)+\kappa(H_{\overline{D}_{2n}}) \geq \phi(n)+1$.

(iii) Let $T^\dagger$ be a minimum separating set of $S(D_{2n})$. Assume that $T^\dagger$ contains an element of order 2. Then the equivalence class $[b] \subseteq T^\dagger$. Note that $e \in T^\dagger$ and $[b]$ contains at least n elements. From this, $\kappa(S(D_{2n})) = |T^\dagger| \geq |[b]| + 1 \geq n + 1$. Since $\kappa(S(\langle a\rangle)) \leq n - 1$, removing at most n from $S(D_{2n})$ disconnects the same, which is a contradiction to the minimality of $T^\dagger$. □

Theorem 4.2. *Let $n \in \mathbb{N}$ having the prime factorization $n = 2^{\alpha_0}p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_m^{\alpha_m}$, where $m \in \mathbb{N}$ and $p_1 < p_2 < \cdots < p_m$ are odd primes, $\alpha_i \in \mathbb{N}, 0 \leq i \leq m$. Then there exists a minimal separating set T of $S(D_{2n})$ with*

$$|T| = \frac{n}{2^{\alpha_0}} + \phi(n) + \phi(\frac{n}{2^{\alpha_0}})[2^{\alpha_0-1} - 2]. \tag{4.1}$$

and hence $\kappa(S(D_{2n})) \leq |T|$. *Also, this bound is tight.*

Proof. Write $\Gamma = H_{\overline{D}_{2n}}$. Define

$$\overline{T} = \bigcup \left\{[a^{2^{\alpha_0} d}] : \ d \text{ is a non-trivial divisor of } \frac{n}{2^{\alpha_0}}\right\} \bigcup \left\{[a^{2^i}] : 1 \leq i \leq \alpha_0 - 1\right\}$$

It can be seen that $\overline{T}$ is a separating set of Γ, since no element of $[a^{2^{\alpha_0}}]$ is adjacent to any element of $\overline{D}_{2n} \setminus \{\overline{T} \cup [a^{2^{\alpha_0}}]\}$. Now we prove that $\overline{T}$ is a minimal separating set of Γ by showing that $\Gamma \setminus X$ is connected for any proper subset X of $\overline{T}$. Without loss of generality assume that $\overline{T} \setminus X = [a^{2^{\alpha_0} d}]$ for some non trivial divisor d of $\frac{n}{2^{\alpha_0}}$. Let $x, y \in \Gamma \setminus X$ be arbitrary. We have the following cases;

Case 1: When $x \in [a^{\frac{n}{2^i d_1}}]$ and $y \in [a^{\frac{n}{2^j d_2}}]$, where d_1, d_2 are divisors of $\frac{n}{2^{\alpha_0}}$. Then $P =: \langle x, a^{\frac{n}{2}}, y\rangle$ is a path between x and y.

Case 2: When $x \in [a^{2^{\alpha_0}}], y \in [a^{\frac{n}{2^i d}}]$, then $P =: \langle x, a^{2^{\alpha_0} d}, a^d, a^{\frac{n}{2}}, y\rangle$ is a path between x and y.

Thus, $\overline{T}$ is a minimal separating set of Γ. Take $T = \overline{T} \cup [a] \cup [a^n]$, then T is a minimal separating set of $S(D_{2n})$ and

$$\begin{aligned}
|T| =& |\overline{T}| + |[a]| + |[a^n]| \\
=& \sum_{d | \frac{n}{2^{\alpha_0}}, d \neq 1, \frac{n}{2^{\alpha_0}}} |[a^{2^{\alpha_0} d}]| + \sum_{i=1}^{\alpha_0 - 1} \phi(\frac{n}{2^{\alpha_0 - i}}) + \phi(n) + 1 \\
=& \frac{n}{2^{\alpha_0}} + \phi(n) + \phi(\frac{n}{2^{\alpha_0}})[2^{\alpha_0 - 1} - 2].
\end{aligned}$$

Finally, we show that this bound is tight when $m = 2, \alpha_1 = \alpha_2 = 1$. That is, when $n = 2p_1p_2$.

Let $n = 2p_1p_2$ where $p_1 < p_2$ are primes. Then the equivalence classes of $\overline{D}_{2n}$ with respect to $\sim$ are precisely $[a^{2p_1}]$, $[a^{2p_2}]$, $[a^2]$, $[a^{p_1p_2}]$, $[a^{p_1}]$, $[a^{p_2}]$ with cardinality $p_2 - 1$, $p_1 - 1$, $(p_1 - 1)(p_2 - 1)$, $2p_1p_2 + 1$, $p_2 - 1$ and $p_1 - 1$ respectively, and each of these equivalence classes is a clique in $H_{\overline{D}_{2n}}$. Thus,

$$\Gamma = \Delta_{\overline{D}_{2n}}[K_{(p_2-1)}, K_{(p_1-1)}, K_{(p_1-1)(p_2-1)}, K_{2p_1p_2+1}, K_{(p_2-1)}, K_{(p_1-1)}],$$

as given in Figure 4.2. It is clear that deletion of any one of the cliques in Γ does not disconnect Γ. However, deletion of any two cliques in Γ that are not adjacent disconnect Γ. This along with Theorem 5.9 imply that a minimal separating set of Γ is precisely the union of any two non adjacent cliques.

Also, we have the inequality $|[a^{p_1p_2}]| > |[a^2]| > |[a^{2p_1}]| = |[a^{p_1}]| > |[a^{2p_2}]| = |[a^{p_2}]|$. Consequently, $\{[a^{2p_2}] \cup [a^{2p_1}]\}$ is a separating set with minimum cardinality among all pairs of non adjacent of equivalence classes. Hence $[a^{2p_2}] \cup [a^{2p_1}]$ is a minimum separating set of Γ and so $\kappa(H_{\overline{D}_{2n}}) = (p_1 - 1)(p_2 - 1)$. Thus $\kappa(S(D_{2n})) = \phi(n) + \phi(p_1) + \phi(p_2) + 1 = \frac{n}{2}$. □

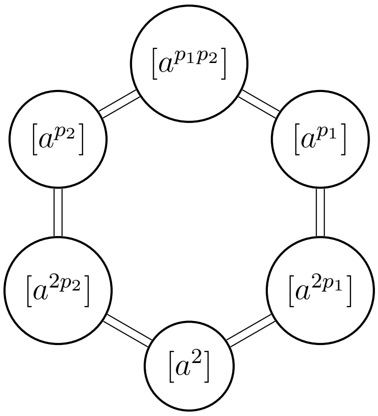


FIGURE 4.2: Connectivity in $H_{\overline{D}_{2n}}$, for $n = 2p_1p_2$.

Theorem 4.3. *For a positive integer n, $\kappa(S(D_{2n})) = \phi(n) + 1$ if and only if $n = 2p$, where p is an odd prime number.*

Proof. Assume that $n = 2p$ where p is an odd prime.

By Theorem 4.1 (ii), we have $\kappa(S(D_{2n})) \geq \phi(n) + 1$. Since $H_{\overline{D}_{2n}}$ is disconnected, we get that we have $\kappa(S(D_{2n})) \leq \phi(n) + 1$ and hence we have $\kappa(S(D_{2n})) = \phi(n) + 1$.

Conversely, assume that, for any integer n, $\kappa(S(D_{2n})) = \phi(n) + 1$. For $m \geq 2$, let $n = 2^{\alpha_0} p_1^{\alpha_1} \cdots p_m^{\alpha_m}$, where $\alpha_0 \geq 2$ and $\alpha_1, \ldots, \alpha_m \geq 1$. Without loss of generality, one can take that $m = 2$ and so $n = 2^{\alpha_0} p_1^{\alpha_1} p_2^{\alpha_2}$.

Let $x, y \in V(H_{\overline{D}_{2n}})$ be two arbitrary vertices with $o(x) = 2^{\beta_0} p_1^{\beta_1} p_2^{\beta_2}$ and $o(y) = 2^{\gamma_0} p_1^{\gamma_1} p_2^{\gamma_2}$ where $1 \leq \beta_i, \gamma_j \leq \alpha_i$ for each i, j, $1 \leq i, j \leq 2$. Let $z = a^{\frac{n}{lcm(p_1, p_2)}} \in V(H_{\overline{D}_{2n}})$. Then $\gcd(o(x), o(z)) = p_i$ and $\gcd(o(y), o(z)) = p_j$. Let u, v be vertices in $V(H_{\overline{D}_{2n}})$ be elements of order p_i and p_j respectively. This gives that $P := \langle x, u, z, v, y \rangle$ is a path between x and y which implies that $H_{\overline{D}_{2n}}$ is connected.

Similarly, one can prove that $H_{\overline{D}_{2n}}$ is connected when $o(x) = 2^{\beta_0}$ and $o(y) = p_1^{\gamma_1} p_2^{\gamma_2}$.

If $n = 2^k p^\ell$ where $\ell, k > 0$ are integers and either $k \geq 2$ or $l \geq 2$. Then one can choose an element of order $2p$ and applying arguments as above, we can see that $H_{\overline{D}_{2n}}$ is connected.

In all the above cases, $\kappa(S(D_{2p})) > \phi(n) + 1$, which is a contradiction.

If $n = 2p_1p_2$, then by Theorem 4.2, $\kappa(S(D_{2n})) = \frac{n}{2} > \phi(n) + 1$, again a contradiction.

Hence $n = 2p$ where p is an odd prime integer. ∎

Corollary 4.4. *Let $n \in \mathbb{N}$ having the prime factorization $n = 2^{\alpha_0} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}$, where $m \in \mathbb{N}$ and $p_1 < p_2 < \cdots < p_m$ are odd primes, $\alpha_i \in \mathbb{N}, 0 \leq i \leq m$. Then*

$$\phi(n) + 1 \leq \kappa(S(D_{2n})) \leq |T|.$$

where $|T|$ denotes the cardinality of minimal separating set of the minimal separating set obtained in Theorem 4.2.

Proof. Result follows from Theorems 4.1, 4.2 and 4.3. Further, if $n = 2p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_m^{\alpha_m}$, then for $\alpha_0 = 1$ in equation (4.1), we have

$$\phi(n) + 1 \leq \kappa(S(D_{2n})) \leq |T| \leq \frac{n}{2}.$$

□

Remark 4.5. For an even integer n, $S(D_{2n})$ contains two blocks H_n and $S(\langle a\rangle)$, where H_n is the clique containing all n involution in D_{2n}. From this one may claim that the connectivity of $S(D_{2n})$ can be determined once the connectivity of $S(\langle a\rangle)$ is known. In particular, one may claim that $\kappa(S(D_{2n})) = \kappa(S(\mathbb{Z}_n))$ for every even integer n. However, this claim is not true. For consider the graph $S(\mathbb{Z}_{12})$. Then $T = \{e, a, a^5, a^6, a^7, a^{11}\}$ is a minimum separating set of $S(\mathbb{Z}_{12})$ and so, $\kappa(S(\mathbb{Z}_{12})) = 6$. Note that the set $T^{'} = \{e, a, a^5, a^7, a^{11}, a^4, a^8\}$ is a minimum separating set of the graph $S(D_{24})$ and so $\kappa(S(D_{24})) = 7 \neq \kappa(S(\mathbb{Z}_{12}))$.

In Theorems 4.2 and 4.3, while calculating the vertex connectivity of $S(D_{2n})$ for various values of $n = 2^{\alpha_0}p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_m^{\alpha_m}$, we noticed that the class $[a^{2^{\alpha_0}}]$ is not fully contained in any minimal separating set of $S(D_{2n})$. However, this is not the case always. Motivated by this, we present the next two theorems where we find bounds for the connectivity depending on the presence of the class $[a^2]$ in the minimal separating set.

Theorem 4.6. *Let $n = 2p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_m^{\alpha_m}$, where $m \in \mathbb{N}$ and $p_1 < p_2 < \cdots < p_m$ are odd primes, $\alpha_i \in \mathbb{N}, 1 \leq i \leq m$. If T is a minimal separating set of $S(D_{2n})$ such that $[a^2] \nsubseteq T$, then $|T| \geq \frac{n}{2}$.*

Proof. Let T be a minimal separating set of $S(D_{2n})$. By Theorem 4.1 (ii), we have $T = \overline{T} \cup [a] \cup [a^n]$ where $\overline{T}$ is a minimal separating set of $H_{\overline{D}_{2n}}$. Hence it is enough to consider a minimal separating set $\overline{T}$ of $H_{\overline{D}_{2n}}$.

Case 1. If $[a^{\frac{n}{2}}] \subset \overline{T}$, then we have $|T| \geq |[a^{\frac{n}{2}}]| = n + 1 > \frac{n}{2}$.

Case 2. Assume that $[a^{\frac{n}{2}}] \nsubseteq \overline{T}$. We claim that, for every non-trivial divisor d of $\dfrac{n}{2} = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_m^{\alpha_m}$ either $[a^{\frac{n}{d}}] \subseteq \overline{T}$ or $[a^{\frac{n}{2d}}] \subseteq \overline{T}$. Suppose not assume that $[a^{\frac{n}{d}}] \nsubseteq \overline{T}$ and $[a^{\frac{n}{2d}}] \nsubseteq \overline{T}$ for a non-trivial divisor d of $\dfrac{n}{2}$. Let $x, y \in V(H_{\overline{D}_{2n}}) \setminus \overline{T}$.

Case 2.1. If $x \in [a^{\frac{n}{d_1}}]$, $y \in [a^{\frac{n}{d_2}}]$ for some non-trivial divisors d_1, d_2 of $\frac{n}{2}$, then $P := \langle x, a^2, y\rangle$ is a path between x and y in $H_{\overline{D}_{2n}}$ which is a contradiction to $\overline{T}$ is a separating set of $H_{\overline{D}_{2n}}$.

Case 2.2. If $x \in [a^{\frac{n}{2d_1}}]$, $y \in [a^{\frac{n}{2d_2}}]$, then $P := \langle x, a^{\frac{n}{2}}, y\rangle$ is a path between x and y, again a contradiction.

Case 2.3. If $x \in [a^{\frac{n}{d_1}}], y \in [a^{\frac{n}{2d_2}}]$, then $P := \langle x, a^2, a^{\frac{n}{d}}, a^{\frac{n}{2d}}, a^{\frac{n}{2}}, y\rangle$ is a path between x and y, again a contradiction.

Thus, for every non-trivial divisor d of $\dfrac{n}{2} = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_m^{\alpha_m}$ either $[a^{\frac{n}{d}}] \subseteq \overline{T}$ or $[a^{\frac{n}{2d}}] \subseteq \overline{T}$ which implies that $|\overline{T}| \geq \displaystyle\sum_{d|\frac{n}{2}, d\neq 1, \frac{n}{2}} \phi(d) = \frac{n}{2} - \phi(n) - 1$ and so $|T| \geq \dfrac{n}{2}$. □

Corollary 4.7. *Let $n \in \mathbb{N}$ having the prime factorization $n = 2p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_m^{\alpha_m}$, where $m \in \mathbb{N}$ and $p_1 < p_2 < \cdots < p_m$ are odd primes, $\alpha_i \in \mathbb{N}, 1 \leq i \leq m$. If $T^{\dagger}$ is a minimum separating set of $S(D_{2n})$ such that $[a^2] \nsubseteq T$, then*

$$\kappa(S(D_{2n})) = \frac{n}{2}.$$

Proof. Result follows from Corollary 4.2 and Theorem 4.6. □

Theorem 4.8. *Let $n = 2p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_m^{\alpha_m}$ where $\alpha_i \in \mathbb{N}$ and $2 < p_1 < p_2 \cdots < p_m$ are primes. For each $i, 1 \leq i \leq m$, there exists a minimal separating set T_i of $S(D_{2n})$ such that $[a^2] \subset T_i$ with $|T_i| = \displaystyle\sum_{d|\frac{n}{2p_i^{\alpha_i}}} \phi(d) + 2\phi(n)$ and $\kappa(S(D_{2n})) \leq |T_m|$.*

Proof. Consider the graph $H_{\overline{D}_{2n}}$.
Let $\overline{T_i} = \bigcup \left\{ [a^{2d}] : \text{ } d \text{ is a non-trivial divisor of } \frac{n}{2p_i^{\alpha_i}} \right\} \cup [a^2] \cup [a^{p_i^{\alpha_i}}]$. Observe that there is no path between the vertices $a^{2p_j^{\alpha_j}}$ and $a^{2p_k^{\alpha_k}}$, for $j \neq k$ and $1 \leq j, k \leq m$ in $H_{\overline{D}_{2n}} \setminus \overline{T_i}$. Hence, for each i, $\overline{T_i}$ is a separating set of $H_{\overline{D}_{2n}}$ and $|\overline{T_i}| = \sum_{d | \frac{n}{2p_i^{\alpha_i}}} \phi(d) + \phi(\frac{n}{2}) - 1$. Next we show the minimality of $\overline{T_i}$. Let x, y be any two vertices from $H_{\overline{D}_{2n}} \setminus X$ where $X = \overline{T_i} \setminus \{z\}$, where $z \in [a^{2d'}]$ for d' is a non-trivial divisor of $\frac{n}{2p_i^{\alpha_i}}$. We show that x and y are connected in $H_{\overline{D}_{2n}} \setminus X$. The following cases arises: Let d_1 and d_2 be the divisors of $\frac{n}{2}$.

Case 1. If $x \in [a^{2p_i^{\alpha_i}}]$ and $y \in [a^{d_1}]$, then $P := \langle x, a^{\frac{n}{d'}}, a^{\frac{n}{2d'}}, a^{\frac{n}{2}}, y \rangle$ is a required path between x and y.

Case 2. If $x \in [a^{2p_i^{\alpha_i}}]$ and $y \in [a^{2d_1}]$, then $P := \langle x, a^{\frac{n}{d'}}, a^{\frac{n}{2d'}}, a^{\frac{n}{2}}, a^{d_1}, y \rangle$.

Case 3. If $x \in [a^{d_1}], y \in [a^{d_2}]$,then $P := \langle x, a^{\frac{n}{2}}, y \rangle$.

Case 4. If $x \in [a^{d_1}], y \in [a^{2d_2}]$, then $P := \langle x, a^{\frac{n}{2}}, a^{d_2}, y \rangle$.

Case 5. If $x \in [a^{2d_1}], y \in [a^{2d_2}]$, then $P := \langle x, a^{d_1}, a^{\frac{n}{2}}, a^{d_2}, y \rangle$.

Similarly, we can show that $H_{\overline{D}_{2n}} \setminus \{\overline{T_i} \setminus \{z \in [a^2]\}\}$ and $H_{\overline{D}_{2n}} \setminus \{\overline{T_i} \setminus \{z \in [a^{p_i^{\alpha_i}}]\}\}$ is connected. Finally, take $T_i = \overline{T_i} \cup [a] \cup [a^n]$, then for each i, $1 \leq i \leq m$, T_i is minimal separating set of $S(D_{2n})$ such that $[a^2] \subset T_i$ with $|T_i| = \sum_{d | \frac{n}{2p_i^{\alpha_i}}} \phi(d) + \phi(\frac{n}{2}) + \phi(n) = \sum_{d | \frac{n}{2p_i^{\alpha_i}}} \phi(d) + 2\phi(n)$. Further, $|T_m| \leq |T_{m-1}| \leq \cdots \leq |T_1|$. Hence $\kappa(S(D_{2n})) \leq |T_m|$. $\square$

4.1.2 Edge connectivity

In [44], it is shown that $\kappa^{'}(\Gamma) = \delta(\Gamma)$ for any graph Γ whose diameter is at most 2. For any finite group G, the vertex e is adjacent to all the other vertices in $S(G)$ and so $diam(S(G)) \leq 2$. From this, we have $\kappa^{'}(S(G)) = \delta(S(G))$ for any group G. Hence, to find the edge connectivity of $S(D_{2n})$, it is enough to find its minimum degree. With this observation, let us find the minimum degree for special values of n. Note that any two elements of the group G having same order has the same degree in the corresponding graph $S(G)$. However, converse is not true. For example, when n is even, $deg_{S(D_{2n})}(e) = deg_{S(D_{2n})}(a) = 2n-1$ in $S(D_{2n})$ where as $e, a \in D_{2n}$ are not of the same order. In the following theorem, we obtain sharp upper bounds for the minimum degree of $S(D_{2n})$ for some special values of n.

Theorem 4.9. *Let n be a positive integer and $n \neq 2^k, k \in \mathbb{N}$. Then $\delta(S(D_{2n})) \leq n-1$. Also, equality holds if and only if $n = p^k$ for some odd prime p and $k \in \mathbb{N}$.*

Proof. Since $n \neq 2^k$, there exists a non-negative integer ℓ and an odd integer m such that $n = 2^\ell m$. Let $x = a^{2^\ell}$. Then $o(x) = m$ and so $deg_{S(D_{2n})}(x) \leq n-1$ in $S(D_{2n})$ and so $\delta(S(D_{2n})) \leq deg_{S(D_{2n})}(x) \leq n-1$.

To prove the other part of the statement, let $n = p^k$ for some odd prime p and $k \in \mathbb{N}$. Then $deg_{S(D_{2n})}(a^{\frac{n}{r}}) = n-1$ for every positive divisor $r(1 < r < n)$ of n and so $\delta(S(D_{2n})) = n-1$. Conversely, assume that $\delta(S(D_{2n})) = n-1$. Suppose n has at least two odd distinct prime divisors say p_1 and p_2. Then $n = p_1^{\alpha_1} p_2^{\alpha_2} m$ where $m \geq 1$ is an integer and relatively prime to both p_1 and p_2. Let $n_1 = p_2^{\alpha_2} m$, $n_2 = p_1^{\alpha_1} m$, $x = a^{n_1}$ and $y = a^{n_2}$ are non-adjacent vertices in $S(D_{2n})$ having odd orders, implying that $\delta(S(D_{2n})) < n-1$, a contradiction. Thus n has at most one odd prime divisor.

If $n = 2^k p^l$ and p is an odd prime, then it should be noted that $deg_{S(D_{2n})}(x) \leq n-1$ for every element $x \neq e$ of odd order and for every element y of even order, $deg_{S(D_{2n})}(y) \geq n$ in $S(D_{2n})$. Consequently, $\delta(S(D_{2n}))$ is attained by the vertex of odd order. Thus, it is enough to show that $deg_{S(D_{2n})}(x^{2^k p^{l-\alpha}}) - deg_{S(D_{2n})}(x^{2^k p^{l-\beta}}) \geq 0$ for any $\alpha \leq \beta$, $0 \leq \alpha, \beta \leq l$.

$$\begin{aligned}
&deg_{S(D_{2n})}(x^{2^k p^{l-\alpha}}) - deg_{S(D_{2n})}(x^{2^k p^{l-\beta}}) \\
&\qquad = \left(\sum_{i=1}^{l} \phi(p^i) + \sum_{i=1}^{k} \sum_{j \geq \alpha} \phi(2^i p^j)\right) - \left(\sum_{i=1}^{l} \phi(p^i) + \sum_{i=1}^{k} \sum_{j \geq \beta} \phi(2^i p^j)\right) \\
&\qquad = \sum_{i=1}^{k} \sum_{j=\alpha}^{\beta-1} \phi(2^i p^j) \geq 0.
\end{aligned}$$

Hence x^{2^k} is of minimum degree among all the vertices of $S(D_{2n})$ and

$$deg_{S(D_{2n})}(x^{2^k}) = \delta(S(D_{2n})) = \sum_{i=1}^{l} \phi(p^i) + \sum_{i=1}^{k} \phi(2^i p^l) = 2^k \phi(p^l) + p^{l-1} - 1 \neq n-1,$$

a contradiction. Hence n is a power of odd prime. □

In the following theorem, we present a lower bound for the edge connectivity of $S(D_{2n})$ for any integer $n \in \mathbb{N}$.

Theorem 4.10. *Let $n \in \mathbb{N}$ and p be the largest prime divisor of n. Then*

$$\kappa'(S(D_{2n})) = \delta(S(D_{2n})) \geq \frac{n}{p}.$$

Proof. Let $x \in S(D_{2n})$ be a non-identity element of odd order m. Then $deg_{S(D_{2n})}(x)$ in $S(D_{2n})$ is given by

$$deg_{S(D_{2n})}(x) = m + \sum_{km|n, k\neq 1} \phi(km) - 1$$

Thus, we have,

$$deg_{S(D_{2n})}(x) \geq m + \phi(n) - 1.$$

Recall that, if n is a natural number and p is the greatest prime divisor of n, then $\phi(n) \geq \frac{n}{p}$ [45, Lemma C]. Hence $deg_{S(D_{2n})}(x) \geq m - 1 + \frac{n}{p}$. Since $m - 1 > 0$, this gives $\delta(S(D_{2n})) \geq \frac{n}{p}$. □

Now we find an exact expression for $\delta(S(D_{2n}))$, for some special values of n.

Theorem 4.11. *Let $n = 2^k p_1^{\alpha_1} p_2^{\alpha_2}$ where $p_1 < p_2$ are odd prime numbers and the integers $k, \alpha_1 > 0$ and $\alpha_2 \geq 0$. Then $\delta(S(D_{2n})) = p_1^{\alpha_1 - 1}(1 - 2^k p_2^{\alpha_2}) + 2^k p_1^{\alpha_1} p_2^{\alpha_2} - 1$ and it is attained by $deg_{S(D_{2n})}(a^{2^k p_2^{\alpha_2}})$.*

Proof. We will make the four claims about the degree of the vertices based on the order of the element in D_{2n}.

Claim 1. Among all vertices of the form $a^{2^k p_1^{\alpha_1 - \beta_1} p_2^{\alpha_2}}$, for $0 \leq \beta_1 \leq \alpha_1$, the vertex $a^{2^k p_2^{\alpha_2}}$ has the minimum degree.

For $0 \leq \gamma_1 \leq \beta_1 \leq \alpha_1$,

$$\begin{aligned}
&deg_{S(D_{2n})}(a^{2^k p_1^{\alpha_1 - \gamma_1} p_2^{\alpha_2}}) - deg_{S(D_{2n})}(a^{2^k p_1^{\alpha_1 - \beta_1} p_2^{\alpha_2}}) \\
&\quad = \left\{ p_1^{\alpha_1} - 1 + \sum_{i \geq \gamma_1}^{\alpha_1} \sum_{j=1}^{\alpha_2} \phi(p_1^i p_2^j) + \sum_{i \geq \gamma_1}^{\alpha_1} \sum_{j=1}^{\alpha_2} \sum_{r=1}^{k} \phi(2^r p_1^i p_2^j) + \sum_{r=1}^{k} \sum_{i \geq \gamma_1}^{\alpha_1} \phi(2^r p_1^i) \right\} \\
&\qquad - \left\{ p_1^{\alpha_1} - 1 + \sum_{i \geq \beta_1}^{\alpha_1} \sum_{j=1}^{\alpha_2} \phi(p_1^i p_2^j) + \sum_{i \geq \beta_1}^{\alpha_1} \sum_{j=1}^{\alpha_2} \sum_{r=1}^{k} \phi(2^r p_1^i p_2^j) + \sum_{r=1}^{k} \sum_{i \geq \beta_1}^{\alpha_1} \phi(2^r p_1^i) \right\}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=\gamma_1}^{\beta_1-1} \phi(p_1^i) \sum_{j=1}^{\alpha_2} \phi(p_2^j) + \sum_{i=\gamma_1}^{\beta_1-1} \phi(p_1^i) \sum_{r=1}^{k} \phi(2^r) + \sum_{i=\gamma_1}^{\beta_1-1} \phi(p_1^i) \sum_{j=1}^{\alpha_2} \phi(p_2^j) \sum_{r=1}^{k} \phi(2^r) \\
&= \sum_{i=\gamma_1}^{\beta_1-1} \phi(p_1^i)[(p_2^{\alpha_2}-1)+(2^k-1)+(p_2^{\alpha_2}-1)(2^k-1)] \\
&\geq 0.
\end{aligned}
$$

Thus, among all vertices of the form $a^{2^k p_1^{\alpha_1-\beta_1} p_2^{\alpha_2}}$, we find that $a^{2^k p_2^{\alpha_2}}$ has the minimum degree.

Claim 2. Similar to proof of Claim 1, one can show that $deg_{S(D_{2n})}(a^{2^k p_1^{\alpha_1} p_2^{\alpha_2-\gamma_2}}) \geq deg_{S(D_{2n})}(a^{2^k p_1^{\alpha_1} p_2^{\alpha_2-\beta_2}})$, for $0 \leq \gamma_2 \leq \beta_2 \leq \alpha_2$ and hence among all the vertices of the form $a^{2^k p_1^{\alpha_1} p_2^{\alpha_2-\beta_2}}$, for $0 \leq \beta_2 \leq \alpha_2$, the vertex $a^{2^k p_1^{\alpha_1}}$ has the minimum degree.

Now let us compare the minimum degree obtained for vertices in Claims 1 and 2.

Claim 3. $deg_{S(D_{2n})}(a^{2^k p_1^{\alpha_1}}) \geq deg_{S(D_{2n})}(a^{2^k p_2^{\alpha_2}})$.

For,

$$
\begin{aligned}
&deg_{S(D_{2n})}(a^{2^k p_1^{\alpha_1}}) - deg_{S(D_{2n})}(a^{2^k p_2^{\alpha_2}}) \\
&\quad = \left\{ p_2^{\alpha_2} - 1 + \sum_{i=1}^{\alpha_1} \phi(p_1^i p_2^{\alpha_2}) + \sum_{r=1}^{k} \phi(2^r p_2^{\alpha_2}) + \sum_{r=1}^{k} \sum_{i=1}^{\alpha_1} \phi(2^r p_1^i p_2^{\alpha_2}) \right\} \\
&\qquad - \left\{ p_1^{\alpha_1} - 1 + \sum_{j=1}^{\alpha_2} \phi(p_1^{\alpha_1} p_2^j) + \sum_{r=1}^{k} \phi(2^r p_1^{\alpha_1}) + \sum_{r=1}^{k} \sum_{j=1}^{\alpha_2} \phi(2^r p_1^{\alpha_1} p_2^j) \right\} \\
&\quad = p_2^{\alpha_2} - p_1^{\alpha_1} + \phi(p_2^{\alpha_2})[2^k(p_1^{\alpha_1}-1)+2^k-1] - \phi(p_1^{\alpha_1})[2^k(p_2^{\alpha_2}-1)+2^k-1] \\
&\quad = \{2^k p_1^{\alpha_1-1} p_2^{\alpha_2-1}(p_2-p_1) + p_2^{\alpha_2-1} - p_1^{\alpha_1-1}\} \geq 0.
\end{aligned}
$$

Thus, among all the vertices of the form $a^{2^k p_1^{\alpha_1}}$ and $a^{2^k p_2^{\alpha_2}}$, we get the minimum degree is attained by $deg_{S(D_{2n})}(a^{2^k p_2^{\alpha_2}})$.

Finally, now we shall an any arbitrary element of the form $a^{2^k p_1^{\alpha_1-\alpha} p_2^{\alpha_2-\beta}}$ with elements of the form $a^{2^k p_2^{\alpha_2}}$ for $0 \leq \alpha \leq \alpha_1, 0 \leq \beta \leq \alpha_2$.

Claim 4. $deg_{S(D_{2n})}(a^{2^k p_1^{\alpha_1-\alpha} p_2^{\alpha_2-\beta}}) \geq deg_{S(D_{2n})}(a^{2^k p_2^{\alpha_2}})$ for $0 \leq \alpha \leq \alpha_1, 0 \leq \beta \leq \alpha_2$.

For,

$$\begin{aligned}
& deg_{S(D_{2n})}(a^{2^k p_1^{\alpha_1-\alpha} p_2^{\alpha_2-\beta}}) - deg_{S(D_{2n})}(a^{2^k p_2^{\alpha_2}}) \\
&= \left\{ p_1^{\alpha} p_2^{\beta} - 1 + \sum_{i\geq\alpha}^{\alpha_1} \sum_{j\geq\beta}^{\alpha_2} \phi(p_1^i p_2^j) + \sum_{r=1}^{k} \sum_{i\geq\alpha}^{\alpha_1} \sum_{j\geq\beta}^{\alpha_2} \phi(2^r p_1^i p_2^j) - \phi(p_1^{\alpha} p_2^{\beta}) \right\} \\
&\quad - \left\{ p_1^{\alpha_1} - 1 + \sum_{j=1}^{\alpha_2} \phi(p_1^{\alpha_1} p_2^j) + \sum_{r=1}^{k} \phi(2^r p_1^{\alpha_1}) + \sum_{r=1}^{k} \sum_{j=1}^{\alpha_2} \phi(2^r p_1^{\alpha_1} p_2^j) \right\} \\
&= \left\{ 2^k (p_1^{\alpha_1} - \sum_{i=0}^{\alpha-1} \phi(p_1^i))(p_2^{\alpha_2} - \sum_{j=0}^{\beta-1} \phi(p_2^j)) + p_1^{\alpha-1} p_2^{\beta} + p_1^{\alpha} p_2^{\beta-1} - p_1^{\alpha-1} p_2^{\beta-1} \right\} \\
&\quad - \left\{ 2^k p_1^{\alpha_1} p_2^{\alpha_2} - 2^k p_1^{\alpha_1-1} p_2^{\alpha_2} + p_1^{\alpha-1} \right\} \\
&= \left\{ 2^k (p_1^{\alpha_1} - p_1^{\alpha-1})(p_2^{\alpha_2} - p_2^{\beta-1}) + p_1^{\alpha-1} p_2^{\beta} + p_1^{\alpha} p_2^{\beta-1} - p_1^{\alpha-1} p_2^{\beta-1} \right\} \\
&\quad - \left\{ 2^k p_1^{\alpha_1} p_2^{\alpha_2} - 2^k p_1^{\alpha_1-1} p_2^{\alpha_2} + p_1^{\alpha-1} \right\} \\
&= \left\{ 2^k p_1^{\alpha_1-1} p_2^{\beta-1}(1-p_2) + 2^k p_1^{\alpha_1-1} p_2^{\beta-1}(p_2^{\alpha_2-\beta+1} - 1) + p_1^{\alpha_1-1} p_2^{\beta} \right. \\
&\quad \left. + p_1^{\alpha} p_2^{\beta-1} - p_1^{\alpha-1} p_2^{\beta-1} - p_1^{\alpha_1-1} \right\} \\
&\geq p_1^{\alpha_1-1} p_2^{\beta} + p_1^{\alpha} p_2^{\beta-1} - p_1^{\alpha-1} p_2^{\beta-1} - p_1^{\alpha_1-1} \\
&= p_1^{\alpha_1-1}(p_2^{\beta} - 1) + p_1^{\alpha-1} p_2^{\beta-1}(p_1 - 1) \geq 0.
\end{aligned}$$

From all the above claims, we find that the element $a^{2^k p_2^{\alpha_2}}$ has the minimum degree.

$$\begin{aligned}
\delta(S(D_{2n}) =& deg_{S(D_{2n})}(a^{2^k p_2^{\alpha_2}}) \\
&= \{ p_1^{\alpha_1} - 1 + \sum_{j=1}^{\alpha_2} \phi(p_1^{\alpha_1} p_2^j) + \sum_{r=1}^{k} \phi(2^r p_1^{\alpha_1}) + \sum_{r=1}^{k} \sum_{j=1}^{\alpha_2} \phi(2^r p_1^{\alpha_1} p_2^j) \}.
\end{aligned}$$

Hence the proof. $\square$

Note that in the above theorem we considered n as an even integer. In the following, we present a similar result for n is an odd integer. In fact we consider n is either a product of three distinct primes or product of powers of two distinct primes. Since the proof is similar to the proof of Theorem 4.11, we omit the proof for brevity.

Theorem 4.12. *Let $n \in \mathbb{N}$ and $p_1 < p_2 < p_3$ be prime numbers.*

(i) *If $n = p_1p_2p_3$, then $\delta(S(D_{2n})) = \phi(n) + p_1p_2 - 1$ and it is attained through the element $a^{p_1p_2}$.*

(ii) *Let $n = p_1^{\alpha_1}p_2^{\alpha_2}$ and $\alpha_1, \alpha_2 \in \mathbb{N}$. Then $\delta(S(D_{2n})) = p_1^{\alpha_1} + p_1^{\alpha_1-1}(p_1-1)(p_2^{\alpha_2}-1)-1$ and it is attained through the element $a^{p_2^{\alpha_2}}$.* □

Theorem 4.13. *The superpower graph $S(D_{2n})$ is critically edge connected if and only if $n = 2^k$.*

Proof. If $n \neq 2^k$, then there exists an edge ε in $S(D_{2n})$ which is incident to vertices b and ab. By Theorem 4.9, $\kappa'(S(D_{2n})) = \kappa'(S(D_{2n} \setminus \{\varepsilon\}))$, hence $S(D_{2n})$ is not critically edge connected. If $n = 2^k$ then $S(D_{2n}) = K_{2n}$ and hence $S(D_{2n})$ is critically edge connected . □

4.1.3 Hamiltonian property and its variations

In this section, we explore the Hamiltonian-like properties of $S(D_{2n})$.

To check whether $S(D_{2n})$ is Hamiltonian, we would like to first identify the elements of D_{2n} which will go in a specific equivalence class. Then combining with the fact that each equivalence class of $S(D_{2n})$ induces a clique, which will help us to construct the required Hamiltonian cycle in $S(D_{2n})$.

Theorem 4.14. *Let $n \geq 3$ be any integer. Then the following are equivalent:*

(i) *The superpower graph $S(D_{2n})$ is Hamiltonian;*

(ii) *n is an even integer;*

(iii) *The superpower graph $S(D_{2n})$ is 1-Hamiltonian.*

Proof. $(i) \Rightarrow (ii)$. Assume that $S(D_{2n})$ is Hamiltonian. If n is an odd integer, then $H_{D^*_{2n}}$ is disconnected and number of components of $H_{D^*_{2n}}$ is 2 which is more than the number of vertices in a minimum separating set. Hence $S(D_{2n})$ is not Hamiltonian when n is odd, a contradiction.

$(ii) \Rightarrow (i)$. Assume that n is an even integer. If t denotes the total number of non-trivial divisors of n, then we have t disjoint cliques in $H_{\overline{D}_{2n}}$. One can check that $t < \phi(n) + 1$. Since each of t cliques has a spanning path whose initial and terminating vertices are adjacent to every vertex of $[a] \cup [a^n]$, we get a Hamiltonian cycle in $S(D_{2n})$ as follows (see Figure 2).

Let $d_1 < d_2 < \cdots < d_t$ be the ordering of the non-trivial divisors of n. Start from any vertex v_1 of $[a] \cup [a^n]$, go to any vertex of $[a^{d_1}]$ and traverse along the spanning path in $[a^{d_1}]$ and come back to another unused vertex v_2 of $[a] \cup [a^n]$. From v_2 go to any vertex of $[a^{d_2}]$ and repeat the process until all the cliques $[a^d]$ are covered. Since $t < \phi(n) + 1$, there are sufficient number of vertices in $[a] \cup [a^n]$ to connect each of disjoint cliques. Finally, complete the cycle by using the remaining unused vertices of $[a] \cup [a^n]$ and reach back to v_1.

$(ii) \Rightarrow (iii)$. Assume that n is an even integer. As discussed earlier, D_{2n} can be expressed as the disjoint union of $[a^d]$ for every divisor d of n. Let $g \in V(S(D_{2n}))$, if $g \in [a^d]$ for some non-trivial divisor d of n, then $S([a^d]) \setminus \{g\}$ remains a clique and so it has a spanning path whose initial and terminal vertices can be joined by two different vertices of $[a] \cup [a^n]$. Now, the proof follows along the same lines discussed

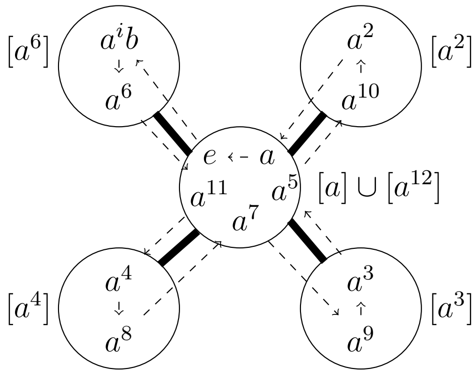


FIGURE 4.3: Hamiltonian cycle in $S(D_{24})$

above. If $g \in [a] \cup [a^n]$, then t, total number of non-trivial divisors of n is less than $\phi(n)+1$. Thus, $S(D_{2n}) \setminus \{g\}$ contains a Hamiltonian cycle implying that $S(D_{2n})$ is 1-Hamiltonian.

$(iii) \Rightarrow (ii)$. Follows from the fact that $S(D_{2n})$ fails to be Hamiltonian when n is an odd integer. □

Corollary 4.15. *For any integer $n \geq 3$, the superpower graph $S(D_{2n})$ contains a subgraph isomorphic to a wheel W_{2n-1} on $2n-1$ vertices if and only if n is an even integer.*

We have seen that when n is an even integer, $S(D_{2n})$ contains cycles of length $2n$ and $2n-1$. On the other hand, $S(D_{2n})$ contains cycles of each length from 3 to $\phi(n)+1$, since $[a] \cup [a^n]$ is a clique. Now the question is does $S(D_{2n})$ contains cycles each of length from $\phi(n)+2$ to $2n-2$. In the following theorem, we answer this question in affirmative.

Theorem 4.16. *For any integer $n \geq 3$, the superpower graph $S(D_{2n})$ is pancyclic if and only if n is an even integer.*

Proof. Let d_1 be a non-trivial divisor of n. By Theorem 4.14, for any $x_1 \in [a^{d_1}]$, $S(D_{2n}) \setminus \{x_1\}$ is Hamiltonian. This further implies that $S(D_{2n})$ contains a cycle of length $2n-1$. Again, choose $\{x_2\} \in [a^{d_1}]$ (if exists), otherwise choose $\{x_2\} \in [a^{d_2}]$ for some non-trivial divisor $d_2 \neq d_1$ of n. It is immediate that $S(D_{2n}) \setminus \{x_1, x_2\}$ is Hamiltonian, and so $S(D_{2n})$ contains a cycle of length $2n-2$. By continuing this way one can cycles of length $\phi(n)+2$ to $2n-2$. Thus $S(D_n)$ contains cycles of length ℓ for each ℓ, $1 \leq \ell \leq 2n$ and hence $S(D_{2n})$ is pancyclic. □

Theorem 4.17. *For any edge ε in the superpower graph $S(D_{2n})$, there exists a Hamiltonian cycle in $S(D_{2n})$ containing ε if and only if n is an even integer.*

Proof. Assume that n is an even integer and $\varepsilon = uv \in E(S(D_{2n}))$ be an arbitrary edge. By making four possible cases depending on the partite sets containing u and v and by following similar lines of proof of the Theorem 4.14, we obtain the required result. □

Even though $S(D_{2n})$ is not Hamiltonian, when n is odd, we prove below that $S(D_{2n})$ contains a Hamiltonian path.

Theorem 4.18. *For any odd integer n, $S(D_{2n})$ contains a Hamiltonian path.*

Proof. Let n be an odd integer. Then the identity element is a cut vertex of $S(D_{2n})$. Now, by considering a Hamiltonian path in $S(\langle a \rangle)$ ending at e and joining this with the spanning path of the clique corresponding to $[b]$, one can obtain the required Hamiltonian path. □

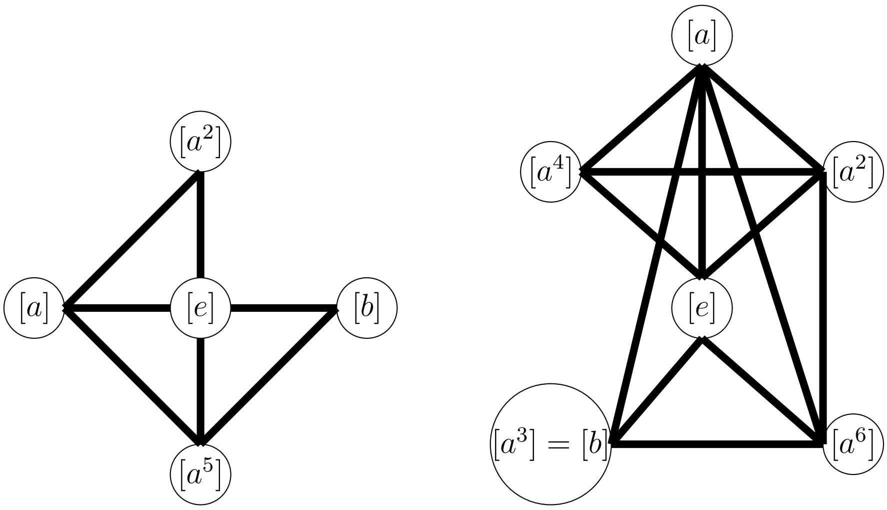
4.2 Superpower Graphs of Dicyclic Groups

Recall that $T_{4n} = \{\langle a, b\rangle \mid a^{2n} = e, a^n = b^2, ab = ba^{-1}\}$ denotes the *dicyclic group* of order $4n$. In this section, we use the notations $H_{T^*_{4n}}$ for the subgraph of $S(T_{4n})$

induced by $T_{4n}^* = T_{4n} \setminus \{e\}$ and $H_{\overline{T}_{4n}}$ for the subgraph of $S(T_{4n})$ induced by $\overline{T}_{4n} = T_{4n}^* \setminus \{x \in T_{4n} : o(x) = 2n\}$. By the representation in Equation (1.1), it can be seen that the superpower graph of dicyclic group is given by

$$S(T_{4n}) = \Delta_{T_{4n}}[K_{2n}, K_{\phi(d_1)}, \ldots, K_{\phi(d_m)}] = \Delta_{T_{4n}}[K_1, K_{\phi(2n)}, H_{\overline{T}_{4n}}],$$

where $d_1, d_2, \cdots, d_m$ are all positive divisors of $2n$. It is also important to note that when n is an odd integer, then the equivalence class $[b]$ which contains all elements of order 4 of the group T_{4n} is of cardinality $2n$ and for even integer n, $[b] = [a^{\frac{n}{2}}]$ contains $2n+2$ elements. See the superpower graphs of dicyclic graphs when $n = 5$ and $n = 6$ as shown in Figure 4.4.



(a) $S(T_{20})$ (b) $S(T_{24})$

FIGURE 4.4: Superpower graphs of T_{20} and T_{24}. [Every circled node is an equivalence class which represents a complete graph. Bold lines between two equivalence classes indicates that every vertex in one class is joined to every vertex in the other.]

4.2.1 Vertex connectivity

In this section, we give the sharp bounds for the vertex connectivity of superpower graph $S(T_{4n})$.

First, let us compare the connectivity of super power graph of dicyclic graph and the cyclic group. Is it true that $\kappa(S(T_{4n})) = \kappa(S(\mathbb{Z}_{2n}))$? The answer to this question is not in affirmative as can be seen from the following example.

Example 4.1. *Consider the graph $S(\mathbb{Z}_{24})$ and the set $T = \{e, a, a^5, a^6, a^7, a^{11}, a^{12}, a^{13}, a^{17}, a^{18}, a^{19}, a^{23}\}$. It can be verified that T is a minimum separating set of $S(\mathbb{Z}_{24})$ and so $\kappa(S(\mathbb{Z}_{24})) = 12$. However, the set $T' = \{e, a, a^2, a^4, a^5, a^7, a^{10}, a^{11}, a^{13}, a^{14}, a^{17}, a^{19}, a^{20}, a^{22}, a^{23}\}$ is a minimum separating set of the graph $S(T_{48})$ implying that $\kappa(S(T_{48})) = 15$.*

Motivated by this, we study the parameter $\kappa(S(T_{4n}))$. We begin with a result that shows the equivalence class $[b] \subset T_{4n}$ cannot be a part of a minimum separating set of $S(T_{4n})$.

Theorem 4.19. *If $T^\dagger$ is a minimum separating set of $S(T_{4n})$, then the equivalence class $[b] \nsubseteq T^\dagger$.*

Proof. Let $T^\dagger$ be a minimum separating set of $S(T_{4n})$ and the equivalence class $[b] \subseteq T^\dagger$. Then $\kappa(S(T_{4n})) = |T^\dagger| \geq 2n + 1$, since the identity element $e \in T^\dagger$ and $[b]$ contains $2n$ elements. But, $\kappa(H_{\overline{T}_{4n}}) < 2n - \phi(2n) - 2$. That is, removing at most $2n - \phi(2n) - 2 + (\phi(2n) + 1) = 2n - 1$ elements from $S(T_{4n})$ disconnects the same, which is a contradiction to the minimality of $T^\dagger$. $\square$

Next, we give the exact value of the connectivity of $S(T_{4n})$, when n is an odd integer.

Theorem 4.20. *For any positive integer $n > 1$, $\kappa(S(T_{4n})) = 2$ if and only if n is an odd integer.*

Proof. Let $n > 1$ be an odd integer. Recall that $S(T_{4n}) = \Delta_{T_{4n}}[K_{2n}, K_{\phi(d_1)}, \ldots, K_{\phi(d_m)}]$, where $d_1, d_2, \ldots, d_m$ are all positive divisors of $2n$. Clearly, the identity element e must be in a minimum separating set, since e is adjacent to all other vertices of $S(T_{4n})$. Take $T^\dagger = \{a^n, e\}$, then $S(T_{4n}) \setminus T^\dagger$ is disconnected, since there is no path between the vertices a and $a^{\frac{n}{2}}$. Hence $\kappa(S(T_{4n})) = 2$.

Conversely, if n is an even integer, then all vertices corresponding to the class $[a] \cup [a^{2n}]$ are adjacent to all other vertices of the graph $S(T_{4n})$. Also, $|[a] \cup [a^{2n}]| = \phi(2n) + 1 > 2$. Consequently, $\kappa(S(T_{4n})) > 2$ and hence n is an odd integer. $\square$

To find the upper bound of connectivity of the superpower graph of dicyclic group for an even integer n, we partition the set of even integers into two collections, namely (i) when n is twice an odd integer and (ii) when n is any arbitrary even integer not in (i). In the following theorem, we first present the tight upper bound for the vertex connectivity of $S(T_{4n})$ when n is two times an odd integer.

Theorem 4.21. *Let $n \in \mathbb{N}$ having the prime factorization $n = 2p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_m^{\alpha_m}$, where $m \in \mathbb{N}$ and $p_1 < p_2 < \cdots < p_m$ are odd primes, $\alpha_i \in \mathbb{N}, 1 \leq i \leq m$. Then there exists a minimal separating set T of $S(T_{4n})$ with $|T| = n$, and $\kappa(S(T_{4n})) \leq n$ and equality is attained when $n = 2p_1p_2$.*

Proof. Consider the graph $\Gamma = H_{\overline{T_{4n}}}$. Define

$$\overline{T} = \bigcup\{[a^{p_1^{\alpha_1-\beta_1}\cdots p_m^{\alpha_m-\beta_m}}] \mid 0 \leq \beta_i \leq \alpha_i,\ \beta_i \in \mathbb{N}\cup\{0\} \text{ and not all } \beta_i \text{ zero}\} \cup \{[a^n]\}.$$

It can be seen that $\overline{T}$ is a separating set of Γ, since no element $[a^{\frac{n}{2}}]$ is adjacent to any element of $\overline{T_{4n}} \setminus \{\overline{T} \cup [a^{\frac{n}{2}}]\}$. That is, $\overline{T}$ divides Γ into two components namely

$S([a^{\frac{n}{2}}])$ and $S(\overline{T_{4n}} \setminus \{\overline{T} \cup [a^{\frac{n}{2}}]\})$. Now we prove that $\overline{T}$ is a minimal separating set of Γ by showing that $\Gamma \setminus X$ is connected for any proper subset X of $\overline{T}$. Without loss of generality assume that $\overline{T} \setminus X = [a^d]$ for some non trivial divisor d of $\frac{n}{2}$. Let $x, y \in \Gamma \setminus X$ be arbitrary and d_1, d_2 are divisors of $\frac{n}{2}$. We have the following cases:

Case 1: When $x \in [a^{2d_1}]$ and $y \in [a^{2d_2}]$. Then $\langle x, x^2, a^4, y^2, y\rangle$ is a path connecting x and y.

Case 2: When $x \in [a^{2d_1}]$ and $y \in [a^{4d_2}]$. Then $\langle x, x^2, a^4, y\rangle$ is a path connecting x and y.

Case 3: When $x \in [a^{4d_1}]$ and $y \in [a^{4d_2}]$. Then $\langle x, a^4, y\rangle$ is a path connecting x and y.

Case 4: When $x \in [a^{\frac{n}{2}}], y \in [a^{2^i d_1}]$, $i \in \{1, 2\}$. Then $\langle x, a^d, a^{2d}, a^{4d}, a^4, a^{4d_1}, y\rangle$ is a path connecting x and y.

Take $T = \overline{T} \cup [a] \cup [a^{2n}]$. Then T is a minimal separating set of $S(T_{4n})$. Further,

$$\begin{aligned}
|T| =& |\overline{T}| + |[a]| + |[a^{2n}]| \\
=& \sum_{d|\frac{n}{2}, d\neq 1, \frac{n}{2}} |[a^d]| + 1 + \phi(2n) + 1 \\
=& \phi(4)\left[\frac{n}{2} - \phi(\frac{n}{2}) - 1\right] + 1 + \phi(2n) + 1 \\
=& n.
\end{aligned}$$

Now, we claim that this bound coincides with $\kappa(S(T_{4n}))$ when $m = 2, \alpha_1 = \alpha_2 = 1$, that is, when $n = 2p_1p_2$.

Notice that the equivalence classes of $\overline{T_{4n}}$ with respect to $\sim$ are precisely $[a^{2p_1p_2}]$, $[a^{p_1p_2}]$, $[a^{p_1}]$, $[a^{p_2}]$, $[a^{2p_1}]$, $[a^{2p_2}]$, $[a^2]$, $[a^{4p_1}]$, $[a^{4p_2}]$ and $[a^4]$ with cardinalities 1, $2n+2$, $2(p_2-1)$, $2(p_1-1)$, (p_2-1), (p_1-1), $(p_1-1)(p_2-1)$, (p_2-1), (p_1-1) and

$(p_1-1)(p_2-1)$, respectively, and forms cliques in Γ; See Figure 4.5. Thus,

$$\Gamma = \Delta_{\overline{T}_{4n}}[K_1, K_{2n+2}, K_{2(p_2-1)}, K_{2(p_1-1)}, K_{(p_2-1)}, K_{(p_1-1)}, K_{(p_1-1)(p_2-1)}, K_{(p_2-1)}, K_{(p_1-1)}, K_{(p_1-1)(p_2-1)}].$$

It is clear from Figure 4.5 that deletion of at most two cliques in Γ does not disconnect it. However, deletion of three or more cliques in Γ can disconnect the same. Hence a minimal separating set of Γ is precisely the union of any three or more equivalence classes. Also, there are only two ways in which we can remove exactly three equivalence classes to disconnect Γ, namely $\{[a^{p_2}] \cup [a^{p_1}] \cup [a^{2p_1p_2}]\}$ and $\{[a^{4p_2}] \cup [a^{4p_1}] \cup [a^2]\}$. In the rest of the cases, we need to remove at least four equivalence classes. Consequently, $\overline{T} = \{[a^{p_2}] \cup [a^{p_1}] \cup [a^{2p_1p_2}]\}$ is the set of classes having minimum cardinality. Thus, $\overline{T}$ is a minimum separating set of $H_{\overline{T}_{4n}}$ and $|\overline{T}| = 2p_1 + 2p_2 - 3$. Finally, by letting $T = \overline{T} \cup [a] \cup [a^{2n}]$, we get that T is a minimum separating set for $S(T_{4n})$ and $|T| = n$. $\square$

Note that we can not extend the above result obtained for $n = 2^2 \times 3 \times 5 = 60$, as the size of minimal separating set will be $6(3-1)(5-1) + 6(3+5) - 10 = 86 > n$. However, using another technique, we obtain a minimal separating set of size $\frac{n}{2^2} + \phi(\frac{n}{2^2})(2^{2+1} - 2) = 63 < 86$ in the following theorem.

Theorem 4.22. *Let $n \in \mathbb{N}$ having the prime factorization $n = 2^{\alpha_0} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}$, where $p_1 < p_2 < \cdots < p_m$ are odd primes and $\alpha_0 > 1$, $\alpha_i \in \mathbb{N}$, $1 \le i \le m$. Then there exists a minimal separating set T of $S(T_{4n})$ with $|T| = \frac{n}{2^{\alpha_0}} + \phi(\frac{n}{2^{\alpha_0}})(2^{\alpha_0+1} - 2)$ and the vertex connectivity*

$$\kappa(S(T_{4n})) \le \frac{n}{2^{\alpha_0}} + \phi(\frac{n}{2^{\alpha_0}})(2^{\alpha_0+1} - 2). \tag{4.2}$$

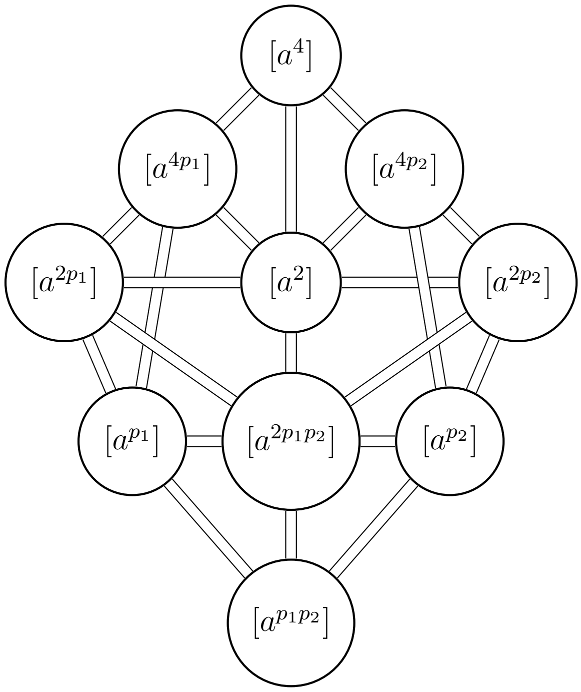


FIGURE 4.5: Connectivity in $H_{\overline{T}_{4n}}$

Proof. Write $\Gamma = H_{\overline{T}_{4n}}$. Then it can be seen that Γ is connected, since for $x, y \in V(\Gamma)$, if $x \in [a^{\frac{n}{d_1}}], y \in [a^{\frac{n}{d_2}}]$, then $\langle x, a^{\frac{n}{2d_1}}, a^n, a^{\frac{n}{2d_2}}, y\rangle$ is a path between x and y, where d_1, d_2 are non trivial divisors of n. Let

$$\overline{T} = \bigcup\{[a^{2^{\alpha_0+1}d}] | d \text{ is a non trivial divisor of } \frac{n}{2^{\alpha_0}}\} \cup \left(\bigcup_{i=0}^{\alpha_0}\{[a^{2^i}]\}\right).$$

It can be seen that an element of $[a^{2^{\alpha_0+1}}]$ must be of order $\frac{n}{2^{\alpha_0}}$, while an element of $V(\Gamma) \setminus (\overline{T} \cup [a^{2^{\alpha_0+1}}])$ will be of the order $\dfrac{n}{2^{\gamma_0}p_1^{\gamma_1}\cdots p_m^{\gamma_m}}, 1 < \gamma_0 < \alpha_0 + 1, 0 \leq \gamma_i < \alpha_i$. Thus, no element of the class $[a^{2^{\alpha_0+1}}]$ is adjacent to an element of $V(\Gamma)\setminus(\overline{T}\cup[a^{2^{\alpha_0+1}}])$ implying that $\overline{T}$ is a separating set of Γ. Now, we show that $\overline{T}$ is a minimal separating set by showing that $\Gamma \setminus X$ is connected for any non empty proper subset X of T. Without loss of generality assume that $T \setminus X = \{[a^{2^{\alpha_0+1}p_1^{\gamma_1}\cdots p_m^{\gamma_m}}]\}$. For $x, y \in \Gamma \setminus X$, following cases arise:

Case 1: For $x \in [a^{2^{\alpha_0+1}}]$ and $y \in [a^{2^i d}]$, $0 \leq i \leq \alpha_0$.

Then $P := \langle x, a^{2^{\alpha_0+1}p_1^{\gamma_1}\cdots p_m^{\gamma_m}}, a^{2^{\alpha_0}p_1^{\gamma_1}\cdots p_m^{\gamma_m}}, a^n, y\rangle$ is a path between x and y.

Case 2: For $x \in [a^{2^i d_1}], y \in [a^{2^j d_2}]$, where d_1, d_2 are non trivial divisors of $\frac{n}{2^{\alpha_0+1}}$ and $i \neq j, 0 \leq i, j \leq \alpha_0$. Then $P := \langle x, a^n, y\rangle$ is a path between x and y.

Letting $T = \overline{T} \cup [a] \cup [a^{2n}]$, we then have T is a minimal separating set of $S(T_{4n})$ whose cardinality is given by

$$\begin{aligned}
|T| =& |\overline{T}| + \phi(2n) + 1 \\
=& \sum_{d|\frac{n}{2^{\alpha_0}}, d\neq 1, \frac{n}{2^{\alpha_0}}} |[a^{2^{\alpha_0+1}d}]| + \sum_{i=1}^{\alpha_0} |[a^{2^i}]| + \phi(2n) + 1. \\
=& \frac{n}{2^{\alpha_0}} - 1 + \phi(\frac{n}{2^{\alpha_0}}) \sum_{i=2}^{\alpha_0} \phi(2^i) + \phi(2n) + 1 \\
=& \frac{n}{2^{\alpha_0}} + \phi(\frac{n}{2^{\alpha_0}})(2^{\alpha_0+1} - 2).
\end{aligned}$$

□

In the next theorem, we present the sharp lower bound for the vertex connectivity of $S(T_{4n})$ and also prove it is tight.

Theorem 4.23. *For even integer n, $\kappa(S(T_{4n})) \geq \phi(2n) + 2$ and equality holds if and only if $n = 2p$, where p is an odd prime number.*

Proof. It is important to note that $S(T_{4n})$ has $\phi(2n) + 1$ vertices of degree $4n - 1$ corresponding to the classes of elements $[a] \cup [a^{2n}]$ in the group T_{4n}. Also, for $x, y \in H_{\overline{T}_{4n}}$, if $x \in [a^{\frac{n}{d_1}}], y \in [a^{\frac{n}{d_2}}]$, then $P := \langle x, a^{\frac{n}{2d_1}}, a^n, a^{\frac{n}{2d_2}}, y\rangle$ is a path between x and y, where d_1, d_2 are non trivial divisors of n. Thus, to disconnect the graph $S(T_{4n})$, we need to remove at least $\phi(2n) + 2$ vertices. Consequently, $\kappa(S(T_{4n})) \geq \phi(2n) + 2$.

When $n = 2p$, we have $H_{\overline{T}_{4n}} \setminus [a^n]$ is disconnected, since there is no path between the vertices a^4 and a^p. Thus, $\kappa(S(T_{4n})) = \phi(2n) + 2$.

Conversely, assume that n is not a square free even number with at least three prime divisors, that is, $n = 2^{\alpha_0} p_1^{\alpha_1} \cdots p_m^{\alpha_m}$, $m \geq 2$. We will show that $H_{\overline{T}_{4n}} \setminus [a^n]$ is connected. Let $x, y \in V(H_{\overline{T}_{4n}} \setminus [a^n])$ be arbitrary with $o(x) = 2^{\beta_0} p_1^{\beta_1} \cdots p_m^{\beta_m}$ and $o(y) = 2^{\gamma_0} p_1^{\gamma_1} \cdots p_m^{\gamma_m}$ where $0 \leq \beta_i, \gamma_i \leq \alpha_i$ for each i, $0 \leq i \leq m$. Choose an element $z = a^{\frac{2n}{lcm(p_i, p_j)}} \in V(H_{\overline{T}_{4n}} \setminus [a^n])$, where p_i, p_j are the least odd prime divisors of $o(x), o(y)$ respectively for which $\beta_i \neq 0$, $\gamma_j \neq 0$. Two cases arises:

Case 1: When $p_i = p_j$ Then $gcd(o(x), o(z)) = p_i$ and $gcd(o(y), o(z)) = p_i$. Then, there is an x, y-path $P := \langle x, z, y \rangle$ between x and y which implies that $H_{\overline{T}_{4n}} \setminus [a^n]$ is connected.

Case 2: When $p_i \neq p_j$. Then $gcd(o(x), o(z)) = p_i$ and $gcd(o(y), o(z)) = p_j$. Choose $u = a^{\frac{2n}{p_i}}, v = a^{\frac{2n}{p_j}} \in V(H_{\overline{T}_{4n}} \setminus [a^n]))$. Then, there is an x, y-path $P := \langle x, u, z, v, y \rangle$ between x and y which implies that $H_{\overline{T}_{4n}} \setminus [a^n]$ is connected.

Similarly, one can prove that $H_{\overline{T}_{4n}}$ is connected when $o(x) = 2^{\beta_0}$, $\beta_0 > 1$ and $o(y) = p_1^{\gamma_1} \cdots p_m^{\gamma_m}$.

When $n = 2p_1p_2$, then by Theorem 4.21, $\kappa(S(T_{4n})) = n > \phi(2n) + 2$. So, we are left with the case when $n = 2^k p^l$, where $l, k > 0$ are integers and either $k \geq 2$ or $l \geq 2$. Then, choose an element $a^{\frac{n}{2p}}$ and applying similar argument given above, we can see that $H_{\overline{T}_{4n}} \setminus [a^n]$ is connected. Hence $n = 2p$. $\square$

Corollary 4.24. *Let $n \in \mathbb{N}$ having the prime factorization $n = 2p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}$, where $p_1 < p_2 < \cdots < p_m$ are odd primes and $\alpha_i \in \mathbb{N}, 1 \leq i \leq m$. Then*

$$\phi(2n) + 2 \leq \kappa(S(T_{4n})) \leq n.$$

Proof. Result follows from Theorem 4.21 and Theorem 4.23. □

Corollary 4.25. *Let $n \in \mathbb{N}$ having the prime factorization $n = 2^{\alpha_0} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}$, where $p_1 < p_2 < \cdots < p_m$ are odd primes and $\alpha_0 > 1$, $\alpha_i \in \mathbb{N}, 0 \leq i \leq m$, $m \in \mathbb{N}$. Then*

$$\phi(2n) + 2 < \kappa(S(T_{4n})) \leq \frac{n}{2^{\alpha_0}} + \phi(\frac{n}{2^{\alpha_0}})(2^{\alpha_0+1} - 2).$$

Proof. Results follows from Theorem 4.22 and Theorem 4.23. □

4.2.2 Hamiltonian property and its variations

It can be observe that all the results in the Subsection 4.1.3 which we have proved for $S(D_{2n})$ can be obtained with the same technique for $S(T_{4n})$. So we are stating them without proof.

Theorem 4.26. *Let n be any positive integer. Then*

(i) *$S(T_{4n})$ is Hamiltonian for any $n \geq 3$.*

(ii) *$S(T_{4n})$ is 1-Hamiltonian for any $n \geq 4$.*

(iii) *$S(T_{4n})$ is pancyclic for every integer $n \geq 3$.*

(iv) *For any edge ε in the superpower graph $S(T_{4n})$, there exists a Hamiltonian cycle in $S(T_{4n})$ containing ε, $n \geq 3$.*

(v) *$S(T_{4n})$ contains a Hamiltonian path.* □

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