Chapter 3

Superpower Graphs of Finite Abelian Groups

In this chapter and rest of the thesis, we will explore an interesting class of graphs derived from groups, namely the Superpower graphs. Recall from Definition 1.4, that for a finite group G, the superpower graph $S(G)$ is an undirected simple graph with vertex set G and any two distinct vertices are adjacent in $S(G)$ if and only if the order of one divides the order of the other in G. We first set the preliminaries on Abelian groups required for proving our results.

Theorem 3.1 ([43], Theorem 8.2)**.** *Let G_1 and G_2 be two finite cyclic groups. Then their external direct product $G_1 \times G_2$ is cyclic if and only if orders $o(G_1)$ and $o(G_2)$ are relatively prime.* □

Theorem 3.2 ([43], Theorem 11.1)**.** *Every finite Abelian group is the direct product of cyclic groups of prime-power order. Moreover, the number of terms in the direct product and the orders of the cyclic groups are uniquely determined by the group.* □

The above theorem actually gives for a finite Abelian group G, $G \cong \mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \times \cdots \times \mathbb{Z}_{p_m^{\alpha_m}}$, where the cyclic groups $\mathbb{Z}_{p_1^{\alpha_1}}$, $\mathbb{Z}_{p_2^{\alpha_2}}$ and $\mathbb{Z}_{p_m^{\alpha_m}}$ are uniquely determined by G. The following is a known characterization of finite Abelian groups through elementary divisors of the same.

Theorem 3.3 ([43], Chapter 11). *Let G be a finite Abelian group of order $n = p_1^{\alpha_1}.p_2^{\alpha_2}.\cdots p_m^{\alpha_m}$, where each p_i is prime and $\alpha_i \in \mathbb{N}$. Then $G \simeq \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}$, where $n_1 \geq n_2 \geq \cdots \geq n_k, n_j | n_i$ for each $j \geq i, 1 \leq j \leq k$ and $n_1 n_2 \cdots n_k = o(G)$.* □

Lemma 3.4 ([43], Chapter 11). *Let G be a finite Abelian group having an element x of maximum nontrivial order. Then $o(g)$ divides $o(x)$ $\forall$ $g \in G$.* □

In Section 3.1, we first understand the structure of a superpower graph defined on any finite group. In Section 3.2, we identify the set of dominant vertices of $S(G)$ followed by results on vertex connectivity in Section 3.3. In Section 3.4, we study the Hamiltonian-related properties for $S(G)$ when defined on a finite Abelian group.

3.1 Structure of Superpower Graph of any Finite Group

Asma Hamzeh *et al.*[15] proved that, for any group G of order n, the superpower graph $S(G)$ is complete if and only if $n = p^k$ for some prime number p and $k \in \mathbb{N}$.

Theorem 3.5 ([15], Theorem 2.3). *Let G be a finite group. The superpower graph $S(G)$ is complete if and only if G is a p-group.* □

Recall that, it was proved in [22], [19] that $S(G)$ can be considered as a join of complete graphs as stated in Equation (1.1) of Chapter 1.

$$S(G) = \Delta_G[K_{\Omega_{r_1}}, K_{\Omega_{r_2}}, \cdots, K_{\Omega_{r_m}}],$$

where K_n denote the complete graph on n vertices.

In view of this, we give the following lemma.

Lemma 3.6. *Let G be a group and p be a prime number such that p^α divides $o(G)$ for some $\alpha \in \mathbb{N}$. Then K_{p^α} is an induced subgraph of the superpower graph $S(G)$.*

Proof. Let p be a prime number such that p^α divides order of G. By Sylow's theorem, G has a subgroup H of order p^α. Note that any two elements of H are adjacent in $S(G)$, since their orders are some powers of p. Thus, the elements of H induces a K_{p^α} in $S(G)$. □

3.2 Dominant Set

It is a well known fact that dominant vertices play an important role in characterization of graphs. In fact, if a graph contains a dominant vertex, then it is connected and diameter is at most two. Thus, it is interesting to find out the set of all dominant vertices in $S(G)$. Note that for any group G, the identity element, e, of G is a dominant vertex in $S(G)$. So, we say a graph $S(G)$ is *dominatable*, if it contains dominant vertices other than identity, that is $|\operatorname{dom}(S(G))| > 1$. In the following theorem, we find the number of dominant vertices in $S(G)$ for any finite Abelian non p-group G.

Theorem 3.7. *Let G be a finite Abelian non p-group of order n and* $\operatorname{dom}(S(G))$ *be the set of all dominant vertices in the superpower graph $S(G)$ of G. Then the size of dominant set,* $|\operatorname{dom}(S(G))| = t\phi(n_1)+1$*, where n_1 is the largest order of an element in the group and t is the number of distinct cyclic subgroups of order n_1 in G.*

Proof. Let G be a finite Abelian group of order n and let n_1 be the largest order of an element in G. For a divisor d of n, let $w_d = \{x \in G : o(x) = d\}$. From Lemma 3.4, vertices of $w_{n_1}(G) \cup w_1(G)$ are adjacent to every other vertex of $S(G)$ and hence $w_{n_1}(G) \cup w_1(G) \subseteq \operatorname{dom}(S(G))$. On the other hand, let $x \notin \operatorname{dom}(S(G))$. This gives that $1 < o(x) < n_1 \leq o(G)$. Further by Lemma 3.4, $o(x) \mid n_1$. Let y be an element in G such that $o(y) = \frac{n_1}{o(x)}$. It is trivial that such an element y exists always and x is not adjacent to y. So x is not an element of $w_{n_1}(G) \cup w_1(G)$. Hence $\operatorname{dom}(S(G) \subseteq w_{n_1}(G) \cup w_1(G)$. Thus $w_{n_1}(G) \cup w_1(G) = \operatorname{dom}(S(G))$. Also, $|w_{n_1}(G) \cup w_1(G)| = t\phi(n_1) + 1$, where t is the number of distinct cyclic subgroups of order n_1 in G. Therefore $|\operatorname{dom}(S(G))| = t\phi(n_1) + 1$. □

As mentioned in the proof of Theorem 3.7, $S(G)$ always contains a dominant vertex other than identity and hence we have the following corollary.

Corollary 3.8. *For any finite Abelian group G, the superpower graph $S(G)$ is dominatable.*

In view of the above theorem, we also define $H_{\overline{G}}$ as the induced subgraph induced by the subset $\overline{G} = G^* \setminus w_{n_1}(G) = G \setminus (\{e\} \cup w_{n_1}(G))$ of G.

3.3 Vertex Connectivity

In this section, we will study the connectivity of superpower graphs defined on finite Abelian groups. We first begin by giving the tight lower bound for the vertex connectivity of $S(G)$ for any finite Abelian group G which extend the results [38, Theorem 2.7] and [15, Theorems 2.11].

Theorem 3.9. *Let G be a finite Abelian group. Then $\kappa(S(G)) \geq t\phi(n_1) + 1$, where n_1 is the largest order of an element in G and t is the number of cyclic subgroups of order n_1 in G. Further, $\kappa(S(G)) = t\phi(n_1) + 1$ if and only if $G = \mathbb{Z}_{pq}^r$ or $\mathbb{Z}_{pq}^{r_1} \times \mathbb{Z}_p^{s_1}$ or $\mathbb{Z}_{pq}^{r_2} \times \mathbb{Z}_q^{s_2}$ where p and q are two distinct primes and $r, r_1, r_2 \geq 1$ and $s_1, s_2 \geq 0$ are integers.*

Proof. Let G be a finite Abelian group. By Theorem 3.7, to disconnect $S(G)$, we need to remove at least all vertices of $\text{dom}(S(G))$. This implies that $\kappa(S(G)) \geq |\text{dom}(S(G))| = t\phi(n_1) + 1$.

Next we prove the second part of the statement. Assume that $G = \mathbb{Z}_{pq}^r$ or $\mathbb{Z}_{pq}^{r_1} \times \mathbb{Z}_p^{s_1}$ or $\mathbb{Z}_{pq}^{r_2} \times \mathbb{Z}_q^{s_2}$ where p and q are two distinct primes and $r, r_1, r_2 \geq 1$ and $s_1, s_2 \geq 0$ are integers. Note that the largest order of elements in G is $n_1 = pq$. Let $\overline{G} = G \setminus (\{e\} \cup w_{n_1}(G))$. Since there is no path between elements of order p and q, the induced subgraph $H_{\overline{G}}$ is disconnected. Thus $\kappa(S(G)) = t\phi(n_1) + 1$.

Conversely, assume that $\kappa(S(G)) = t\phi(n_1)+1$. Suppose $o(G)$ be such that the largest order n_1 of elements in G satisfies $n_1 = p^{\alpha_1} q^{\beta_1}$ and α_1, β_1 are integers, $\alpha_1, \beta_1 \geq 1$ and either $\alpha_1 > 1$ or $\beta_1 > 1$. We will prove that $\kappa(S(G)) > t\phi(n_1) + 1$ by showing that $H_{\overline{G}}$ is connected. For $u, v \in V(H_{\overline{G}})$, take $w, x, z \in V(H_{\overline{G}})$ with $o(w)$ is the least prime divisor of $o(u)$ and $o(z)$ is the least prime divisor of $o(v)$ and $o(x)$ is the product of least prime divisors of $o(u)$ and $o(v)$. Then $P := \langle u, w, x, z, v \rangle$ is a path between u and v. Also, by the same way it can be shown that $o(G)$ has at most two prime divisors with maximum order of an element as $n_1 = pq$ in G. Otherwise, $H_{\overline{G}}$ is connected. Similarly, $H_{\overline{G}}$ is connected if $o(G)$ has at least three prime divisors. Now, only possible groups having two prime factors with largest order as $n_1 = pq$ will be of the form $\mathbb{Z}_{pq}^r$ or $\mathbb{Z}_{pq}^{r_1} \times \mathbb{Z}_p^{s_1}$ or $\mathbb{Z}_{pq}^{r_2} \times \mathbb{Z}_q^{s_2}$ where p and q are two distinct primes and $r, r_1, r_2 \geq 1$ and $s_1, s_2 \geq 0$ are integers. □

Let G be a finite Abelian group of order $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}, m \geq 2$. Assume that $G \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}$, where n_i for $1 \leq i \leq m$ are elementary divisors of G. As stated in Theorem 3.3, $n_j | n_i$ for each $j \geq i, n_i \in \mathbb{N}, 1 \leq j \leq k$ and $n_1 n_2 \cdots n_k = n$. Let $n_1 = p_1^{\beta_1} p_2^{\beta_2} \cdots p_s^{\beta_s}, 1 \leq s \leq m, \beta_i \geq 0, 1 \leq i \leq s$ be the prime decomposition of n_1. Let $a_0 = p_s^{\beta_s}, a_1 = \frac{n_1}{a_0}$ and $\pi(G) = \{a_0, a_1, a_2, \cdots, a_r\}$ be the set of all orders of elements in G.

In the following theorem, we find the tight upper bound for the vertex connectivity of $S(G)$ using the notations defined above.

Theorem 3.10. *Let G be a finite Abelian group of order $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}, m \geq 2$. Then there exists a minimal separating set T of $S(G)$ with*

$$|T| = \sum_{(a_i | a_1 \quad or \quad a_1 | a_i, a_i \neq a_1)} t_i \phi(a_i),$$

where t_i is the number of cyclic subgroups of order a_i in G. Also, this bound is tight.

Proof. Consider the graph $S(G)$ and define a set T in $S(G)$ as follows:

$$T = \{w_{a_i}(G) | \quad a_i | a_1 \quad or \quad a_1 | a_i, \quad 2 \leq i \leq r\},$$

Clearly, T is a separating set of $S(G)$, since there is no path between any vertices of the cliques $w_{a_0}(G)$ and $w_{a_1}(G)$. Let A and B are two connected component of $S(G) \setminus T$ such that $w_{a_0}(G) \in A$ and $w_{a_1}(G) \in B$. Now, we prove that this set is a minimal separating set by showing that for any non empty subset $T^\dagger$ of T, there is a path, connecting $u \in w_{a_0}(G)$ and $v \in w_{a_1}(G)$ in $S(G)\setminus T^\dagger$. Without loss of generality assume that $T \setminus T^\dagger = \{x\}$ with $x \in w_{a_r}(G)$. Since either $a_r|a_1$ or $a_1|a_r$, there exists a path $P_1(u, x)$, connecting u and x in $A \cup w_{a_r}(G)$. Similarly, let $y \in B$ such that $o(y) = a_r p_s^{\beta_s}$, then there exists a path $P_2(y, v)$, connecting y to v in B. Now, consider the path $P =: \langle P_1(u, x), x, y, P_2(y, v)\rangle$ which connect u to v in $S(G) \setminus T^\dagger$. Thus, T is a minimal separating set of $S(G)$. If t_i, number of cyclic subgroups of order a_i in G, $1 \leq i \leq r$. Then

$$|T| = \sum_{(a_i|a_1 \quad or \quad a_1|a_i, a_i \neq a_1)} t_i \phi(a_i).$$

Now, we show that the obtained bound is in fact tight. That is, the bound attained above serves as the $\kappa(S(G))$ for the groups $G \simeq \mathbb{Z}_{pq}$, $G \simeq \mathbb{Z}_{pqr}$ or $G \simeq \mathbb{Z}_{pqr} \times \mathbb{Z}_r \times \mathbb{Z}_r \times \cdots \times \mathbb{Z}_r$, where $p < q < r$ are primes. When $G \simeq \mathbb{Z}_{pq}$, then by Theorem 3.9, we have that $\kappa(S(G)) = \phi(1) + \phi(pq) = |T|$. Let $G \simeq \mathbb{Z}_{pqr} \times \mathbb{Z}_r \times \cdots \times \mathbb{Z}_r$. By Theorem 3.9, $H_{\overline{G}}$ is connected. To get the vertex connectivity of $S(G)$, it is enough to find the vertex connectivity of $H_{\overline{G}}$, since if $\overline{T}$ is a minimum separating set of $H_{\overline{G}}$, then $T = \overline{T} \cup \text{dom}(S(G))$ is a minimum separating set of $S(G)$. Notice that the equivalence classes of $\overline{G}$ with respect to $\sim$ are precisely $w_p(G)$, $w_q(G)$, $w_r(G)$, $w_{pq}(G)$, $w_{pr}(G)$, $w_{qr}(G)$ and each of these equivalence classes is a clique in $H_{\overline{G}}$. Thus,

$$H_{\overline{G}} = \Delta_{\overline{G}}[K_{w_p(G)}, K_{w_q(G)}, K_{w_r(G)}, K_{w_{pq}(G)}, K_{w_{pr}(G)}, K_{w_{qr}(G)}],$$

as given in Figure 3.1. It is clear that deletion of any one of the cliques in $H_{\overline{G}}$ does not disconnect $H_{\overline{G}}$. However, deletion of any two cliques in $H_{\overline{G}}$ that are not adjacent can disconnect the same. This imply that a minimal separating set of $H_{\overline{G}}$

is precisely the union of any two non adjacent cliques. Also, we have the inequality $|w_p(G)| < |w_q(G)| < |w_r(G)|$ and $|w_{pq}(G)| < |w_{pr}(G)| < |w_{qr}(G)|$. Heuristically, we first add the smallest clique namely $w_p(G)$ into the minimum separating set $\overline{T}$, we find that next best possible non adjacent clique having minimum cardinality is $w_q(G)$. Thus, $\overline{T} = w_p(G) \cup w_q(G)$ is a minimum separating set of $H_{\overline{G}}$ implying that $T = w_p(G) \cup w_q(G) \cup w_{pqr}(G) \cup w_1(G)$ is a minimum separating set of $S(G)$ having the cardinality $\phi(pqr) + \phi(p) + \phi(q) + \phi(1)$.

A similar proof holds when $G \simeq \mathbb{Z}_{pqr}$. □

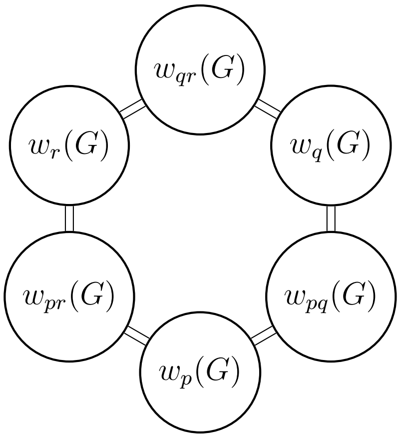

FIGURE 3.1: Connectivity in $H_{\overline{G}}$.

3.4 Hamiltonian-Like Properties

In [18], it was proved that the power graph $\mathscr{P}(G)$ of any cyclic group of order at least three is Hamiltonian [see Theorem 4.13]. In [15], it was proved that $S(G) = \mathscr{P}(G)$ if and only if G is a finite cyclic group. Thus $S(G)$ is Hamiltonian for any cyclic group of order at least three. Natural question arises that can we extend this result for finite Abelian groups? Unfortunately, the same question has got a negative answer in the case of $\mathscr{P}(G)$. For example, $\mathscr{P}(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is not Hamiltonian. However, we can extend this result to $S(G)$ and the same is proved in the following theorem.

Theorem 3.11. *For any finite Abelian group G of order at least three, the superpower graph $S(G)$ is Hamiltonian.*

Proof. Let G be a finite Abelian group of order n and n_1 be the largest order of an element in the group G. Let $\{d_1, d_2, \ldots, d_\ell, n_1\}$ be the divisors of n such that $1 < d_1 < \cdots < d_\ell < n_1 \leq n$. By Lemma 3.4, $d_i | n_1 \ \forall \ i(1 \leq i \leq \ell)$. For any divisor d of n, the induced subgraph H_d induced by the vertex subset $w_d(G) = \{x \in G : o(x) = d\}$ is a clique in $S(G)$. Now let us see a Hamiltonian cycle in $S(G)$ as follows:

Start from the vertex $v_1 \in \text{dom}(S(G))$. From v_1, go to any vertex of the clique H_{d_1} and traverse all vertices of H_{d_1}. Now we have a Hamilton path containing all the vertices in $H_{d_1} \cup \{v_1\}$. Note that the terminal vertex of this Hamiltonian path is adjacent to a vertex $v_2 \in \text{dom}(S(G))$ and $v_2 \neq v_1$. From v_2, go to any vertex of the clique H_{d_2} and traverse all vertices of H_{d_2}. Now the terminating vertex of the resulting Hamiltonian path is adjacent to a vertex vertex $v_3 \in \text{dom}(S(G))$ and $v_3 \notin \{v_1, v_2\}$. Repeat this until all the cliques $H_{d_i}(G), 1 \leq i \leq \ell$ are covered. Since $\ell < |\text{dom}(S(G))|$, there are sufficient number of vertices in $\text{dom}(S(G))$ to connect all disjoint cliques. Finally, complete the cycle by joining all the uncovered vertices of $\text{dom}(S(G))$ by path to v_1. The entire process of identifying a Hamiltonian cycle is given in Figure 3.2. □

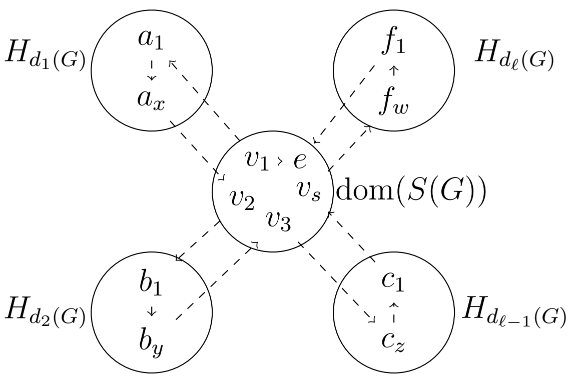

FIGURE 3.2: Hamiltonian cycle in $S(G)$

Recall that a graph Γ is called *1-Hamiltonian* if it is *Hamiltonian* and all of its 1-vertex-deleted subgraphs are *Hamiltonian.* Now, we prove that the superpower graph $S(G)$ is 1-Hamiltonian.

Corollary 3.12. *For any finite Abelian group G with $o(G) \geq 4$, the superpower graph $S(G)$ is 1-Hamiltonian.*

Proof. Let n_1 be the largest order of an element in the group G and let $g \in G$. If $o(g) = d < n_1$, then g is a vertex in the clique induced by w_d for the divisor d of $o(G)$. Further $H_d \setminus \{g\}$ remains as a clique and so it has a spanning path whose initial and terminal vertices can be joined by two different vertices of $\text{dom}(S(G))$. Now, the proof can be completed as in the case of Theorem 3.11.

If $o(g) = n_1$, then $g \in \text{dom}(S(G))$. As seen in the proof of Theorem 3.11, there are sufficient number of vertices in $\text{dom}(S(G)) \setminus \{g\}$ to connect all the disjoint cliques corresponding to all proper divisors of $o(G)$. Hence the required Hamiltonian cycle can be obtained as in Theorem 3.11. Thus, $S(G) \setminus \{g\}$ contains a Hamiltonian cycle implying that $S(G)$ is 1-Hamiltonian. □

Corollary 3.13. *For any finite Abelian group G of order at least three, the superpower graph $S(G)$ is pancyclic.*

Proof. Let $o(G) = n \geq 3$ and n_1 be the largest order of an element in the group G. Let $\{d_1, d_2, \ldots, d_\ell, \}$ be the set of all nontrivial divisors of n_1 with $d_1 < d_2 < \cdots < d_\ell < n_1 \leq n$. By Theorem 3.7, the subgraph induced by $\text{dom}(S(G))$ is a clique of size $t\phi(n_1) + 1$ and so we have cycles of lengths from 3 to $t\phi(n_1) + 1$ in $S(G)$. Also, from Theorem 3.11 we know that $S(G)$ contains a cycle of length n.

For any $g_1 \in V(S(G))$, by Corollary 3.12, $S(G) \setminus \{g_1\}$ is Hamiltonian and thus $S(G)$ contains a cycle of length $n - 1$. Note that, in the proof of Corollary 3.12, we see that as long as we keep choosing a vertex $g \in w_{d_1} \subset V(G) \setminus \text{dom}(S(G))$, obtaining a cycle containing remaining vertices is immediate. Choose $g_2 \in w_{d_1}(G)$ (if exists), otherwise choose $\{g_2\} \in w_{d_2}(G)$ for some non-trivial divisor $d_2 \neq d_1$ of n and we immediately get that $S(G) \setminus \{g_1, g_2\}$ is Hamiltonian. So $S(G)$ contains a cycle of length $n-2$. Recursively deleting the vertices of w_{d_i} for each i, $1 \leq i \leq l$, we can get cycles of length $n-2$ to $t\phi(n_1)+2$. Thus $S(G)$ contains cycles of all length ℓ for $3 \leq \ell \leq n$ and hence $S(G)$ is pancyclic. □

It is not always true that there exists a Hamiltonian path between any pair of vertices in a graph even if it a Hamiltonian. However, this happens in the case of the superpower graph $S(G)$ of any finite Abelian group G and hence we have the following result.

Corollary 3.14. *For any finite Abelian group G, the superpower graph $S(G)$ is Hamiltonian connected.*

Proof. Let $u, v \in V(S(G))$ be two distinct vertices in $S(G)$. Without loss of generality, one can take $u = v_1 \in w_{d_1}(G)$ and $v = v_2 \in w_{d_2}(G)$ where d_1 and d_2 are two non-trivial distinct divisors of $o(G)$. Start from the vertex v_1 and traverse along the spanning path in $H_{d_1}(G)$ and join it with a vertex v_3 of $\text{dom}(S(G))$. From v_3 go to any vertex of $H_{d_3}(G)$ and repeat the process until all vertices of the cliques $H_{d_i}(G) \cup \text{dom}(S(G)), 3 \leq i \leq \ell$ belongs to the path such that $v_\ell \in \text{dom}(S(G))$ is the last vertex of this path. Now, join v_ℓ to a vertex $x \neq v_2$ of $H_{d_{2}(G)}$. Upon completing the path from x to v_2 in $H_{d_2}(G)$, we obtain the required Hamiltonian path between u and v in $S(G)$. □

Corollary 3.15. *For any finite Abelian group G, the superpower graph $S(G)$ is pan connected.*

Proof. Let $u, v \in V(S(G))$ be two distinct non-adjacent vertices in $S(G)$. Then there always exists a path of every length from 2 to n. Now the required path can be obtained by inserting the vertices from $H_{d_i} \cup \text{dom}(S(G)), 1 \leq i \leq \ell$ in such a way that any two vertices of cliques H_{d_i}, H_{d_j} can be joined through a vertex of $\text{dom}(S(G))$. □

Now a natural question arises that what will be the effect on Hamiltonian of the graph $S(G)$, if we remove all dominant vertices from it? In the following theorem, we are giving the answer to this question.

Theorem 3.16. *Let G be a finite Abelian non p-group and n_1 be the largest order of an element in the group G. Let $w_{n_1}(G) = \{x \in G : o(x) = n_1\}$. Then the induced subgraph $H_{\overline{G}}$ of the superpower graph $S(G)$ induced by $\overline{G} = G \setminus (\{e\} \cup w_{n_1}(G))$ is Hamiltonian if and only if n_1 is not a product of two distinct primes.*

Proof. Assume that $H_{\overline{G}}$ is Hamiltonian. If $n_1 = pq$ for two distinct primes p and q, then $H_{\overline{G}} = H_p \cup H_q$ is disconnected as there is no path connecting the vertices of H_p and H_q, a contradiction.

Conversely, assume that n_1 is not a product of two distinct primes. i.e., $n_1 = p_1^{\beta_1} p_2^{\beta_2} \cdots p_m^{\beta_m}$ and $\beta_i \in \mathbb{N}, m \geq 2$ and $p_1 < \cdots < p_m$ are distinct primes. Since G is not a p-group, $m \geq 2$. By the assumption on n_1, we have either $\beta_1 > 1$ or $\beta_2 > 1$ or $m \geq 3$. Since G is not a p-group and by the assumption on n_1, the largest order of an element in the set $\overline{G} = G \setminus (\{e\} \cup w_{n_1}(G))$ will be $\dfrac{n_1}{p_1}(= \overline{n_1}$, say).

Let $w_{\overline{n_1}}(\overline{G}) = \{b_1, \cdots, b_s\}$ be the set of all elements of order $\overline{n_1}$. Let $\{\overline{d_1}, \ldots, \overline{d_\ell}\}$ be the set of all non trivial divisors of $\overline{n_1}$ with $1 < \overline{d_1} < \overline{d_2} < \cdots < \overline{d_\ell} < \overline{n_1}$. Let $w_{\overline{d_i}}(\overline{G})$ be the set of all elements of order $\overline{d_i}$ in $\overline{G}$ and let $H_{\overline{d_i}}$ be the subgraph induced by $w_{\overline{d_i}}(\overline{G})$. For each $i, 1 \leq i \leq \ell$, let $P_{H_{\overline{d_i}}} = \langle v_i, u_i \cdots, x_i \rangle$ be the Hamiltonian path in $H_{\overline{d_i}}$. Then the induced subgraph H of $H_{\overline{G}}$ on the vertices of $\bigcup\limits_{1\leq i\leq \ell} w_{\overline{d_i}} \cup w_{\overline{n_1}}$ is Hamiltonian, since $C = \langle b_1, P_{H_{\overline{d_1}}}, b_2, P_{H_{\overline{d_2}}}, b_3, \cdots, b_\ell, P_{H_{\overline{d_\ell}}}, b_{\ell+1}, \cdots, b_s\rangle$ is a Hamiltonian cycle in H.

It remains for us to include remaining vertices from $H_{\overline{G}} \setminus H$ into C appropriately to get Hamiltonian cycle in $H_{\overline{G}}$. Based on the condition on $\overline{n_1}$, we observe that the only possible subsets of different orders in $\overline{G} \setminus \{\bigcup\limits_{1\leq i\leq \ell} w_{\overline{d_i}}(\overline{G}) \cup w_{\overline{n_1}}(\overline{G})\}$ are of the form $w_{p_1{}^{\beta_1}}(\overline{G})$ and $w_{p_1^{\beta_1}r}(\overline{G})$, where $r = \overline{d_i}$, for some $i, 1 < i \leq \ell$. If cliques $H_{p_1^{\beta_1}}$, $H_{p_1{}^{\beta_1}\overline{d_j}}$ exists in $H_{\overline{G}} \setminus H$, then the spanning paths $P(v_1^{'}, u_1^{'})$ of $H_{p_1^{\alpha_1}}$ and $P(v_j^{'}, u_j^{'})$ for $1 < j \leq \ell$ of $H_{p_1{}^{\alpha_1}d_j}$ are inserted into the spanning path of $H_{\overline{d_1}}$ and $H_{\overline{d_j}}$ respectively, as shown in Figure 3.3. That is, the required Hamiltonian cycle $C_{H_{\overline{G}}}$ in $H_{\overline{G}}$ is given by $\langle b_1, P_1, b_2, P_2, \cdots, b_\ell, P_l, b_{\ell+1}, \cdots, b_s\rangle$, where $P_j = \langle v_j, v_j^{'}, P(v_j^{'}, u_j^{'}), u_j, P(u_j, x_j)\rangle$(if it exists). □

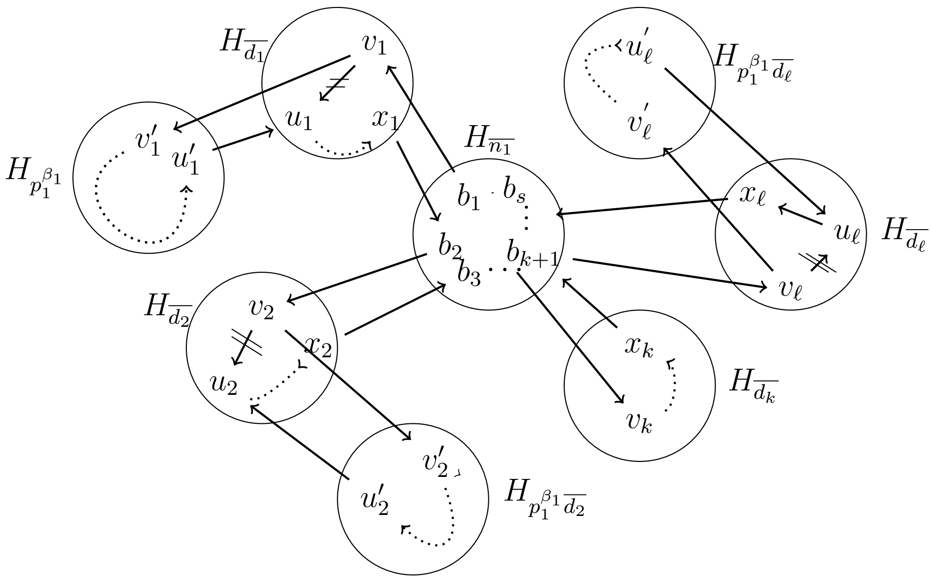

FIGURE 3.3: Hamiltonian cycle in $H_{\overline{G}}$

Corollary 3.17. *For any positive integer* n, $H_{\overline{\mathbb{Z}_n}}$ *is Hamiltonian if and only if* n *is neither a power of prime number nor a product of two primes.*

In [15], it was proved that $S(\mathbb{Z}_n) = \mathscr{P}(\mathbb{Z}_n)$ for every positive integer n. Now, observe that the $\mathrm{dom}(S(\mathbb{Z}_n)) = \mathrm{dom}(\mathscr{P}(\mathbb{Z}_n))$, since the dominant vertices of both these graphs are the only generators of $\mathbb{Z}_n$. This observation gives us the following interesting property of $P(\overline{\mathbb{Z}_n})$ from $H_{\overline{\mathbb{Z}_n}}$.

Corollary 3.18. *For any positive integer n, $H_{\overline{\mathbb{Z}_n}} = \mathscr{P}(\overline{\mathbb{Z}_n})$ is Hamiltonian if and only if n is neither a power of prime number nor a product of two distinct primes, where $P(\overline{\mathbb{Z}_n})$ is induced subgraph of $\mathscr{P}(\mathbb{Z}_n)$ by removing $\mathrm{dom}(\mathscr{P}(\mathbb{Z}_n))$ from it.*

Corollary 3.19. *Let G be a finite Abelian non p-group such that maximum order of an element is not a product of two primes. Then $H_{\overline{G}}$ is 2-connected.*

In view of the above corollary, we also give an improved lower bound for vertex connectivity of $S(G)$ combining all the results proved so far.

Corollary 3.20. *Let G be an finite Abelian non p-group of order n. If largest order n_1 of an element in G is not a product of two distinct primes, then*

$$t\phi(n_1) + 3 \leq \kappa(S(G)) \leq |T|,$$

where t denotes the number of cyclic subgroups of order n_1 in G and T is the minimal separating set of $S(G)$.

Proof. Note that the required lower bound follows from Corollary 3.19 and Theorem 3.9 while the upper bound follows from Theorem 3.10. □

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