Chapter 1

Introduction

1.1 Motivation

The origin of graph theory can be traced back to a Swiss Mathematician named Leonhard Euler solving the Konigsberg seven bridges problem in the 18th century. Euler's solution for the seven bridges problem was a new type of geometry called "Geometry of Position," later renamed Graph Theory. In 1736, Euler was the first Mathematician who published the first article on graph theory due to his efforts to solve the Konigsberg bridge problem. A branch of graph theory in which algebraic methods are applied to problems on graphs is algebraic graph theory. It involves the use of group theory and the study of graph invariants. Using a graphical representation of a group, researchers have discovered new regular graphs with more nodes for a given diameter and number of edges per node than were previously known. This allows the construction of larger networks while meeting design criteria of a fixed number of nearest neighbors and a fixed maximum communication time between arbitrary nodes. Some other motivating factors of this topic are, firstly, it provides a method of visualizing a group and also its implementation in a computer; secondly, it gives important classes of graphs having applications in several branches of science and engineering; thirdly, it connects two important branches of modern mathematics-groups and graphs. In this context, the most well-known class of graphs is the Cayley graph. Cayley graphs were first presented in 1878, are well established, and are hugely considered to have numerous applications. In particular, Cayley graphs of finite groups are used as routing networks in parallel computing due to their basic properties such as regular and vertex-transitive. Recently, the study of the graphical representation of an algebraic structure, especially a semi-group or a group has become an energizing research topic over a couple of decades, prompting many intriguing outcomes and questions. Some other well-studied algebraic graphs are non-commuting graphs, conjugate graphs, zero divisor graphs, divisibility graphs, and prime graphs, to name a few. The concept of power graphs and superpower graphs of semi-groups and/or groups is a very recent development in the domain of graphs from groups.

In this thesis, we mainly focus on analyzing the properties of graphs obtained from groups namely the complement of a conjugate graph and superpower graph and their application. Before that, we begin with some basic definitions and concepts required for further reading this thesis.

1.2 Group-Theoretical Concepts

A non-empty set together with an associative binary operation defined on it is called a semi-group. A *group* G is a nonempty set together with a binary operation $*$ on G such that the following properties hold:

- (i) $*$ is associative on G.

- (ii) there exists an element $e \in G$ such that for any $a \in G$, $a * e = e * a = a$.

- (iii) for any $a \in G$, there exists $b \in G$ such that $a * b = b * a = e$. This element b is called the inverse of a and denoted by a^{-1}.

A group G is called *commutative or Abelian*, if $a*b = b*a$ for all $a, b \in G$, otherwise it is called *non-commutative or non-Abelian*. A subset H of a group G is called a *subgroup* of G if H itself is a group under the same binary operation of G. A group having finite number of elements in it is called a *finite group*. The number of elements of a finite group G is called the *order of* G and is denoted by $o(G)$. For any $a \in G$ the order of a, denoted by $o(a)$ is the least positive integer r such that $a^r = e$. For a prime p, a finite group G is called a $p-$*group* if the order of each element of G is a power of p. Further, it is well-known that a non-trivial finite group G is a $p-$group if and only if $o(G) = p^r$ for some positive integer r. Let G be a finite group with identity element e. A *generating set* S of the group G is a subset of G such that every element of G can be expressed as a combination (under the group operation) of elements of S and in this case we write $G = \langle S \rangle$. If S is a generating set of a group G then the elements of S are called *generators* of G and we say that S *generates* G. If a singleton set $\{a\}$ is a generating set of a group G then G is called a *cyclic group* and we write $G = \langle a \rangle = \{a^m : m \in \mathbb{Z}\}$. For each positive divisor d of n, define $w_d(G) = \{x \in G : \ o(x) = d\}$. The *exponent* $\exp(G)$ of G is the least common multiple of the orders of the elements of G, in other words, the smallest positive integer m such that $x^m = e$ for all $x \in G$. We say that G *has an element of exponent order* if there exists $x \in G$ with $o(x) = \exp(G)$. Let $\pi(G)$ be the set of orders of elements of G. Thus, G has an element of exponent order if and only if $\pi(G)$ is the set of all divisors of m, for some integer m.

Throughout this thesis, we consider groups of finite order n, that is, a group G with $o(G) = n$ with identity e.

For any subset $X \subseteq G$, $X^* = X \setminus w_1(G) = X \setminus \{e\}$ and $\overline{G} = G^* \setminus w_{\exp(G)}(G)$. Center of G is denoted by $Z(G) = \{u \in G \mid uv = vu, \forall v \in G\}$. Let $G_1 \times \cdots \times G_m$ denote the direct products of finite groups $G_1, \ldots, G_m$.

Define a relation $\sim$ on G as follows: Two elements $x, y \in G$ are related, if x and y are of the same order in G. Clearly, this relation forms an equivalence relation on G, and so it partitions G into disjoint equivalence classes. The corresponding equivalence class containing $x \in G$ is denoted by $[x]$. Similarly, another relation $\sim$ on G can be defined as follows: For $u, v \in G$, $u \sim v$, if there exists an element $w \in G$ such that $v = w^{-1}uw$ forms an equivalence relation on G. This relation is also known as the ***conjugate relation*** in [2]. The corresponding equivalence class of conjugate relation on $u \in G$ is denoted by $cl(u)$, and is known as ***conjugacy class***(conjugate class) of u in G. Note that $cl(u) = \{u\} \Leftrightarrow u \in Z(G)$. Let $K(G)$ represents the ***number of conjugacy classes*** in G. It is immediate that, if G is an Abelian Group of order n, then, $K(G) = n = o(G)$.The class equation of the group G defined by $o(G) = \sum o(cl(u))$, where summation is taken over the representative member from each ***conjugacy class***.

Let us recall that, for a positive integer n, $\phi(n)$ denotes the Euler's phi function which counts the positive integers up to n that are relatively prime to n. If $n = p_1^{\alpha_1} p_2^{\alpha_m} \cdots p_m^{\alpha_m}$, is the prime decomposition of n, where p_i is a prime number and α_i is a positive integer for every $i(1 \leq i \leq m)$, then $\phi(n) = \prod_{i=1}^{m} (p_i^{\alpha_i} - p_i^{\alpha_i - 1}) = n \prod_{i=1}^{m} (1 - \frac{1}{p_i})$. Also, for any positive integer n, $\sum_{d|n} \phi(d) = n$, [3]. Whenever, we consider the prime factorization of a positive integer $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}$, it is assumed that $m \geq 2$, $p_1 < p_2 < \cdots < p_m$ are primes and $\alpha_i \in \mathbb{N}$, $\forall i$, $1 \leq i \leq m$, where $\mathbb{N}$ denotes the set of all natural numbers.

We use the following standard groups:

- $\mathbb{Z}_n = \{\overline{0}, \overline{1}, \cdots, \overline{n-1}\} = \{a^n = e, a, a^2, \cdots, a^{n-1}\}$ denotes the *finite cyclic group* of order n.
- $D_{2n} = \{\langle a, b\rangle \mid a^n = b^2 = e,\ aba = b\}$ denotes the *dihedral group* of order $2n$;
- $T_{4n} = \{\langle a, b\rangle \mid a^{2n} = e, a^n = b^2, ab = ba^{-1}\}$ denotes the *dicyclic group* of order $4n$. If n is a power of 2, this group is the *generalized quaternion group.*
- $Q_8 = \{\langle a, b\rangle \mid a^4 = 1, b^2 = a^2, b^{-1}ab = a^{-1}\}$ denotes the *quaternion group.*
- S_n and A_n denote the *symmetric group* and the *alternating group* on the set of n symbols respectively.

1.3 Graph-Theoretical Concepts

A graph is denoted by $\Gamma = (V(\Gamma), E(\Gamma))$ consisting of a finite, non-empty set $V(\Gamma)$ of vertices, together with a set $E(\Gamma)$ of unordered pairs of vertices called edges. If $V(\Gamma)$ has finite number of elements, then Γ is called a *finite graph*; otherwise, it is called infinite graph. If $V(\Gamma)$ has a single element, then Γ is called a *trivial graph*; otherwise, nontrivial graph.

For $\varepsilon \in E(\Gamma)$, if $\varepsilon = uv$, where $u, v \in V(\Gamma)$, we say that u and v are adjacent vertices, and ε is said to be incident with u and v, or we say that ε joins the vertex u and v. If two or more edges join the same pair of vertices then these edges are called *multiple edges.* An edge with identical ends $\varepsilon = uv$ is called a *loop.* The *degree of* *a vertex* u in a simple graph is denoted by $deg_\Gamma(u)$ is the number of edges incident on u. A graph is said to be a *simple graph* if it has no multiple edges or loops. A graph Γ is called *planar* if it can be drawn on a plane in such a way that no edges cross each other. The *complement* of a graph Γ denoted by $\overline{\Gamma}$, is the graph on the same vertex set as Γ and two vertices are adjacent in $\overline{\Gamma}$ if and only if they are not adjacent in Γ.

A graph Γ is called a *complete graph* if for any two distinct vertices u and v of Γ there is an edge between them. A complete graph on n vertices is denoted by K_n. A subgraph Γ_1 of a graph Γ is called a *clique* if Γ_1 induces a complete graph in Γ. A vertex of a graph Γ is called a ***dominant vertex*** if it is adjacent to every other vertex of Γ and $\mathrm{dom}(\Gamma)$ denote the set of all dominant vertices in Γ. An *k-partite* graph Γ is a graph whose vertex set can be decomposed into k independent subsets. A k-partite graph Γ is called *complete k-partite* if and only if any two vertices in different parts are adjacent.

In a graph Γ, let n and m denote the number of vertices and edges, respectively. A *walk* is an alternating sequence of vertices and edges $\langle v_0, \varepsilon_0, v_1, \varepsilon_1, \cdots, v_k, \varepsilon_k\rangle$ such that $\varepsilon_i = v_{i-1}v_i \ \forall i$, $1 \leq i \leq k$, and the number k denotes the *length of the walk*. If all the vertices and edges of a walk are distinct, we call it a *path*. A graph is *connected* if there is a path between any two vertices. For $k \geq 3$, a walk $\langle v_0, \varepsilon_0; v_1, \varepsilon_1 \cdots v_k, \varepsilon_k\rangle$ in which no vertices and edges are repeated is called a *cycle* if $v_0 = v_k$, it is denoted by C_k where k is the length of the cycle.

If a graph Γ has a path P (or a cycle C) which contains all the vertices of Γ, then P (or C) is called the *Hamiltonian path* (or *Hamiltonian cycle*). A *Hamiltonian graph* Γ is a graph which contains a Hamiltonian cycle. A graph Γ is called randomly Hamiltonian, if a Hamiltonian cycle is formed by starting at any vertex of Γ and then consecutively moving to any adjacent vertex. A graph Γ is called *1-Hamiltonian* if it is Hamiltonian and all of its 1-vertex-deleted subgraphs are Hamiltonian. A simple graph Γ on n vertices is called *pancyclic* if for every l, $3 \leq l \leq n$, there exists a cycle of length l in Γ. If a graph Γ has a closed walk W which traverse every edge of Γ exactly once, then W is called an *Euler circuit* and Γ is called an *Eulerian graph*.

A *connected component* of Γ is a maximal connected subgraph of Γ. A subset $S \subset V$ of a graph is called *separating set* for Γ if $\Gamma \setminus S$ increases the number of connected components of Γ. A separating set S of a connected graph Γ is said to be a *minimal separating set* if $\Gamma \setminus X$ is connected for any proper subset X of S. A minimal separating set of minimum cardinality is a *minimum separating set*, and the cardinality $\kappa(\Gamma)$ of this minimum separating set is called the *vertex connectivity* of Γ. A subset $F \subset E$ of a graph is called a *disconnecting* set for Γ if $\Gamma \setminus F$ increases the number of connected components of Γ. If $F \subset E$ is of minimum cardinality $\kappa^{'}(\Gamma)$ such that $\Gamma \setminus F$ is disconnected or trivial graph, then $\kappa^{'}(\Gamma)$ is called the *edge connectivity* of the graph Γ. A graph Γ is called *k-critically edge connected* if $\kappa^{'}(\Gamma) = k$ and $\kappa^{'}(\Gamma \setminus \varepsilon) = k - 1$ for each edge$\varepsilon \in \Gamma$.

Given two graphs, the product of the graphs is a well-studied and interesting concept. Studying the structural properties of the product graph with the help of the given graphs have received wide attention among the researchers. In the literature, various types of products are defined such as Cartesian product, strong product, tensor product, complete product, corona product, to name a few. Among these, we will be considering the tensor product of two graphs in this thesis. The *tensor product*, $\Gamma_1 \times \Gamma_2$, of the graphs Γ_1 and Γ_2 is a graph with vertex set $V(\Gamma_1) \times V(\Gamma_2)$ and any two vertices (a_1, b_1) and (a_2, b_2) are adjacent in $\Gamma_1 \times \Gamma_2$ if and only if $a_1a_2 \in E(\Gamma_1)$ and $b_1b_2 \in E(\Gamma_2)$.

1.4 Graphs Associated With Groups

Graphs associated with groups and other algebraic structures have been actively investigated, since they have valuable applications are related to automata theory which can be seen in [4],[5] and the books [6],[7]. Some of the most popular graphs in this area, Cayley graph [8], [9], [10], commuting graph [11] , conjugate graphs [12], complement of conjugate graphs [13], power graph [14] and superpower graphs [15] to name a few. Among these algebraic graphs, we introduce here a few which are of interest and will be used in the thesis.

The conjugate graph associated to finite non-Abelian group G was introduced by Erfanian and Tolue in [12] as follows:

Definition 1.1. [12] Let G be a finite non-Abelian group having center $Z(G)$. Then the conjugate graph Γ_G^c associated to G is a graph with vertex set $G \setminus Z(G)$ and two vertices are adjacent in Γ_G^c if and only if they are conjugate in the group G.

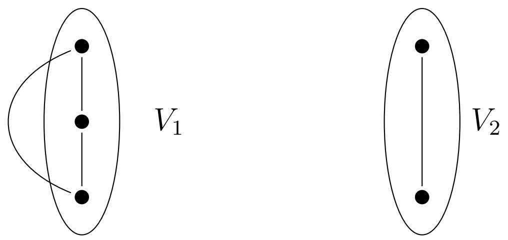

FIGURE 1.1: Conjugate graph $\Gamma_{S_3}^c$

Following this, the complement of conjugate graph was defined by Abdussakir *et al.* in [13] as follows:

Definition 1.2. [13] The *complement of conjugate graph* of a group G, denoted by $\overline{C(G)}$ is a graph with vertex set G and any two vertices are adjacent in $\overline{C(G)}$ if and only if they are not conjugate in the group G.

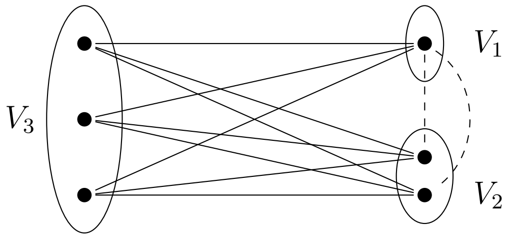

FIGURE 1.2: Complement of conjugate graph $\overline{\mathrm{C}(\mathrm{S}_3)}$

The concept of *directed power graph* $\overrightarrow{\mathscr{P}}(G)$ of a group G, introduced by Kelarev and Quinn[16], is a digraph with vertex set G and for any $a, b \in G$, there is a directed edge from a to b in $\overrightarrow{\mathscr{P}}(G)$ if and only if $a^k = b$, where $k \in \mathbb{N}$. For a semi-group, it was first considered in [14] and further studied in [17]. All of these papers used the brief term 'power graph' to refer to the directed power graph, with the understanding that the undirected power graph is the underlying undirected graph of the directed power graph. Motivated by this, Chakrabarty *et al.* [18] introduced the concept of an *undirected power graph* $\mathscr{P}(G)$ of a group G, which was defined as follows:

Definition 1.3. Given a group G, the power graph $\mathscr{P}(G)$ of G is the simple undirected graph with vertex set G and two vertices $a, b \in G$ are adjacent in $\mathscr{P}(G)$ if and only if $b \neq a$ and $b^k = a$ or $a^k = b$, $k \in \mathbb{N}$.

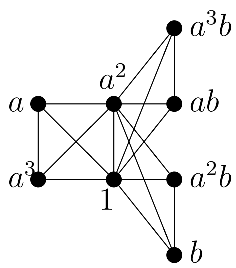

FIGURE 1.3: The power graph of Q_8

Recently, undirected superpower graph $S(G)$ of a finite group G, was first presented by Hamzeh and Ashrafi [19] (which they call it as 'order super graph of the power graph of a finite group'), and defined as given below:

Definition 1.4. Let G be a finite group. Then superpower $S(G)$ is a graph with the vertex set G and two distinct vertices $a, b \in G$ are adjacent in $S(G)$ if and only if either $o(a)|o(b)$ or $o(b)|o(a)$ in the group G.

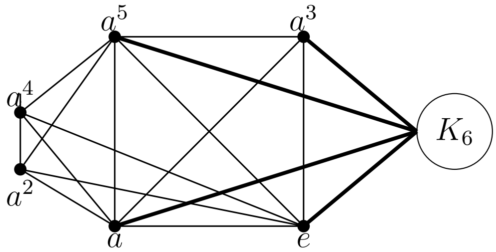

FIGURE 1.4: The superpower graph $S(D_{12})$ [bold line between a vertex and a clique indicates that the vertex is adjacent to every vertex in the clique].

1.5 Literature Review

The investigation of graphs associated with algebraic structures are very important, as graphs like these enrich both algebra and graph theory. Also, these graphs have practical applications and are related to automata theory which can be seen in [4, 5, 6, 7]. The study of graphical representation of semi-group and group has become an energizing research topic over a recent couple of decades, prompting many intriguing outcomes and questions. In this context, some of the most studied graphs are the Cayley graph [8, 9, 10], conjugate graph [12] and power graph [20, 21] superpower graph [15, 19, 22].

Conjugate Graphs: In [12], Erfanian and Tolue introduced the conjugate graph Γ_G^c associated to a non-Abelian group with vertex set $G\setminus Z(G)$ such that two distinct vertices join by an edge if they are conjugate. They proved that if $\Gamma_G^c = \Gamma_S^c$, where S is a finite non-Abelian simple group which satisfy Thompson's conjecture, then $G \sim S$. Further, if central factors of two non-Abelian groups H and G are isomorphic and $|Z(G)| = |Z(H)|$, then H and G have isomorphic conjugate graphs. Following this, Abdussakir *et al.* [13], studied the spectrum and energy of detour matrix of complement of conjugate graph of dihedral groups. They denoted conjugate graph and complement of conjugate graph of a finite group G by $C(G)$ and $\overline{C(G)}$ respectively.

Power Graphs: The concept of *directed power graph* $\overrightarrow{\mathscr{P}}(G)$ of a group G, introduced by Kelarev and Quinn[16], it was first considered in [14] and further studied in [17]. All of these papers used the brief term 'power graph' to refer to the directed power graph, with the understanding that the undirected power graph is the underlying undirected graph of the directed power graph. Motivated by this, Chakrabarty *et al.* [18] introduced the concept of an *undirected power graph* $\mathscr{P}(G)$ of a group G,. After that, the undirected power graph became the main focus of study by several authors in [16, 20, 23, 24, 25, 26, 27, 28, 29, 30, 31]. While on the other side, many researchers have modified and generalized the concept of power graphs in several ways, such as proper power graph [32], reduced power graph [33] and quotient graph [34]. In order to find the relationship between power graph and commuting graph [35], Aalipour *et al.*[36], introduced enhanced power graph of a group and found that this graph lies between these two graphs. In 2021, we wrote an extensive survey on power graph which contains recent development in this area [21]. For more details on power graph, reader can consider the survey paper [20].

Superpower Graphs: The notion of the superpower graph $S(G)$ of finite group G is a very recent development in the domain of graphs obtained from groups, and it was firstly introduced by Hamzeh and Ashrafi [15] in 2018, where they refer to it as the order super graph $S(G)$ of the power graph $\mathscr{P}(G)$ of a finite group. They first explore the relationship between the power graph $\mathscr{P}(G)$, and the superpower graph $S(G)$, wherein they compute some of the structural properties of superpower graph and see how much they differ from power graphs. In fact, they proved that $S(G)$ and

$\mathscr{P}(G)$ are identical if and only if G is a finite cyclic group. Further they posed the question that what is the structure of G when $|\pi(G)| = \alpha(S(G))$ where $\pi(G)$ is the set of all orders of elements in G and $\alpha(S(G))$ is the independence number of $S(G)$. Following this, Ma and Su [37], studied the independence number $\alpha(S(G))$ and answered this question. Having defined the superpower graph of a group, Hamzeh and Ashrafi [19] studied about the groups that can be represented as the automorphism group of the graphs $\mathscr{P}(G)$ and $S(G)$. They also proved that the automorphism group of this graph could be written as a combination of Cartesian and wreath products of some symmetric groups. In the same article, the authors compute the automorphism group of the superpower graph of the dihedral group D_{2n}, dicyclic group T_{4n}. Also, Hamzeh and Ashrafi [22] have obtained the characteristic polynomial of the power graph and the superpower graph of certain finite groups. Consequently, the spectrum and Laplacian spectrum of these graph for dihedral, semi-dihedral, cyclic, and dicyclic groups were computed. In [38], the authors have proved some results on 2-connectivity, Eulerian and Hamiltonian properties of the superpower graph for some special classes of groups. Recall that $PSL(2,p)$ and $PGL(2,p)$ are the sporadic quotient groups of the simple linear and general linear groups respectively. In [39], Asboei and Salehi proved that the the groups $PSL(2,p)$, $PGL(2,p)$ and sporadic simple groups uniquely determine their superpower graphs. That is, for any finite group G, if $S(G) \simeq S(PSL(2,p))$ or $S(PGL(2,p))$, then $G \simeq PSL(2,p)$ or $PGL(2,p)$ respectively. They also proved that if M is a sporadic simple group, then $G \simeq M$ if $S(M) \simeq S(G)$.

Quotient Graph of Superpower Graphs: Another interesting notion that will be used in this thesis is the concept of quotient graph. Let $\mathcal{P}$ be a partition of the vertex set V of the graph $\Gamma(V, E)$. The *quotient graph* Γ^q of Γ is the graph with vertex set is $\mathcal{P}$ and two vertices $A, B \in \mathcal{P}$ are adjacent in Γ^q if and only if there exist vertices $u \in A$ and $v \in B$ such that u and v are adjacent in Γ. The notion of *order graph* $\mathcal{O}(G)$ of G was introduced by Bubboloni *et al.* [34], whose vertex set is $V(\mathcal{O}(G)) = \{o(a) \mid a \in G\}$ and the edge set is $E(\mathcal{O}(G)) = \{rs|r, s \in V(\mathcal{O}(G))$ and either $r|s$ or $s|r\}$. Now, recall the definition of *undirected superpower graph* $S(G)$ of a finite group G and the equivalence relation defined on the group G, it can be seen that the vertex set $V(S(G))$ can be partitioned into disjoint equivalence classes which are precisely the elements of same order and the induced subgraph corresponding to each equivalence class forms a clique in $S(G)$. Thus, $\mathcal{O}(G)$ is isomorphic to a quotient graph of $S(G)$ in which the set $V(S(G))$ is partitioned into elements with the same order.

Suppose A is a simple graph and $\Gamma = \{\Gamma_a\}_{a\in A}$ is a set of graphs labeled by vertices of A. Following Sabidussi [40], the *A-join of* Γ is the graph Δ with the following vertex and edge sets: $V(\Delta) = \{(x, y)|x \in V(A)$ and $y \in V(\Gamma_x)\}$ and $E(\Delta) = \{(x, y)(x', y')|\ xx' \in E(A)$ or else $x = x'$ and $yy' \in E(\Gamma_x)\}$. If A is a p-vertex labeled graph then the A-join of $\Delta_1, \Delta_2, \cdots \Delta_p$ is denoted by $A[\Delta_1, \Delta_2, \cdots, \Delta_p]$. Let $\pi(G) = \{r_1, r_2, \cdots, r_m\}$ be the set of all order elements of the group G and $E_{r_i}(G)$ be the set(or equivalence class) of all elements of order r_i in G with $\Omega_{r_i} = |E_{r_i}|$. Define a graph Δ_G with vertex set $\pi(G)$ and edge set $E(\Delta_G) = \{xy|\ x, y \in \pi(G), x|y$ or $y|x\}$. It was proved in [22], [19] that

$$S(G) = \Delta_G[K_{\Omega_{r_1}}, K_{\Omega_{r_2}}, \cdots, K_{\Omega_{r_m}}], \tag{1.1}$$

where K_n denote the complete graph on n vertices.

1.6 Objective and Scope

This thesis aims to study the structural properties, especially the connectivity and Hamiltonian properties of graphs obtained from finite groups. It is well known that multi-partite graphs have significance in the field of graph theory. In 2017, Joseph Varghese proved that a complete multi-partite graph is Hamiltonian if and only if the cardinality of each partite set is less than or equal to the sum of the cardinality of all other partite sets. A natural question that arises as an extension of his work is that, can we decompose the given set of vertices into k independent sets in such a manner the graph obtained from this partition will be complete multi-partite and Hamiltonian? We explore the structural properties of the complement of conjugate graph $\overline{C(G)}$ of a finite group G and answer the above question in the affirmative. In [15], Hamzeh and Ashrafi computed the sharp bound for the vertex connectivity for $S(\mathbb{Z}_n)$ of a finite cyclic group. Later on, they proved that the power graph and superpower graph are identical for finite cyclic groups. Motivated by this, we studied the connectivity properties and computed the sharp bound of $S(D_{2n})$ and $S(T_{4n})$ to see how they differ from their corresponding power graph. In [38, Theorem 2.7], authors showed that $S(G)$ is 2-connected for any finite Abelian group. A natural extension is to find the size of the minimum separating set required to disconnect $S(G)$. Motivated by this, we give sharp lower and upper bounds for the superpower graph $S(G)$ of an Abelian group. We identify the dominant set and prove that after removing all dominant vertices from $S(G)$, the remaining graph is still Hamiltonian and hence 2-connected. We have extended all the results proved earlier for $S(G)$ to graphs defined on finite groups having an element of exponent order. We also compute the edge connectivity and Hamiltonian-like properties for $S(D_{2n})$ of non-Abelian non-simple group D_{2n}.

1.7 Outline of the Thesis

In this thesis, we highlight the rich interplay between the two topics viz groups and graphs obtained from finite groups, particularly the Complement of conjugate graphs and Superpower graphs.

As a first work, in ***Chapter 2***, we extend Joseph Varghese's results by studying the Hamiltonian and Eulerian properties of graphs generated with the group-theoretic concept of conjugacy relation on a finite group, referred to as the complement of a conjugate graph $\overline{C(G)}$. We also explore the other variations of the Hamiltonian property, namely, 1-Hamiltonian, pancyclicity, to name a few. Further, we present several structural characterizations of the considered graph class.

Next, we focus on the superpower graphs, $S(G)$, defined on finite Abelian groups in ***Chapter 3***. We begin by first characterizing the structure of such graphs. We study the relation between the superpower graph of direct product of Abelian groups and the tensor product of their respective superpower graphs. Further, we identify the dominant vertices in the graph and use that to find the tight bounds for the vertex connectivity of $S(G)$. In addition, we also identify those groups for which the bounds are attained by their superpower graphs. Further, we also establish the conditions for this class of graphs to be Hamiltonian along with other structural properties such as maximal dominating set, 1-Hamiltonian, Hamiltonian-connected, pancyclic and panconnected to name a few.

Next in ***Chapter 4***, we target the class of the dihedral and dicyclic groups among the class of non-Abelian groups and explore the properties of their superpower graphs namely $S(D_{2n})$ and $S(T_{4n})$. It is well-known that these groups hold a fundamental place in group theory and they also appear as a subgroup of many groups. Our motive is to study the structural properties of the superpower graph of these groups and see how they differ from their corresponding power graphs. In this process, we compute sharp bounds for the vertex connectivity of $S(D_{2n})$ and $S(T_{4n})$ and identify the value of n for which the superpower graphs of respective groups attains the bound. In another attempt, we also obtain bounds for the edge connectivity of $S(D_{2n})$ and compute the edge connectivity of $S(D_{2n})$ for some special values of n through the minimum degree $S(D_{2n})$. We also investigate and characterize Hamiltonian-like properties for the groups and their superpower graphs under consideration.

Chapter 5 is of two-fold: Firstly, we study the structural characterizations of superpower graph defined on any finite group. Here, we establish some of the properties such as perfectness, Eulerian, relation between comparability graph and super power graph, identify the separating sets and discuss the connection between the underlying quotient graph and superpower graph. Secondly, we extend our results on the properties such as dominant sets, connectivity in superpower graphs for arbitrary finite groups which were not considered before. To this end, we focus on superpower graphs defined on any finite non-Abelian groups, particularly, those groups having an element of exponent order. We first begin by structurally characterizing this collection of superpower graphs and establishing tight bounds for vertex connectivity. In addition, we identify graphs that attain the obtained bounds. Further, we also explore various Hamiltonian-like properties for this collection of superpower graphs.