Chapter 2

Complement of Conjugate Graph of Finite Groups

It is well known that multi-partite graphs have its own significance in the field of graph theory. Joseph Varghese [1] has proved the necessary and sufficient condition for a complete multi-partite graph to be Hamiltonian. As an extension of his work we pose the following questions:

1. Does there exists a condition on a group of n elements, such that the graph constructed from this group will always be a complete multipartite graph containing a Hamiltonian cycle?
2. Under what conditions the graph obtained in this manner will be Eulerian?
3. Can we characterize the group G and an underlying relation defined on G so as to be associated with the given graph?

In this chapter, we answer all these questions in affirmative by proving that the graphs generated with the group-theoretic concept of conjugacy relation on a finite group, referred to as the complement of a conjugate graph $\overline{C(G)}$. We also explore the other variations of the Hamiltonian property, namely, 1-Hamiltonian, pancyclicity, etc. Further, we present several structural characterizations of the considered graph class.

The structure of this chapter is as follows: In Section 2.1, we present various structural characterizations of $\overline{C(G)}$. In Section 2.2, we answer the questions posed above on the Hamiltonian-like properties of $\overline{C(G)}$.

2.1 Structural Characterizations

This section characterizes the structure of $\overline{C(G)}$ by using the class equation of the underlying group G.

Theorem 2.1. *If G is a group with the number of conjugacy class, $K(G) = m$, then $\overline{C(G)}$ is a complete $m-$partite graph.*

Proof. Let $n = n_1 + n_2 + \cdots + n_m$ be the class equation of the group G. According to the definition of $\overline{C(G)}$, each conjugacy class partitions the vertex set into an independent set, that is, since for any $u, v \in V(\overline{C(G)})$ are adjacent if and only if they belong to different conjugacy classes in the group G, $V(\overline{C(G)}) = V_1 \cup V_2 \cup \cdots \cup V_m$ forms a partition, where each V_i is a conjugacy class with $|V_i| = n_i$, $1 \leq i \leq m$. Hence $\overline{C(G)}$ is a complete $m-$partite graph $K_{n_1,n_2,\ldots,n_m}$.

Further, the identity element $e \in G$ forms a partite set with one element and will be adjacent to all other vertices in $\overline{C(G)}$. Similarly, any $u \in Z(G)$, forms a partite set of one element and will be adjacent to every vertex in $\overline{C(G)}$, since $cl(u) = \{u\} \Leftrightarrow u \in Z(G)$. □

Corollary 2.2. $|V_i|$ *divides* $|V(\overline{C(G)})|$, *for each* i, $1 \leq i \leq m$.

Proof. Since $|V_i|$ is the cardinality of conjugacy class of the group G and we know that cardinality of the conjugacy class of a group divides order of the group. Hence the result follows. □

Remark 2.3. If G is a group of odd order, then $\overline{C(G)}$ is a graph with odd number of partite sets and each partite set contains odd number of vertices.

Remark 2.4. Observe that $\overline{C(G)}$ is a connected graph with diameter ≤ 2, since the identity element e of G is adjacent to every vertex of $\overline{C(G)}$.

In view of the above theorem, for the rest of the chapter, we have $\overline{C(G)} = K_{n_1,n_2,\ldots,n_m}$ with the partition $V(\overline{C(G)}) = V_1 \cup V_2 \cdots \cup V_m$ where each V_i is the conjugacy class of the group G with n_i elements, $1 \leq i \leq m$. We also have that $1 = n_1 \leq n_2 \leq \cdots \leq n_m \leq \frac{n}{2}$. Further, we can specify the size of the partite sets for certain collection of groups.

Corollary 2.5. *$\overline{C(G)}$ is a complete-(q^2+q-1)-partite graph of the form $K_q + K_{\underbrace{q,q,\ldots,q}_{q^2-1\ times}}$, for any non-Abelian group G of order q^3, where q is a prime number.*

Proof. The class equation of any non-Abelian group G of order q^3 is given by

$$o(G) = q^3 = \underbrace{1+1+\cdots+1}_{q \text{ times}} + \underbrace{q+q+\cdots+q}_{q^2-1 \text{ times}}.$$

Thus, $\overline{C(G)} = K_{\underbrace{1,1,\ldots,1}_{q \text{ times}},\underbrace{q,q,\ldots,q}_{q^2-1 \text{ times}}}$. Hence $\overline{C(G)}$ can be written as a join of two graphs K_q and $K_{\underbrace{q,q,\ldots,q}_{q^2-1 \text{ times}}}$, that is, $\overline{C(G)} = K_q + K_{\underbrace{q,q,\ldots,q}_{q^2-1 \text{ times}}}$. □

Corollary 2.6. $\overline{\mathrm{C(D_{2n})}}$ *is a complete m-partite graph for the Dihedral group D_{2n} of order $2n$, where*

$$m = \begin{cases} \frac{n}{2}+3, & \text{when } n \text{ is even.} \\ \frac{n+3}{2}, & \text{when } n \text{ is odd.} \end{cases}$$

Proof. Since the class equation of Dihedral group D_{2n} is given by

$$2n = \begin{cases} 1+1+\underbrace{2+2+\cdots+2}_{\frac{n}{2}-1 \text{ times}}+\frac{n}{2}+\frac{n}{2}, & \textit{when } n \textit{ is even} \\ 1+\underbrace{2+2+\cdots+2}_{\frac{n-1}{2} \text{ times}}+n, & \textit{when } n \textit{ is odd.} \end{cases}$$

Hence the result follows. □

Theorem 2.7. *A group G is an Abelian group of order $n \Leftrightarrow \overline{C(G)}$ is a complete graph K_n.*

Proof. If G is an Abelian group of order n, then $K(G) = n$ and the result follows from Theorem 2.1. Conversely, suppose that $\overline{C(G)}$ is a complete graph on n vertices, then any two vertices $u \neq v$ in $\overline{C(G)}$ are adjacent. This implies that no two distinct members in the group G are conjugate, that is, $cl(u) = \{u\}, \forall u \in G$. Hence for all $u \in G$, $u \in Z(G)$ implying G is an Abelian group. □

Corollary 2.8. $o(Z(G)) = r \geq 1 \Leftrightarrow$ *the induced subgraph on these $r-$elements form a complete graph.*

Corollary 2.9. *For any group G, the clique number of $\overline{C(G)}$ is the number of conjugate classes in G, that is, $\omega(\overline{C(G)}) = K(G)$.*

The next question one would like to ask is what about the converse of Theorem 2.1, that is, can every complete multi-partite graph be realized as a $\overline{C(G)}$, for some group G? In the following, we show that the answer is not in affirmative.

Example 2.1. *Now we present a complete multi-partite graph which cannot be associated to any group G. Consider a complete tripartite graph Γ on 5 vertices with vertex set $V = \{a,b,c,d,e\}$, $V = V_1 \cup V_2 \cup V_3$ with $V_1 = \{a,b\}, V_2 = \{c,d\}$ and*

$V_3 = \{e\}$ as shown in Fig. 2.1. Then, Γ cannot be associated to any group G, since it is well known that any group G of order 5 is Abelian and by Theorem 2.7, $\overline{C(G)}$ must be complete while the considered graph Γ is not complete.

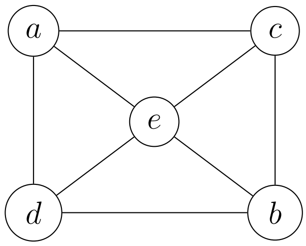

FIGURE 2.1: $\Gamma = K_{2,2,1}$ that cannot be associated to any group G.

Theorem 2.10. *For any group G, number of edges in $\overline{C(G)}$ is $\frac{1}{2}\sum_{i=1}^{m} n_i(n-n_i)$.*

Proof. If $u \in V_i$, $1 \le i \le m$, then $deg_{\overline{C(G)}}(u) = n - n_i$. It follows that $\sum_{u\in V_i} deg(u) = n_i(n-n_i)$. This gives, $\sum_{u\in G} deg_{\overline{C(G)}}(u) = \sum_{i=1}^{m} n_i(n-n_i)$. Since each edge is incident to exactly two vertices, we have total number of edges in $\overline{C(G)}$ is $\frac{1}{2}\sum_{i=1}^{m} n_i(n-n_i)$ □

In the remaining section, we characterize $\overline{C(G)}$ based on the group's structural properties.

Theorem 2.11. *(i) $\overline{C(G)}$ is a complete bipartite graph $\Leftrightarrow G \cong \mathbb{Z}_2$.*

(ii) $\overline{C(G)}$ is a complete tripartite graph $\Leftrightarrow G \cong \mathbb{Z}_3$ or S_3.

(iii) $\overline{C(G)}$ is a tree $\Leftrightarrow G \cong \mathbb{Z}_2$ or $\{e\}$.

Proof. (i) If $\overline{C(G)}$ is a complete bipartite graph, then G is a group with $K(G) = 2$. Up to isomorphism, $\mathbb{Z}_2$ is the only group with two conjugacy classes, hence $G \cong \mathbb{Z}_2$. Conversely, if $G \cong \mathbb{Z}_2$, then associated graph will be $K_{1,1}$, since the class equation of $\mathbb{Z}_2$ is given by $o(\mathbb{Z}_2) = 2 = 1 + 1$.

(ii) If $\overline{C(G)}$ is a complete tripartite graph, then G is a group having $K(G) = 3$. Up to isomorphism, $\mathbb{Z}_3$ and S_3 are the only groups with three conjugacy classes, hence $G \cong \mathbb{Z}_3$ or S_3. Conversely, if $G \cong \mathbb{Z}_3$ or S_3 then the class equations of these groups are given by $o(\mathbb{Z}_3) = 3 = 1+1+1$ and $o(S_3) = 6 = 1+2+3$ and the associated $\overline{C(G)}$ is $K_{1,1,1}$ or $K_{1,2,3}$ respectively.

(iii) If $\overline{C(G)}$ is a tree, then it does not contain any cycle which is possible only when G is a group with at most two conjugacy classes, hence $G \cong \mathbb{Z}_2$ or $G \cong \{e\}$. Converse trivially holds true.

Hence the theorem. □

In the next theorem, we will use the Kuratowski's Theorem [41] which says, "A graph Γ is planar if and only if it has no subgraph homeomorphic to K_5 or $K_{3,3}$".

Theorem 2.12. *$\overline{C(G)}$ is non planar, for any non-Abelian group G.*

Proof. G is *non-Abelian* implies that $K(G) \geq 3$. If $K(G) = 3$, then $G \cong S_3$ and so $\overline{C(G)}$ has a subgraph which is homeomorphic to $K_{3,3}$, as shown in Fig. 2.2.

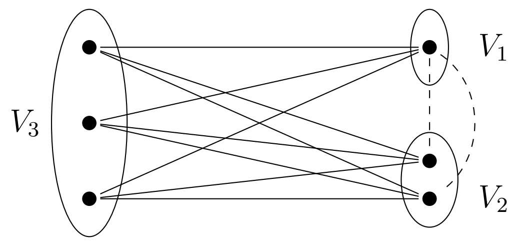

FIGURE 2.2: The bold edges of $\overline{\mathrm{C(S_3)}}$ forms a $K_{3,3}$

Let $K(G) = 4$ and $n = n_1 + n_2 + n_3 + n_4$ be the class equation of the group G. Clearly $n \geq 10$, since G is non-Abelian. Note that, $n_1 = 1$ and there exists some j, $1 < j \leq 4$, such that $n_j > 3$. Thus by keeping the partite set of n_j elements on one hand and rest of the elements on the other hand we can always construct a subgraph which is homeomorphic to $K_{3,3}$ similar to Fig. 2. Hence $\overline{C(G)}$ is non planar, for any non-Abelian group G.

If $K(G) \geq 5$, then we can see $\overline{C(G)}$ has a subgraph which is homeomorphic to K_5 by picking one vertex from five of the different conjugacy classes. □

Theorem 2.13. *$\overline{C(G)}$ is planar $\Leftrightarrow G \cong \mathbb{Z}_2,\ \mathbb{Z}_3,\ \mathbb{Z}_4,\ \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\{e\}$.*

Proof. If $\overline{C(G)}$ is a planar graph, then the contrapositive of Theorem 2.12 implies that G is an Abelian group. Further, $o(G) \leq 4$, since for any group G with $o(G) \geq 5$, $\overline{C(G)}$ has a subgraph homeomorphic to K_5 and cannot be planar by Kuratowski's Theorem. It is well known that the only groups of order less than or equal to 4 are $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\{e\}$. Hence the result follows. By inspection, the converse holds. □

Theorem 2.14. *Two graphs $\overline{\mathrm{C(G_1)}}$ and $\overline{\mathrm{C(G_2)}}$ associated with two groups G_1 and G_2 are isomorphic if and only if there is a one to one correspondence between the conjugacy classes of the groups G_1 and G_2. In particular, $\overline{\mathrm{C(G_1)}} \cong \overline{\mathrm{C(G_2)}} \Leftrightarrow$ Class equations of G_1 and G_2 are identically equal.*

Proof. Let $f : \overline{\mathrm{C(G_1)}} \longrightarrow \overline{\mathrm{C(G_2)}}$ be an *isomorphism preserving adjacency.* If $x \in G_1$, then there exists a conjugacy class, say $cl(a)$, of cardinality $n_i, 1 \leq i \leq m$, such that $x \in cl(a)$. This implies that $x \in V_i$ with $|V_i| = n_i$ in the graph $\overline{\mathrm{C(G_1)}}$, Also $deg_{\overline{C(G)}}(x) = n - n_i$, this implies $deg_{\overline{C(G)}}(f(x)) = n - n_i$, Consequently, $f(x) \in W_i$, where $W_i = f(V_i)$ is a partite set in the graph $\overline{\mathrm{C(G_2)}}$ of cardinality n_i. Thus, corresponding to a conjugacy class with cardinality n_i, for every $1 \leq i \leq m$, in G_1, there is a *conjugacy class* in the group G_2 with same cardinality. Thus the result follows. By simple verification, converse holds. □

2.2 Hamiltonian-Like Properties

In this section, we will study the properties such as *Eulerian, Hamiltonian, randomly Hamiltonian, 1-Hamiltonian, pancyclic* of the complete multi-partite graph, $\overline{C(G)}$. It is well known that not every multi-partite graph contains a Hamiltonian cycle. Due to this, Joseph Varghese[1] gave the necessary and sufficient condition for a complete multi-partite graph to be Hamiltonian.

Lemma 2.2.1. [1] A complete multi-partite graph of at least three vertices is Hamiltonian if and only if the cardinality of no partite set is larger than the sum of the cardinalities of all the other partite sets.

In the following theorem, we use the above lemma to show the existence of Hamiltonian cycle in $\overline{C(G)}$ for any group G.

Theorem 2.15. *$\overline{C(G)}$ is Hamiltonian, for any group G of order $n \geq 3$.*

Proof. In $\overline{C(G)}$, since n_i divides n for each i, $1 \leq i \leq m$, we have the largest possible value of n_i that divides n is $\frac{n}{2}$. Hence, we have that the cardinality of no partite set is larger than the sum of the cardinality of all other partite sets. By Lemma 2.2.1, $\overline{C(G)}$ is a Hamiltonian graph. □

Above theorem assures the existence of a Hamiltonian cycle in $\overline{C(G)}$, for any group G, of order at least three elements. How should one proceed to find such a Hamiltonian cycle, that is after choosing an initial vertex can we get to the next appropriate vertex in the sequence and hence turn it into a Hamiltonian cycle? Chartrand and Kronk [42] proved that a graph Γ on n vertices, $n \geq 3$ is randomly Hamiltonian if and only if it is one of the graph C_n, K_n for $n = 2q$, or $K_{q,q}$. Due to this, we have the following result.

Theorem 2.16. *$\overline{C(G)}$ is randomly Hamiltonian $\Leftrightarrow$ G is an Abelian group of order $n \geq 3$.*

Theorem 2.17. *Suppose G is a group of order $n \geq 3$. $\overline{C(G)}$ is 1-Hamiltonian $\Leftrightarrow$ no conjugacy class of G has $\frac{n}{2}$ elements.*

Proof. If there exists a conjugacy class say V_m, of $\frac{n}{2}$ elements, then n must be an even integer. In $\overline{C(G)} \setminus \{e\}$, we have $\dfrac{n}{2} = |V_m| \nleq \sum_{i=1}^{m-1}|V_i| = (n-1) - \dfrac{n}{2} = \dfrac{n}{2} - 1$, which contradicts Lemma 2.2.1 and hence $\overline{C(G)} \setminus \{e\}$ is not Hamiltonian and hence $\overline{C(G)}$ is not 1-Hamiltonian. Conversely, let us assume, cardinality of each conjugacy class is less than $\frac{n}{2}$. For any $g \in G$, consider the graph $\overline{C(G)} \setminus \{g\}$, we have $|V_i| \leq \dfrac{n}{2} - 1 \leq n - \dfrac{n}{2}$, for each i, $1 \leq i \leq m$. Hence by Lemma 2.2.1, $\overline{C(G)} \setminus \{g\}$ is Hamiltonian implying $\overline{C(G)}$ is 1-Hamiltonian. □

Corollary 2.18. *$\overline{C(G)}$ is 1-Hamiltonian, for any group G of odd order $n > 3$.*

Proof. Cardinality of each conjugacy class is less than $\frac{n}{2}$, since n is an odd integer. Result follows from Theorem 2.17. □

Theorem 2.19. *Let G be a group with conjugacy class, say V_m of $\frac{n}{2}$ elements. $\overline{C(G)} \setminus \{g\}$ is Hamiltonian if and only if $g \in V_m$.*

Proof. If $g \notin V_m$, then $\overline{C(G)} \setminus \{g\}$ is not Hamiltonian, since $|V_m| = \frac{n}{2} \nleq (n-1) - \frac{n}{2} = \frac{n}{2} - 1$. Conversely, if $g \in V_m$, then conclusion follows from Lemma 2.2.1. □

Example 2.2. *In this example, we show why the above theorem does not hold if $g \notin V_m$. Consider the group S_3, let $V_1 = \{e\}, V_2 = \{a, b\}$ and $V_3 = \{c, d, f\}$ be the partition of the vertex set of $\overline{\mathrm{C(S_3)}}$. By simple verification, we can find that $(\overline{\mathrm{C(S_3)}}) \setminus \{g\}$ is not Hamiltonian $\forall$ $g \in V_1 \cup V_2$.*

Theorem 2.20. *If G is a group of order $n \geq 3$ with no conjugacy class having $\frac{n}{2}$ elements, then for any edge $\varepsilon \in E(\overline{C(G)})$, there exists a Hamilton cycle H in $\overline{C(G)}$ containing the edge ε.*

Proof. If $o(G) = 3$, then $\overline{C(G)}$ is a $K_{1,1,1}$ (triangle). So in this case, result is trivially true. Let us assume that $o(G) > 3$. By Theorem 2.17, $\overline{C(G)} \setminus \{g\}$ is Hamiltonian $\forall g \in G$. In particular, $\overline{C(G)} \setminus \{e\}$ is Hamiltonian. Let C be the Hamiltonian cycle in $\overline{C(G)} \setminus \{e\}$ with vertices labeled by $u_1, u_2, \ldots, u_{n-1}$ about C. There are three possibilities for the given edge ε depending on whether it is part of a C or not, that is, ε is of the form

1. $\varepsilon \in C$, that is, $\varepsilon = u_i u_{i+1}$ for some i, $1 \leq i \leq n-1$.
2. ε is a chord on C, that is $\varepsilon = u_i u_j$ for some $j \neq i+1$, $1 \leq i < j \leq n-1$
3. $\varepsilon = u_i e$ for some i, $1 \leq i \leq n-1$.

We construct the Hamiltonian cycle H in $\overline{C(G)}$ by properly inserting the required edge ε and the vertex e into C.

Case 1 When $\varepsilon = u_i u_{i+1}$, for some i, $1 \leq i \leq n-1$.

Now the Hamiltonian cycle H in $\overline{C(G)}$ containing ε is given by $H := \langle u_i \; u_{i+1} \cdots u_j \; e \; u_{j+1} \cdots u_i \rangle$ for some $j \neq i$, $1 \leq j \leq n-1$, where the $\cdots$ represent the traversal along the edges of the cycle C.

Case 2 When $\varepsilon = u_i u_j$, for some $1 \leq i < j \leq n-1$, $j \neq i+1$.

We have the Hamilton cycle $H := \langle u_i \; u_j \; u_{j+1} \cdots u_{i-1} \; e \; u_{j-1} \cdots u_i \rangle$.

Case 3 When $\varepsilon = u_i e$, for some i, $1 \leq i \leq n-1$.

We have $H := \langle u_i \; e \; u_{i+1} \cdots u_i \rangle$ is the required Hamiltonian cycle containing ε in $\overline{C(G)}$.

Hence the theorem holds. □

Corollary 2.21. *If G is a group of odd order atleast 5, then for any edge ε in $E(\overline{C(G)})$, there exists a Hamiltonian cycle H in $\overline{C(G)}$ containing the edge ε.*

Proof. Result immediately follows from above theorem, since no group G of odd order $n > 3$ contains the conjugacy class of $\frac{n}{2}$ elements. □

Remark 2.22. Note that Theorem 2.20 fails, when the group G has a conjugacy class with $\frac{n}{2}$ elements. We again refer to Example 2.2. Observe that there is no Hamiltonian cycle in $\overline{C(S_3)}$ containing the edge $\varepsilon = ea$ or eb.

In view of the above remark, we present the following result.

Theorem 2.23. *If G is a group having the conjugacy class with $\frac{n}{2}$ elements, then for any edge ε in $\overline{C(G)}$, either there exists a Hamiltonian cycle H or a Hamiltonian path P in $\overline{C(G)}$ containing ε.*

Proof. Let G have a conjugacy class, say V_m, with $\frac{n}{2}$ elements, hence we have n to be an even integer, say $n = 2q$, for some $q \in \mathbb{N}$. Let $V_m = \{u_1, u_2, \dots, u_q\}$ and let $G \setminus V_m$ be labeled as $\{w_1, w_2, \dots, w_q = e\}$ such that the Hamiltonian cycle C (exists due to Theorem 2.15) consists of vertices alternating between V_m and $G \setminus V_m$ in the given order, that is, $C = \langle u_1\ w_1\ u_2\ w_2 \cdots u_q\ w_q\ u_1 \rangle$. We will construct the Hamiltonian cycle H or Hamiltonian path P containing ε. Depending on the given edge ε, we have the following cases:

Case 1: $\varepsilon \in [V_m, G \setminus V_m]$

Case 1.1: If $\varepsilon \in C$, that is $\varepsilon = u_i w_i$, $1 \le i \le q$, then $H := C$ is the required cycle.

So, next we consider the cases when $\varepsilon \notin C$.

Case 1.2: When $\varepsilon = u_j w_r$, for some $j \neq r$, $1 \le j \le q$,$1 \le r < q$.

From the cycle C, we obtain H as follows:

$$H := \langle u_j \ w_r \ u_{r+1} \cdots u_q \ w_j \ u_{j+1} \cdots u_r \ w_q \ u_1 \ w_1 \cdots u_j \rangle,$$

where $\cdots$ represent the traversal along the edges of the cycle C.

Case 1.3: When $\varepsilon = u_j w_q = u_j e$, $1 \leq j \leq q$.

Here again we have the required cycle given by

$$H := \langle u_j \ w_q \ u_1 \ w_1 \ u_2 \ w_2 \cdots w_{j-1} \ u_{j+1} \ w_{j+1} \cdots w_{q-1} \ u_q \ w_j \ u_j \rangle.$$

Case 2: When $\varepsilon \in G \setminus V_m$, that is, $\varepsilon = w_i w_j, i \neq j, 1 \leq i, j \leq q$.

Suppose ε belongs to a Hamiltonian cycle, then by Pigeon hole principle, since $|V_m| = q = |G \setminus V_m|$, we will force that at least two elements of V_m need to be adjacent, which contradicts to the fact that V_m is an independent set. Hence there does not exists any Hamiltonian cycle containing ε. Hence, we proceed to construct a Hamiltonian path P containing ε.

Case 2.1 When $\varepsilon = w_i w_j, i \neq j, 1 \leq i, j \leq q - 1$.

We have the required path given by

$$P := \langle u_1 \ w_1 \ u_2 \ w_2 \cdots u_i \ w_i \ w_j \ u_{j+1} \ w_{j+1} \cdots u_q \ w_q \ u_{i+1} \ w_{i+1} \cdots u_j \rangle,$$

where $\cdots$ represent the traversal along the edges of the cycle C.

Case 2.2 When $\varepsilon = w_i w_q$, $1 \leq i \leq q$.

We have

$$P := \langle u_1 \ w_1 \ u_2 \ w_2 \cdots u_i \ w_i \ e \ u_{i+1} \ w_{i+1} \cdots u_q \rangle$$

is the required Hamiltonian path containing ε. □

Corollary 2.24. *For any group G of order n, minimum number of distinct Hamiltonian paths in $\overline{C(G)}$ is $n-1$.*

Proof. Result follows from Theorem 2.23, since the vertex e is adjacent to every vertex in $\overline{C(G)}$. □

We know that $\omega(\overline{C(G)}) = m$, which implies that cycle of the length 3 to m are all available in $\overline{C(G)}$. At the other end, we know that $\overline{C(G)}$ has cycles of length n and $n-1$. So, does $\overline{C(G)}$ contain cycles of all the remaining length lying between $m+1$ to $n-1$. We answer this in affirmative in the next theorem.

Theorem 2.25. *For any group G of order $n \geq 3$, $\overline{C(G)}$ is pancyclic.*

Proof. Assume that $n_1 \leq n_2 \leq \cdots \leq n_m$. By Theorem 2.2.1, $\overline{C(G)}$ is Hamiltonian, that is, contains a cycle of length n. Let x_1 be the vertex in the partite set of largest cardinality in the graph $\overline{C(G)}$, then by Theorem 2.19, $\overline{C(G)} \setminus \{x_1\}$ is Hamiltonian. It follows that $\overline{C(G)}$ contains a cycle of length $n-1$. Again, consider the graph $\overline{C(G)} \setminus \{x_1\}$, let x_2 be the vertex in the partite set of largest cardinality in the graph $\overline{C(G)} \setminus \{x_1\}$, then again by Theorem 2.19, $\overline{C(G)} \setminus \{x_1, x_2\}$ is Hamiltonian. This implies that $\overline{C(G)}$ contains a cycle of length $n-2$. Recursively, we apply this procedure of deleting one vertex from the largest partite set of $\overline{C(G)} \setminus \{x_1, x_2, \ldots, x_r\}$, $1 \leq r \leq n-3$ until we get all cycles of length $n-3$ to three in the graph $\overline{C(G)}$. Hence the result follows. □

Theorem 2.26. *Let G be a group of order $n > 3$. Then $\overline{C(G)}$ contains a subgraph isomorphic to a wheel W_n on n-vertices $\Leftrightarrow$ no conjugacy class of G contains $\frac{n}{2}$ elements.*

Proof. Suppose no conjugacy class of the group G contains $\frac{n}{2}$ elements, by Theorem 2.17, and for $g \in Z(G)$, we have $\overline{C(G)} \setminus \{g\}$ contains a Hamiltonian cycle H. Now

$H + \{g\}$ forms a wheel W_n, since g is adjacent to all the vertices of H. Hence $\overline{C(G)}$ contains a subgraph isomorphic to wheel W_n. Conversely, suppose there exists a conjugacy class of size $\frac{n}{2}$ in the group G. For $g \in Z(G)$ and by Theorem 2.19, $\overline{C(G)} \setminus \{g\}$ is non Hamiltonian, since g does not belong to the largest partite set. Also, only vertices in $\overline{C(G)}$ which are adjacent to all other vertices are precisely the members of $Z(G)$. Therefore, $\overline{C(G)}$ does not contain a subgraph isomorphic to W_n. □

Corollary 2.27. $\overline{\mathrm{C}(\mathrm{D}_{2\mathrm{n}})}$ *contains a subgraph isomorphic to* $W_{2n} \Longleftrightarrow n$ *is an even integer.*

Proof. If n is an even integer, then class equation of the group D_{2n} is given by $2n = 1+1+2+2+\cdots+2+\frac{n}{2}+\frac{n}{2}$. Hence, the conclusion follows from above theorem. Conversely, if n is an odd integer, then D_{2n} has conjugacy class of cardinality n which contradicts the assumption of the above theorem. □

We conclude the chapter with a result which characterizes when $\overline{C(G)}$ becomes an Eulerian graph.

Theorem 2.28. $\overline{C(G)}$ *is an Eulerian graph* $\Leftrightarrow G$ *is a group of odd order.*

Proof. If $\overline{C(G)}$ is Eulerian, then it is well known that $deg_{\overline{C(G)}}(u)$ is an even integer $\forall u \in V(\overline{C(G)})$. The identity element has degree $n-1$ in $\overline{C(G)}$ implies that n has to be an odd integer. Conversely, if n is an odd integer, then by Corollary 2.2, $|V_i| = n_i$ is an odd integer for each i, $1 \leq i \leq m$. For $u \in V_i$, for some i, $1 \leq i \leq m$, we have $deg_{\overline{C(G)}}(u) = n - n_i$, which is an even integer. Thus $deg_{\overline{C(G)}}(u)$ is even, $\forall u \in V(\overline{C(G)})$. Hence $\overline{C(G)}$ is an Eulerian graph. □

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