CERTAIN PSEUDO-DIFFERENTIAL OPERATORS AND WAVELET TRANSFORMS INVOLVING FRACTIONAL FOURIER AND FRACTIONAL HANKEL TRANSFORMS

Thesis submitted in partial fulfillment for

the Award of Degree

Doctor of Philosophy

by

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July 2023

Chapter 5

Characterizations of Continuous Fractional Bessel Wavelet Transform and its Applications

5.1 Introduction

The continuous and discrete Bessel wavelet transforms were investigated by Pathak and Dixit [32] by using Haimo's Hankel transform theory (see, for details, [17]). Using the concept of the Bessel wavelet transform, Parseval relation, inversion formula and other associated results were obtained by [32]. Taking the concepts of Pathak and Dixit [32], many researchers wrote many papers on various functional spaces associated with Bessel wavelet transform. More recently, Srivastava *et al.* [16] studied a certain family of fractional wavelet transforms by applying the theory of the fractional Hankel transform. In the present works of Pathak *et al.* [32, 41], Upadhyay and Khatterwani [29] and Srivastava, Khatterwani and Upadhyay [16], our main objective in the present chapter, is to develop the fractional Bessel wavelet transform and its various properties associated with the fractional Hankel transform and the fractional Hankel convolution. In this chapter, the continuous and discrete fractional Bessel wavelet transform are discussed and found the Parseval formula, the inversion formula for the fractional Bessel wavelet transform, and its boundedness properties. Some applications related to the fractional Bessel wavelet transform in weighted Sobolev-type space were dealt by exploiting the theory of the fractional Hankel transform. Lastly, it is shown that the fractional Bessel wavelet transform is time-invariant linear filter and by using the aforesaid theory, the solution of integral equation involving the fractional wavelet transform is also obtained.

The entire chapter is organized in the following manner:

Section 5.1 is introductory, in which we define the *dilation* $D_{a^{\frac{1}{\alpha}}}$ function, the fractional Bessel wavelet $\psi_{b,a^{\frac{1}{\alpha}}}(x)$ and the fractional Bessel wavelet transform $B_\psi f(b,a)$ for $0 < \alpha \leq 1$. In Section 5.2, many properties of the fractional Bessel wavelet transform $B_\psi f(b,a)$, like boundedness properties, Parseval-Goldstein formula, inversion formula, Caldron reproducing formula, and the discrete fractional Bessel wavelet transform were discussed. In Section 5.3, some applications of the fractional Bessel wavelet transform in weighted Sobolev-type space are obtained by exploiting the theory of the fractional Hankel transform. In the last Section 5.4, it is shown that the fractional Bessel wavelet transform is in the form of time-invariant linear filter. The solution of integral equation involving the fractional wavelet transform is given by utilizing the theory of the fractional Hankel transform.

Definition 5.1.1. Let μ be a positive real number and $0 < \alpha \leq 1$. Then for $\phi \in L^p_\sigma(I)$, $1 \leq p < \infty$, the *dilation* $D_{a^{\frac{1}{\alpha}}}$ is defined by

$$D_{a^{\frac{1}{\alpha}}}\phi_\alpha(x,y) := a^{-2\mu-\frac{1}{\alpha}}\phi_\alpha\left(\frac{x}{a^{\frac{1}{\alpha}}}, \frac{y}{a^{\frac{1}{\alpha}}}\right), \tag{5.1.1}$$

where $\phi_\alpha(x, y)$ is the fractional Hankel translation, which is given by (4.2.11).

Definition 5.1.2. Let the function $\phi \in L^p_\sigma(I)$, be given for $1 \leq p < \infty$. If $a > 0$, $b \geqq 0$ and $0 < \alpha \leq 1$, then the fractional Bessel wavelet $\psi_{b,a^{\frac{1}{\alpha}}}(x)$ is defined by

$$\psi_{b,a^{\frac{1}{\alpha}}}(x) := D_{a^{\frac{1}{\alpha}}}\psi_\alpha(b, x) \tag{5.1.2}$$

$$= a^{-2\mu-\frac{1}{\alpha}} \int_0^\infty D_\alpha\left(\frac{b}{a^{\frac{1}{\alpha}}}, \frac{x}{a^{\frac{1}{\alpha}}}, z\right)\psi(z)d\sigma(z). \tag{5.1.3}$$

Definition 5.1.3. By taking the function $\psi \in L^2_\sigma(I)$ and with the help of (5.1.2), the fractional Bessel wavelet transform $B_\psi f(b, a)$ is defined by

$$B_\psi f(b, a) := \langle f(t), \psi_{b,a^{\frac{1}{\alpha}}}(t)\rangle = \int_0^\infty f(t)\, \overline{\psi_{b,a^{\frac{1}{\alpha}}}(t)}\, \mathrm{d}\sigma(t), \tag{5.1.4}$$

for $0 < \alpha \leq 1$.

From (5.1.3), we get

$$B_\psi f(b, a) = a^{-2\mu-\frac{1}{\alpha}} \int_0^\infty \int_0^\infty f(t)D_\alpha\left(\frac{b}{a^{\frac{1}{\alpha}}}, \frac{t}{a^{\frac{1}{\alpha}}}, z\right)\overline{\psi(z)}d\sigma(z)d\sigma(t). \tag{5.1.5}$$

5.2 Properties of the fractional Bessel wavelet transform

In this section, various properties of the fractional Bessel wavelet transform are given by exploiting the theory of fractional Hankel transform.

Theorem 5.2.1. *If $\psi,\ f \in L^2_\sigma(I)$ and $0 < \alpha \le 1$ then the continuous fractional wavelet transform can be expressed as*

$$B_\psi f(b,a) = \int_0^\infty j\left(w^{\frac{1}{\alpha}} b\right)\ (h_{\mu,\alpha} f)(w)\ \left(\overline{h_{\mu,\alpha}\psi}\right)(aw) \times \left(\frac{w^{\frac{1}{\alpha}-1}}{\alpha}\right)\ \mathrm{d}\sigma(w). \tag{5.2.1}$$

Proof. Using (4.2.12), (5.1.5) becomes

$$B_\psi f(b,a) = a^{-2\mu-\frac{1}{\alpha}} \left[\int_0^\infty \int_0^\infty f(t)\left(\int_0^\infty j\left(\frac{b}{a^{\frac{1}{\alpha}}}\xi^{\frac{1}{\alpha}}\right) j\left(\frac{t}{a^{\frac{1}{\alpha}}}\ \xi^{\frac{1}{\alpha}}\right) \times j\left(\xi^{\frac{1}{\alpha}} z\right)\left(\frac{\xi^{\frac{1}{\alpha}-1}}{\alpha}\right)\ \mathrm{d}\sigma(\xi)\right) \overline{\psi(z)}\ \mathrm{d}\sigma(z)\ \mathrm{d}\sigma(t)\right].$$

From (4.2.8) and (4.2.10), above yields

$$B_\psi f(b,a) = a^{-2\mu-\frac{1}{\alpha}} \int_0^\infty j\left(\frac{b}{a^{\frac{1}{\alpha}}}\ \xi^{\frac{1}{\alpha}}\right)\ \overline{(h_{\mu,\alpha}\psi)}(\xi)\ (h_{\mu,\alpha} f)\left(\frac{\xi}{a}\right) \times \left(\frac{\xi^{\frac{1}{\alpha}-1}}{\alpha}\right)\ \mathrm{d}\sigma(\xi). \tag{5.2.2}$$

Finally, setting $\xi = aw$, (5.2.2) becomes

$$B_\psi f(b,a) = \int_0^\infty j\left(w^{\frac{1}{\alpha}} b\right)\ (h_{\mu,\alpha} f)(w)\ \left(\overline{h_{\mu,\alpha}\psi}\right)(aw) \times \left(\frac{w^{\frac{1}{\alpha}-1}}{\alpha}\right)\ \mathrm{d}\sigma(w).$$

□

Theorem 5.2.2. *For a function $\psi \in L^2_\sigma(I)$ and for any signal $f \in L^2_\sigma(I)$, the following relation holds true:*

$$h_{\mu,\alpha}\Big(B_\psi f(b,a)\Big)(w) = (h_{\mu,\alpha}f)(w)\left(\overline{h_{\mu,\alpha}\psi}\right)(aw), \tag{5.2.3}$$

for $0 < \alpha \leq 1$.

Proof. From (5.2.1), we have

$$\begin{aligned} B_\psi f(b,a) = \int_0^\infty & j\left(w^{\frac{1}{\alpha}}b\right)\ (h_{\mu,\alpha}f)(w)\ \left(\overline{h_{\mu,\alpha}\psi}\right)(aw) \\ & \times \left(\frac{w^{\frac{1}{\alpha}-1}}{\alpha}\right)\ \mathrm{d}\sigma(w). \end{aligned}$$

In view of (4.2.10), we get

$$B_\psi f(b,a) = h^{-1}_{\mu,\alpha}\big[(h_{\mu,\alpha}f)(w)\left(\overline{h_{\mu,\alpha}\psi}\right)(aw)\big](b). \tag{5.2.4}$$

Now, by using (4.2.8), (5.2.4) can be written as

$$h_{\mu,\alpha}\Big(B_\psi f(b,a)\Big)(w) = (h_{\mu,\alpha}f)(w)\left(\overline{h_{\mu,\alpha}\psi}\right)(aw).$$

Hence the proof of Theorem 5.2.2 is completed. □

Theorem 5.2.3. *If $f, \psi \in L^2_\sigma(I)$, then the continuous fractional wavelet transform $B_\psi f(b,a)$ can be expressed by*

$$B_\psi f(b,a) = (f \# \bar{\psi}_a)(b). \tag{5.2.5}$$

Proof. From (5.2.1), (5.2.4) and Theorem 4.3.6, we obtain

$$B_\psi f(b,a) = h^{-1}_{\mu,\alpha}[(h_{\mu,\alpha}(f\#\bar{\psi_a})](b),$$

where $\bar{\psi_a}(x) = \frac{1}{a}\bar{\psi}(\frac{x}{a}), \quad a > 0.$

Therefore, we have

$$B_\psi f(b,a) = (f\#\bar{\psi_a})(b).$$

□

Theorem 5.2.4. *Let $f \in L^p_\sigma(I)$ and $\psi \in L^q_\sigma(I)$, then*

$$||B_\psi f(b,a)||_{r,\sigma} \le ||f||_{p,\sigma}||\psi_a||_{q,\sigma}, \quad where \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1. \tag{5.2.6}$$

Proof. For $f \in L^p_\sigma(I)$, $\psi \in L^q_\sigma(I)$ and from (5.2.5) and (4.2.13), we have

$$\begin{aligned} B_\psi f(b,a) &= (f\#\bar{\psi_a})(b) \\ &= \int_0^\infty f_\alpha(b,y)\bar{\psi_a}(y)d\sigma(y). \end{aligned}$$

Therefore, by using the Young's inequality, we get

$$\begin{aligned} |B_\psi f(b,a)| \le &\Big(\int_0^\infty |f_\alpha(b,y)|^p|\psi_a(y)|^q d\sigma(y)\Big)^{\frac{1}{r}} \\ &\times \Big(\int_0^\infty f_\alpha(b,y)|^p d\sigma(y)\Big)^{1-\frac{1}{q}} \\ &\times \Big(\int_0^\infty |\psi_a(y)|^q d\sigma(y)\Big)^{1-\frac{1}{q}}. \end{aligned}$$

Then above implies that

$$\int_0^\infty |B_\psi f(b,a)|^r d(b) \le \int_0^\infty \Big(\int_0^\infty |f_\alpha(b,y)|^p|\psi_a(y)|^q d\sigma(y)\Big)$$

$$\times \Big(\int_0^\infty |f_\alpha(b,y)|^p d\sigma(y) \Big)^{r-\frac{r}{q}}$$
$$\times \Big(\int_0^\infty |\psi_a(y)|^q d\sigma(y) \Big)^{r-\frac{r}{p}} d\sigma(b).$$

Applying the Fubini's theorem and translation symmetry (i.e., $f_\alpha(b,y) = f_\alpha(y,b)$), then we get

$$\begin{aligned}\Big(\int_0^\infty |B_\psi f(b,a)|^r d\sigma(b) \Big)^{\frac{1}{r}} &\leq \Big(\int_0^\infty |f_\alpha(b,y)|^p d\sigma(b) \Big)^{\frac{1}{r}(1+r-\frac{r}{q})} \\ &\quad \times \Big(\int_0^\infty |\psi_a(y)|^q d\sigma(y) \Big)^{\frac{1}{r}(1+r-\frac{r}{q})} \\ &= ||f_\alpha(b,y)||_{p,\sigma} ||\psi_a||_{q,\sigma}.\end{aligned}$$

From (4.3.2), we have

$$||B_\psi f(b,a)||_{r,\sigma} \leq ||f||_{p,\sigma} ||\psi_a||_{q,\sigma}.$$

□

Theorem 5.2.5. *Let $f \in L^p_\sigma(I)$ and $\psi \in L^q_\sigma(I)$ then,*

$$||B_\psi f(b,a)||_{\infty,\sigma} \leq ||f||_{p,\sigma} ||\psi_a||_{q,\sigma}, \qquad 0 < \alpha \leq 1. \tag{5.2.7}$$

Proof. Taking (5.2.5) and (4.2.13), we have

$$|B_\psi f(b,a)| \leq \int_0^\infty |f_\alpha(b,y)||\psi_a(y)| d\sigma(y).$$

From Holder's inequality, we get

$$\begin{aligned}|B_\psi f(b,a)| &\leq \Big(\int_0^\infty |f_\alpha(b,y)|^p d\sigma(y) \Big)^{\frac{1}{p}} \Big(\int_0^\infty |\psi_a(y)|^q d\sigma(y) \Big)^{\frac{1}{q}} \\ &\leq ||f_\alpha(b,y)||_{p,\sigma} ||\psi_a||_{q,\sigma}.\end{aligned}$$

By using (4.3.2), we find that

$$||B_\psi f(b,a)||_{\infty,\sigma} \leq ||f||_{p,\sigma}||\psi_a||_{q,\sigma}.$$

$\square$

Theorem 5.2.6. *Let $\psi \in L^2_\sigma(I)$. Then, for any $f, g \in L^2_\sigma(I)$, the following Parseval-Goldstein formula holds true for the fractional Bessel wavelet transform given by Definition 5.1.3:*

$$\int_0^\infty \int_0^\infty B_\psi f(b,a)\ \overline{B_\psi g(b,a)}\ a^{-2\mu-1}\ \mathrm{d}\sigma(b)\ \mathrm{d}\sigma(a)$$
$$= C_{\psi,\alpha}\langle f, g\rangle, \tag{5.2.8}$$

where

$$C_{\psi,\alpha} := \int_0^\infty \left|(h_{\mu,\alpha}\psi)(aw)\right|^2\ a^{-2\mu-1}\ \mathrm{d}\sigma(a) < \infty. \tag{5.2.9}$$

Proof. From [32, p. 245], in conjunction with Theorem 5.2.2 and (5.2.11), we have

$$\int_0^\infty \int_0^\infty B_\psi f(b,a)\ \overline{B_\psi g(b,a)}\ a^{-2\mu-1}\ \mathrm{d}\sigma(b)\ \mathrm{d}\sigma(a)$$
$$= \int_0^\infty \left(\int_0^\infty B_\psi f(b,a)\overline{B_\psi g(b,a)} d\sigma(b)\right)\ a^{-2\mu-1}\ \mathrm{d}\sigma(a)$$
$$= \int_0^\infty \Bigg(\int_0^\infty h^{-1}_{\mu,\alpha}\Big[(h_{\mu,\alpha}f)(w)\ (\overline{h_{\mu,\alpha}\psi})(aw)\Big](b)$$
$$\times\, h^{-1}_{\mu,\alpha}\Big[(\overline{h_{\mu,\alpha}g})(w)\ (h_{\mu,\alpha}\psi)(aw)\Big](b)\mathrm{d}\sigma(b)\Bigg)\ a^{-2\mu-1}\ \mathrm{d}\sigma(a). \tag{5.2.10}$$

Now, by using the Parseval-Goldstein formula (4.3.1) for the fractional Hankel transform in the $L^2_\sigma(I)$ sense, we find from (5.2.10) that

$$\int_0^\infty \int_0^\infty B_\psi f(b,a)\ \overline{B_\psi g(b,a)}\ a^{-2\mu-1}\ \mathrm{d}\sigma(b)\ \mathrm{d}\sigma(a)$$

$$= \int_0^\infty \left(\int_0^\infty (h_{\mu,\alpha} f)(w) \big(\overline{h_{\mu,\alpha}\psi}\big)(aw) \big(\overline{h_{\mu,\alpha} g}\big)(w)\ (h_{\mu,\alpha}\psi)(w) \right.$$

$$\left. \times \Big(\frac{w^{\frac{1}{\alpha}-1}}{\alpha}\Big) \mathrm{d}\sigma(w) \right) a^{-2\mu-1}\ \mathrm{d}\sigma(a)$$

$$= \int_0^\infty \left(\int_0^\infty |(h_{\mu,\alpha}\psi)(aw)|^2\ a^{-2\mu-1}\ \mathrm{d}\sigma(a) \right) (h_{\mu,\alpha} f)(w) \big(\overline{h_{\mu,\alpha} g}\big)(w)$$

$$\times \left(\frac{w^{\frac{1}{\alpha}-1}}{\alpha} \right)\ \mathrm{d}\sigma(w)$$

$$= C_{\psi,\alpha} \left\langle \frac{w^{\frac{1}{\alpha}-1}}{\alpha}\ (h_{\mu,\alpha} f)(w), \big(h_{\mu,\alpha} g\big)(w) \right\rangle,$$

which, by applying (4.3.1) in the $L^2_\sigma(I)$ sense, we get

$$\int_0^\infty \int_0^\infty B_\psi f(b,a)\ \overline{B_\psi g(b,a)}\ a^{-2\mu-1}\ \mathrm{d}\sigma(b)\ \mathrm{d}\sigma(a)$$

$$= C_{\psi,\alpha} \langle f, g \rangle. \tag{5.2.11}$$

□

Theorem 5.2.7. *Let $\psi \in L^2_\sigma(I)$. Then a signal $f \in L^2_\sigma(I)$ can be reconstructed by means of the following inversion formula:*

$$f(t) = \frac{1}{C_{\psi,\alpha}} \int_0^\infty \int_0^\infty B_\psi f(b,a)\ \psi_{b,a^{\frac{1}{\alpha}}}(t)\ a^{-2\mu-1}\ \mathrm{d}\sigma(b)\ \mathrm{d}\sigma(a), \tag{5.2.12}$$

where $C_{\psi,\alpha}$ is given by (5.2.9) and $0 < \alpha \leq 1$.

Proof. Let $f, g \in L^2_\sigma(I)$, then from the Parseval-Goldstein formula (5.2.11) for the fractional Bessel wavelet transform, we have

$$C_{\psi,\alpha}\langle f, g\rangle = \langle B_\psi f(b,a),\quad B_\psi g(b,a)\rangle$$

$$= \int_0^\infty \int_0^\infty B_\psi f(b,a) \overline{B_\psi g(b,a)} \frac{d\sigma(b) d\sigma(a)}{a^{2\mu+1}}.$$

We obtain the following expression after using (5.1.4)

$$\begin{aligned} C_{\psi,\alpha}\langle f, g\rangle &= \int_0^\infty \int_0^\infty B_\psi f(b,a)\Big(\int_0^\infty \overline{g(t)}\psi_{b,a^{\frac{1}{\alpha}}}(t)d\sigma(t)\Big)\frac{d\sigma(b)d\sigma(a)}{a^{2\mu+1}} \\ &= \int_0^\infty \Big(\int_0^\infty \int_0^\infty B_\psi f(b,a)\psi_{b,a^{\frac{1}{\alpha}}}(t)\frac{d\sigma(b)d\sigma(a)}{a^{2\mu+1}}\Big)\overline{g(t)}d\sigma(t). \end{aligned}$$

By the definition of inner product, we get

$$C_{\psi,\alpha}\langle f, g\rangle \ = \ \Big\langle \int_0^\infty \int_0^\infty B_\psi f(b,a)\psi_{b,a^{\frac{1}{\alpha}}}(t)\frac{d\sigma(b)d\sigma(a)}{a^{2\mu+1}}, \ \ g\Big\rangle,$$

for all $g \in L^2_\sigma(I)$. Then, we have

$$f(t) = \frac{1}{C_{\psi,\alpha}}\int_0^\infty \int_0^\infty B_\psi f(b,a)\psi_{b,a^{\frac{1}{\alpha}}}(t)\frac{d\sigma(b)d\sigma(a)}{a^{2\mu+1}}.$$

□

Theorem 5.2.8. *Let $f, \psi \in L^2_\sigma(I)$, then the following Calderon reproducing formula is obtained*

$$f(x) = \frac{1}{C_{\psi,\alpha}}\int_0^\infty (f\#\bar{\psi}_a\#\psi_a)(x)\frac{d\sigma(a)}{a^{2\mu+1}}, \tag{5.2.13}$$

where $C_{\psi,\alpha}$ is given by (5.2.9).

Proof. For $f, \psi \in L^2_\sigma(I)$, then in view of (5.1.1) and (5.1.2), we get

$$\begin{aligned} \int_0^\infty \int_0^\infty B_\psi f(b,a)\psi_{b,a^{\frac{1}{\alpha}}}(x)\frac{d\sigma(b)d\sigma(a)}{a^{2\mu+1}} &= \int_0^\infty \int_0^\infty B_\psi f(b,a)a^{-2\mu-\frac{1}{\alpha}} \\ &\quad \times \psi_\alpha\left(\frac{x}{a^{\frac{1}{\alpha}}}, \frac{b}{a^{\frac{1}{\alpha}}}\right)\frac{d\sigma(b)d\sigma(a)}{a^{2\mu+1}}. \end{aligned}$$

Again taking (5.1.3), we find

$$\int_0^\infty\int_0^\infty B_\psi f(b,a)\psi_{b,a^{\frac{1}{\alpha}}}(x)\frac{d\sigma(b)d\sigma(a)}{a^{2\mu+1}} = \int_0^\infty\int_0^\infty B_\psi f(b,a)a^{-2\mu-\frac{1}{\alpha}} \times\Big(\int_0^\infty \psi(z)D_\alpha(\frac{x}{a^{\frac{1}{\alpha}}},\frac{b}{a^{\frac{1}{\alpha}}},z)d\sigma(z)\Big) \times\frac{d\sigma(b)d\sigma(a)}{a^{2\mu+1}}.$$

By using (4.2.12), we get

$$\int_0^\infty\int_0^\infty B_\psi f(b,a)\psi_{b,a^{\frac{1}{\alpha}}}(x)\frac{d\sigma(b)d\sigma(a)}{a^{2\mu+1}} = \int_0^\infty\int_0^\infty B_\psi f(b,a)a^{-2\mu-\frac{1}{\alpha}}\Big(\int_0^\infty\psi(z) \times\Big(\int_0^\infty j(w^{\frac{1}{\alpha}}\frac{x}{a^{\frac{1}{\alpha}}})j(w^{\frac{1}{\alpha}}\frac{b}{a^{\frac{1}{\alpha}}})j(w^{\frac{1}{\alpha}}z) \times\frac{w^{\frac{1}{\alpha}-1}}{\alpha}d\sigma(w)\Big)d\sigma(z)\Big)\frac{d\sigma(b)d\sigma(a)}{a^{2\mu+1}}.$$

From the Fubini's theorem, we get

$$\int_0^\infty\int_0^\infty B_\psi f(b,a)\psi_{b,a^{\frac{1}{\alpha}}}(x)\frac{d\sigma(b)d\sigma(a)}{a^{2\mu+1}} = \int_0^\infty\int_0^\infty B_\psi f(b,a)a^{-2\mu-\frac{1}{\alpha}}\Big(\int_0^\infty j(w^{\frac{1}{\alpha}}\frac{x}{a^{\frac{1}{\alpha}}}) \times j(w^{\frac{1}{\alpha}}\frac{b}{a^{\frac{1}{\alpha}}})\Big(\int_0^\infty\psi(z)j(w^{\frac{1}{\alpha}}z)d\sigma(z)\Big) \times\frac{w^{\frac{1}{\alpha}-1}}{\alpha}d\sigma(w)\Big)\frac{d\sigma(b)d\sigma(a)}{a^{2\mu+1}}.$$

By the definition of the fractional Hankel transform (4.2.8), we find

$$\begin{aligned}\int_0^\infty\int_0^\infty B_\psi f(b,a)\psi_{b,a^{\frac{1}{\alpha}}}(x)\frac{d\sigma(b)d\sigma(a)}{a^{2\mu+1}} &= \int_0^\infty\int_0^\infty B_\psi f(b,a)a^{-2\mu-\frac{1}{\alpha}} \\ &\times\Big(\int_0^\infty j(w^{\frac{1}{\alpha}}\frac{x}{a^{\frac{1}{\alpha}}})j(w^{\frac{1}{\alpha}}\frac{b}{a^{\frac{1}{\alpha}}})(h_{\mu,\alpha}\psi)(w) \\ &\times\frac{w^{\frac{1}{\alpha}-1}}{\alpha}d\sigma(w)\Big)\frac{d\sigma(b)d\sigma(a)}{a^{2\mu+1}} \\ &= \int_0^\infty\int_0^\infty j(w^{\frac{1}{\alpha}}\frac{x}{a^{\frac{1}{\alpha}}})(h_{\mu,\alpha}\psi)(w)\Big(\int_0^\infty j(w^{\frac{1}{\alpha}}\frac{b}{a^{\frac{1}{\alpha}}})\end{aligned}$$

$$
\begin{aligned}
&\times B_\psi f(b,a)d\sigma(b)\Big)\frac{w^{\frac{1}{\alpha}-1}}{\alpha}d\sigma(w)a^{-2\mu-\frac{1}{\alpha}}\frac{d\sigma(a)}{a^{2\mu+1}}\\
&=\int_0^\infty\int_0^\infty j(w^{\frac{1}{\alpha}}\frac{x}{a^{\frac{1}{\alpha}}})(h_{\mu,\alpha}\psi)(w)\\
&\times h_{\mu,\alpha}(B_\psi f(b,a))(\frac{w}{a})\frac{w^{\frac{1}{\alpha}-1}}{\alpha}d\sigma(w)a^{-2\mu-\frac{1}{\alpha}}\frac{d\sigma(a)}{a^{2\mu+1}}.
\end{aligned}
$$

Substituting $\frac{w}{a}=u$, we get

$$
\begin{aligned}
\int_0^\infty\int_0^\infty B_\psi f(b,a)\psi_{b,a^{\frac{1}{\alpha}}}(x)\frac{d\sigma(b)d\sigma(a)}{a^{2\mu+1}} &= \int_0^\infty\int_0^\infty j(u^{\frac{1}{\alpha}}x)(h_{\mu,\alpha}\psi)(au)\\
&\times h_{\mu,\alpha}(B_\psi f(b,a))(u)\frac{u^{\frac{1}{\alpha}-1}}{\alpha}d\sigma(u)\frac{d\sigma(a)}{a^{2\mu+1}}.
\end{aligned}
$$

From the Theorem 5.2.3, we get

$$
\begin{aligned}
\int_0^\infty\int_0^\infty B_\psi f(b,a)\psi_{b,a^{\frac{1}{\alpha}}}(x)\frac{d\sigma(b)d\sigma(a)}{a^{2\mu+1}} &= \int_0^\infty\int_0^\infty j(u^{\frac{1}{\alpha}}x)h_{\mu,\alpha}(f\#\bar{\psi}_a)(u)\\
&\times (h_{\mu,\alpha}\psi_a)(u)\frac{u^{\frac{1}{\alpha}-1}}{\alpha}d\sigma(u)\frac{d\sigma(a)}{a^{2\mu+1}}.
\end{aligned}
$$

Using (4.3.6), the above expression yields

$$
\begin{aligned}
\int_0^\infty\int_0^\infty B_\psi f(b,a)\psi_{b,a^{\frac{1}{\alpha}}}(x)\frac{d\sigma(b)d\sigma(a)}{a^{2\mu+1}} &= \int_0^\infty\Big(\int_0^\infty j(u^{\frac{1}{\alpha}}x)h_{\mu,\alpha}(f\#\bar{\psi}_a\#\psi_a)(u)\\
&\times\frac{u^{\frac{1}{\alpha}-1}}{\alpha}d\sigma(u)\Big)\frac{d\sigma(a)}{a^{2\mu+1}}.
\end{aligned}
$$

Next, with the help of (4.2.10), we obtain

$$
\int_0^\infty\int_0^\infty B_\psi f(b,a)\psi_{b,a^{\frac{1}{\alpha}}}(x)\frac{d\sigma(b)d\sigma(a)}{a^{2\mu+1}} = \int_0^\infty(f\#\bar{\psi}_a\#\psi_a)(x)\frac{d\sigma(a)}{a^{2\mu+1}}.
$$

From the Theorem 5.2.7, we get

$$
f(x)=\frac{1}{C_{\psi,\alpha}}\int_0^\infty(f\#\bar{\psi}_a\#\psi_a)(x)\frac{d\sigma(a)}{a^{2\mu+1}}.
$$

This completes the proof. □

Theorem 5.2.9. *Let $f, \psi \in L^1_\sigma(I) \cap L^2_\sigma(I)$ and $h_{\mu,\alpha} f,\ h_{\mu,\alpha}\psi \in L^1_\sigma(I)$, then by assuming $C_{\psi,\alpha} = 1$, we have following expression*

$$f(x) = \int_0^\infty (f \# \bar{\psi}_a \# \psi_a)(x) \frac{d\sigma(a)}{a^{2\mu+1}}. \tag{5.2.14}$$

Proof.

$$\begin{aligned}
h_{\mu,\alpha}\Big(\int_0^\infty (f \# \bar{\psi}_a \# \psi_a)(x) \frac{d\sigma(a)}{a^{2\mu+1}}\Big)(w) &= \int_0^\infty j(w^{\frac{1}{\alpha}} x)\Big(\int_0^\infty (f \# \bar{\psi}_a \# \psi_a)(x) \frac{d\sigma(a)}{a^{2\mu+1}}\Big) d\sigma(x) \\
&= \int_0^\infty \Big(\int_0^\infty j(w^{\frac{1}{\alpha}} x)(f \# \bar{\psi}_a \# \psi_a)(x) d\sigma(x)\Big) \frac{d\sigma(a)}{a^{2\mu+1}} \\
&= \int_0^\infty h_{\mu,\alpha}\Big((f \# \bar{\psi}_a \# \psi_a)(x)\Big)(w) \frac{d\sigma(a)}{a^{2\mu+1}} \\
&= \int_0^\infty (h_{\mu,\alpha} f)(w) \overline{(h_{\mu,\alpha}\psi_a)(w)} (h_{\mu,\alpha}\psi_a)(w) \frac{d\sigma(a)}{a^{2\mu+1}} \\
&= (h_{\mu,\alpha} f)(w) \int_0^\infty \overline{(h_{\mu,\alpha}\psi)(aw)} (h_{\mu,\alpha}\psi)(aw) \frac{d\sigma(a)}{a^{2\mu+1}} \\
&= (h_{\mu,\alpha} f)(w) \int_0^\infty |(h_{\mu,\alpha}\psi)(aw)|^2 \frac{d\sigma(a)}{a^{2\mu+1}}.
\end{aligned}$$

Since

$$C_{\psi,\alpha} = \int_0^\infty \ |(h_{\mu,\alpha}\psi)(aw)|^2 \ a^{-2\mu-1} \ \mathrm{d}\sigma(a) = 1,$$

therefore by using the inversion formula for the fractional Hankel transform (4.2.10), we have

$$f(x) = \int_0^\infty (f \# \bar{\psi}_a \# \psi_a)(x) \frac{d\sigma(a)}{a^{2\mu+1}}.$$

□

Theorem 5.2.10. *Let $\psi \in L^2_\sigma(I)$. Then the discrete fractional Bessel wavelet transform of a signal $f \in L^2_\sigma(I)$ is given by*

$$B_\psi f(m,n) = \int_0^\infty f(t)\, \overline{\psi_{\alpha,m,n}}(t)\, \mathrm{d}\sigma(t), \qquad (0 < \alpha \leq 1), \tag{5.2.15}$$

where

$$\psi_{\alpha,m,n}(t) = a_0^{-m\left(2\mu+\frac{1}{\alpha}\right)} \psi_\alpha\Big(a_0^{-\frac{m}{\alpha}}\, t, nb_0\Big). \tag{5.2.16}$$

Proof. From (5.1.4), the continuous fractional Bessel wavelet transform is given by

$$B_\psi f(b,a) = \int_0^\infty f(t)\, \overline{\psi_{b,a^{\frac{1}{\alpha}}}(t)}\, \mathrm{d}\sigma(t),$$

where

$$\psi_{b,a^{\frac{1}{\alpha}}}(t) = a^{-2\mu-\frac{1}{\alpha}} \psi_\alpha\left(\frac{t}{a^{\frac{1}{\alpha}}}, \frac{b}{a^{\frac{1}{\alpha}}}\right).$$

For discretizing the parameters a and b, we set

$$a = a_0^m \text{ and } b = a_0^{\frac{m}{\alpha}} nb_0, \quad (m,n \in \mathbb{Z}) \;\; a_0 > 1 \text{ and } b_0 > 0.$$

Then above expressions becomes

$$B_\psi f(m,n) = \int_0^\infty f(t)\, \overline{\psi_{\alpha,m,n}(t)}\, \mathrm{d}\sigma(t),$$

where

$$\begin{aligned} \psi_{\alpha,m,n}(t) &= (a_0^m)^{-2\mu-\frac{1}{\alpha}} \psi_\alpha\left(\frac{t}{a_0^{\frac{m}{\alpha}}}, \frac{a_0^{\frac{m}{\alpha}} nb_0}{a_0^{\frac{m}{\alpha}}}\right) \\ &= a_0^{-m\left(2\mu+\frac{1}{\alpha}\right)} \psi_\alpha\Big(a_0^{-\frac{m}{\alpha}}\, t, nb_0\Big). \end{aligned}$$

□

5.3 Application of the fractional Bessel wavelet transform in Weighted Sobolev type space

In this section, with the help of [36], we are giving applications of the fractional Bessel wavelet transform in weighted Sobolev type space by exploiting the theory of the fractional Hankel transform.

Definition 5.3.1. The convolution product for the fractional Bessel wavelet transform is formally defined by

$$B_\psi(f \otimes g)(b,a) = (B_\psi f)(b,a)(B_\psi g)(b,a). \tag{5.3.1}$$

Now, the relation between the convolution product for the fractional Bessel wavelet transform (5.3.1) and the fractional Hankel convolution (4.2.13) is given below:

Lemma 5.3.2. *Let $f, g, \psi \in L^1_\sigma(I)$, then*

$$\begin{aligned}&\left(\overline{h_{\mu,\alpha}\psi}\right)(aw)(h_{\mu,\alpha}(f\otimes g))(w) = \\ &\quad\Big(\left(\overline{h_{\mu,\alpha}\psi}\right)(a.)(h_{\mu,\alpha}f)(.)\#\left(\overline{h_{\mu,\alpha}\psi}\right)(a.)(h_{\mu,\alpha}g)(.)\Big)(w).\end{aligned} \tag{5.3.2}$$

Proof. In order to prove Lemma 5.3.2, we find from (5.2.3) that

$$h_{\mu,\alpha}\Big(B_\psi(f\otimes g)(b,a)\Big)(w) = \left(\overline{h_{\mu,\alpha}\psi}\right)(aw)(h_{\mu,\alpha}(f\otimes g))(w),$$

which by virtue of (5.3.1), leads us that

$$\left(\overline{h_{\mu,\alpha}\psi}\right)(aw)(h_{\mu,\alpha}(f\otimes g))(w) = h_{\mu,\alpha}\big((B_\psi f)(b,a)(B_\psi g)(b,a)\big)(w).$$

Now, from (5.2.4), we find that

$$\left(\overline{h_{\mu,\alpha}\psi}\right)(aw)(h_{\mu,\alpha}(f\otimes g))(w) = h_{\mu,\alpha}\Big(h^{-1}_{\mu,\alpha}\Big(\left(\overline{h_{\mu,\alpha}\psi}\right)(aw)(h_{\mu,\alpha}f)(w)\Big)(b)$$
$$\times\, h^{-1}_{\mu,\alpha}\Big(\left(\overline{h_{\mu,\alpha}\psi}\right)(aw)(h_{\mu,\alpha}g)(w)\Big)(b)\Big)(w)$$

Finally, by applying (4.3.6), we find

$$\left(\overline{h_{\mu,\alpha}\psi}\right)(aw)(h_{\mu,\alpha}(f\otimes g))(w) = \Big(\left(\overline{h_{\mu,\alpha}\psi}\right)(a.)(h_{\mu,\alpha}f)(.)\#\left(\overline{h_{\mu,\alpha}\psi}\right)(a.)$$
$$\times\,(h_{\mu,\alpha}g)(.)\Big)(w),$$

which evidently completes our demonstration of Lemma 5.3.2. □

Next, motivated by the developments in the earlier work [37, p.142], Equation (1.5), we give the following definition of a weighted Sobolev space.

Definition 5.3.3. Let $k(w)$ be an arbitrary weight function and suppose that $H^{'}_{\mu}(I)$ is the dual of the Zemanian space $H_{\mu}(I)$ for $I=(0,\infty)$. Then a function $\phi\in H^{'}_{\mu}(I)$ is said to belong to the weighted Sobolev space $G^{p}_{\mu,k}(I)$, for $\mu\in\mathbb{R}$ and $1\le p<\infty$, if its fractional Hankel transform $h_{\mu,\alpha}\phi$ corresponding to locally integrable function ϕ over $I=(0,\infty)$ satisfies the following norm:

$$||\phi||_{p,\mu,\sigma,k} = \left(\int_0^\infty |k(w)(h_{\mu,\alpha}\phi)(w)|^p d\sigma(w)\right)^{\frac{1}{p}} < \infty. \tag{5.3.3}$$

In what follows, we first set

$$k(w) = \left(\overline{h_{\mu,\alpha}\psi}\right)(aw),$$

for fixed $a>0$, and we then establish the following result.

Theorem 5.3.4. *Let $f \in G^1_{\mu,k}(I)$, $g \in G^p_{\mu,k}(I)$ and $1 \leq p < \infty$. Then*

$$||f \otimes g||_{p,\mu,\sigma,k} \leq ||f||_{1,\mu,\sigma,k}||g||_{p,\mu,\sigma,k}. \tag{5.3.4}$$

Proof. In view of (5.3.1) and (5.3.3), we have

$$\begin{aligned} ||f \otimes g||_{p,\mu,\sigma,k} &= \Big(\int_0^\infty |k(w)(h_{\mu,\alpha}(f \otimes g))(w)|^p d\sigma(w)\Big)^{\frac{1}{p}} \\ &= \Big(\int_0^\infty | \left(\overline{h_{\mu,\alpha}\psi}\right)(aw)(h_{\mu,\alpha}(f \otimes g))(w)|^p d\sigma(w)\Big)^{\frac{1}{p}}. \end{aligned} \tag{5.3.5}$$

Now, by using (5.3.2), (5.3.5) becomes

$$\begin{aligned} ||f \otimes g||_{p,\mu,\sigma,k} = \Big(\int_0^\infty &|\big(\left(\overline{h_{\mu,\alpha}\psi}\right)(a.)(h_{\mu,\alpha}f)(.) \\ &\# \left(\overline{h_{\mu,\alpha}\psi}\right)(a.)(h_{\mu,\alpha}g)(.)\big)(w)|^p d\sigma(w)\Big)^{\frac{1}{p}}, \end{aligned}$$

and, in view of (4.3.3), we find that

$$\begin{aligned} ||f \otimes g||_{p,\mu,\sigma,k} \leq &|| \left(\overline{h_{\mu,\alpha}\psi}\right)(a.)(h_{\mu,\alpha}f)(.)||_{1,\sigma} \\ &\times || \left(\overline{h_{\mu,\alpha}\psi}\right)(a.)(h_{\mu,\alpha}g)(.)||_{p,\sigma}. \end{aligned}$$

Finally, by making use of (5.3.3), we obtain

$$||f \otimes g||_{p,\mu,\sigma,k} \leq ||f||_{1,\mu,\sigma,k}||g||_{p,\mu,\sigma,k},$$

which proves Theorem 5.3.4. □

Theorem 5.3.5. *Let $f \in G^p_{\mu,k}(I)$ and $g \in G^q_{\mu,k}(I)$, $1 \leq p, q < \infty$ and*

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1.$$

Then

$$||f \otimes g||_{r,\mu,\sigma,k} \leq ||f||_{p,\mu,\sigma,k}||g||_{q,\mu,\sigma,k}. \tag{5.3.6}$$

Proof. The proof of Theorem 5.3.5 follows from (5.3.3) and (4.3.4). □

5.4 Application of the fractional Bessel wavelet transform in integral equation related to time-invariant linear filter

Motivated from the results of [34] and [35], in this section we give applications of the fractional Bessel wavelet transform in time-invariant linear filter and integral equation by exploiting the technique of the fractional Hankel transform.

Definition 5.4.1. Let $f, \bar{f} : \mathbb{R} \to \mathbb{C}$ be signals which are piecewise continuous. Then a filter is a transformation L which maps a signal f into another signal $\bar{f}$.

Filter L is said to be linear if it must satisfy the following properties for any signals f, g and $c \in \mathbb{C}$,

(i) Additivity: $L[f + g] = L[f] + L[g]$.

(ii) Homogeneity: $L[cf] = cL[f]$.

A filter L is said to be time-invariant if for any signal f and non-negative real number a such that

$$L[\tau_a f](t) = [\tau_a(Lf)](t), \tag{5.4.1}$$

where $\tau_a f$ is the fractional Hankel translation of f (defined in (4.2.11)) and is given by

$$(\tau_a f)(t) \equiv f_\alpha(t,a) = \int_0^\infty f(z) D_\alpha(t,a,z) d\sigma(z)$$

and

$$D_\alpha(x,y,z) = \int_0^\infty j(w^{\frac{1}{\alpha}}x) j(w^{\frac{1}{\alpha}}y) j(w^{\frac{1}{\alpha}}z) \frac{w^{\frac{1}{\alpha}-1}}{\alpha} d\sigma(w).$$

Theorem 5.4.2. *Suppose L be a time-invariant linear transformation and w be any fixed non- negative real number, then there exists a function $g \in L^1_\sigma(I)$ such that*

$$L\big(j(w^{\frac{1}{\alpha}}x)\big) = (h_{\mu,\alpha} g)(w) j(w^{\frac{1}{\alpha}}x). \tag{5.4.2}$$

Proof. Let

$$g^w_\alpha(x) = L(j(w^{\frac{1}{\alpha}}x)). \tag{5.4.3}$$

Since L is time-invariant, then for any non-negative real number a such that

$$L(j(w^{\frac{1}{\alpha}}x) j(w^{\frac{1}{\alpha}}a)) = g^w_\alpha(x,a). \tag{5.4.4}$$

If w is fixed non-negative real number then by using the linearity property of L, we obtain

$$L(j(w^{\frac{1}{\alpha}}x) j(w^{\frac{1}{\alpha}}a)) = j(w^{\frac{1}{\alpha}}a) L(j(w^{\frac{1}{\alpha}}x)). \tag{5.4.5}$$

From (5.4.3), (5.4.4) and (5.4.5), we get

$$g^w_\alpha(x,a) = j(w^{\frac{1}{\alpha}}a) g^w_\alpha(x). \tag{5.4.6}$$

Let $x = 0$, (5.4.6) becomes

$$g_\alpha^w(0, a) = j(w^{\frac{1}{\alpha}} a) g_\alpha^w(0).$$

Therefore

$$g_\alpha^w(a) = j(w^{\frac{1}{\alpha}} a) g_\alpha^w(0). \tag{5.4.7}$$

Since a is arbitrary, so we take $a = x$ in (5.4.7), we get

$$g_\alpha^w(x) = j(w^{\frac{1}{\alpha}} x) g_\alpha^w(0). \tag{5.4.8}$$

If we take $g_\alpha^w(0) = h_{\mu,\alpha} g \in L_\sigma^1(I)$ and $g \in L_\sigma^1(I)$, then (5.4.8) becomes

$$L(j(w^{\frac{1}{\alpha}} x)) = (h_{\mu,\alpha} g)(w) j(w^{\frac{1}{\alpha}} x).$$

□

Theorem 5.4.3. *Let X be the space of piecewise continuous signals and L be a time-invariant linear transformation on X. Then there exists an integrable function $g \in L_\sigma^1(I)$ such that*

$$L(f) = f \# g, \tag{5.4.9}$$

for all $f \in X \cap L_\sigma^1(I)$.

Proof. Using (4.2.10), we can write

$$f(x) = \int_0^\infty j(w^{\frac{1}{\alpha}} x)(h_{\mu,\alpha} f)(w) \frac{w^{\frac{1}{\alpha}-1}}{\alpha} d\sigma(x).$$

Then by definition of linear filter, we have

$$(Lf)(x) = L\Big(\int_0^\infty j(w^{\frac{1}{\alpha}}x)(h_{\mu,\alpha}f)(w)\frac{w^{\frac{1}{\alpha}-1}}{\alpha}d\sigma(w)\Big). \tag{5.4.10}$$

The integral of the right side of (5.4.10) can be approximated by the Riemann sum

$$L\Big(\int_0^\infty j(w^{\frac{1}{\alpha}}x)(h_{\mu,\alpha}f)(w)\frac{w^{\frac{1}{\alpha}-1}}{\alpha}d\sigma(w)\Big) \approx L\Big(\sum_k j(w_k^{\frac{1}{\alpha}}x)(h_{\mu,\alpha}f)(w_k)\frac{w_k^{\frac{1}{\alpha}-1}}{\alpha}\Delta w\Big).$$

Therefore right-hand side of the above expression yields

$$L\Big(\sum_k j(w_k^{\frac{1}{\alpha}}x)(h_{\mu,\alpha}f)(w_k)\frac{w_k^{\frac{1}{\alpha}-1}}{\alpha}\Delta w\Big) = \sum_k L\Big(j(w_k^{\frac{1}{\alpha}}x)\Big)(h_{\mu,\alpha}f)(w_k)\frac{w_k^{\frac{1}{\alpha}-1}}{\alpha}\Delta w.$$

By the definition of the Riemann sum, the right-hand side of the above expression can be written in form of integral

$$(Lf)(x) = \int_0^\infty (h_{\mu,\alpha}f)(w)L\Big(j(w^{\frac{1}{\alpha}}x)\Big)\frac{w^{\frac{1}{\alpha}-1}}{\alpha}d\sigma(w).$$

By using the Theorem 5.4.2, we get

$$(Lf)(x) = \int_0^\infty (h_{\mu,\alpha}f)(w)(h_{\mu,\alpha}g)(w)j(w^{\frac{1}{\alpha}}x)\frac{w^{\frac{1}{\alpha}-1}}{\alpha}d\sigma(w).$$

Taking (4.3.6), we have

$$(Lf)(x) = \int_0^\infty (h_{\mu,\alpha}(f\#g))(w)j(w^{\frac{1}{\alpha}}x)\frac{w^{\frac{1}{\alpha}-1}}{\alpha}d\sigma(w).$$

Therefore, finally from (4.2.10), we obtain

$$(Lf)(x) = (f\#g))(x).$$

□

Example 5.1. *Let $\phi(x)$ be a function of compact support. For a signal f, let*

$$(Lf)(t) = (f\#\phi)(t) = \int_0^\infty f_\alpha(t,y)\phi(y)d\sigma(y). \tag{5.4.11}$$

Then the linear operator L is time-invariant.

Proof. By using (4.2.11) and (4.2.13), we get

$$\begin{aligned}
[\tau_a(Lf)](t) &= \int_0^\infty (Lf)(z)D_\alpha(t,a,z)d\sigma(z)\\
&= \int_0^\infty \Big(\int_0^\infty \phi_\alpha(z,x)f(x)d\sigma(x)\Big)D_\alpha(t,a,z)d\sigma(z)\\
&= \int_0^\infty \Big(\int_0^\infty \Big(\int_0^\infty \phi(y)D_\alpha(z,x,y)d\sigma(y)\Big)f(x)d\sigma(x)\Big)\\
&\quad \times D_\alpha(t,a,z)d\sigma(z).
\end{aligned}$$

With the help of the Fubini's theorem, we have

$$\begin{aligned}
[\tau_a(Lf)](t) &= \int_0^\infty \Big(\int_0^\infty \Big(\int_0^\infty f(x)D_\alpha(z,x,y)d\sigma(x)\Big)\phi(y)d\sigma(y)\Big)\\
&\quad \times D_\alpha(t,a,z)d\sigma(z).
\end{aligned}$$

Again, by using (4.2.11), we get

$$[\tau_a(Lf)](t) = \int_0^\infty \Big(\int_0^\infty f_\alpha(z,y)\phi(y)d\sigma(y)\Big)D_\alpha(t,a,z)d\sigma(z).$$

Next, from the Fubini's theorem, above yields

$$[\tau_a(Lf)](t) = \int_0^\infty \Big(\int_0^\infty f_\alpha(z,y)D_\alpha(t,a,z)d\sigma(z)\Big)\phi(y)d\sigma(y).$$

Now, in view of (4.2.11), we have

$$[\tau_a(Lf)](t) = \int_0^\infty f_\alpha(t,a,y)\phi(y)d\sigma(y)$$
$$= \int_0^\infty (\tau_a f)(t,y)\phi(y)d\sigma(y).$$

From (4.2.13) and (5.4.11), we get

$$[\tau_a(Lf)](t) = (\tau_a f \# \phi)(t)$$
$$= [L(\tau_a f)](t).$$

This proves that L is time-invariant. □

Physical interpretation. Assuming that $g(t)$(given in Theorem 5.4.3) is continuous and integrable function on $I = (0,\infty)$ and ϵ is a small positive real number. We define the impulse signal

$$f_\epsilon(t) = \begin{cases} \frac{2^{\mu-1/2}\Gamma(\mu+1/2)(2\mu+1)}{\epsilon^{2\mu+1}}, & t \leq \epsilon, \ \mu > -1/2, \\ 0, & otherwise. \end{cases} \tag{5.4.12}$$

Then, we have $\int_0^\epsilon f_\epsilon(t)d\sigma(t) = 1$. For ϵ is small, f_ϵ represents a strong signal but only lasts a short period of time (such as sound signal generated by hammer blow). Next, we apply L to f_ϵ and using (5.4.9), we obtain the following expression

$$(Lf_\epsilon)(x) = \int_0^\infty g_\alpha(x,y)f_\epsilon(y)d\sigma(y)$$
$$= \int_0^\epsilon g_\alpha(x,y)f_\epsilon(y)d\sigma(y).$$

Since g is continuous, $g_\alpha(x,y)$ is approximately equal to $g(x)$ for $y \leq \epsilon$. Therefore

$$(Lf_\epsilon)(x) \approx g(x)\int_0^\epsilon f_\epsilon(y)d\sigma(y) = g(x).$$

Thus, $g(x)$ is the approximate response from applying L to an input signal f_ϵ that is an impulse on half-plane. For that reason, $g(x)$ is called the impulse response function.

In Theorem 5.4.2, $(h_{\mu,\alpha}g)(w)$ is the amplitude of the response to a "pure frequency" signal $j(w^{\frac{1}{\alpha}}x)$, so $(h_{\mu,\alpha}g)$ is called the system function.

Theorem 5.4.4. *Let $f, \psi_a \in L^1_\sigma(I)$ and L be a time-invariant linear transformation, then the fractional Bessel wavelet transform can be expressed as*

$$(B_\psi f)(t,a) = (Lf)(t). \tag{5.4.13}$$

Proof. By using (5.2.5), we have

$$B_\psi f(t,a) = (f\#\bar{\psi}_a)(t).$$

With the help of (5.2.1) and (5.4.2), we get

$$\begin{aligned} B_\psi f(t,a) &= \int_0^\infty (h_{\mu,\alpha}f)(w)(h_{\mu,\alpha}\bar{\psi}_a)(w)j(w^{\frac{1}{\alpha}}t)\frac{w^{\frac{1}{\alpha}-1}}{\alpha}d\sigma(w) \\ &= \int_0^\infty (h_{\mu,\alpha}f)(w)L(j(w^{\frac{1}{\alpha}}t))\frac{w^{\frac{1}{\alpha}-1}}{\alpha}d\sigma(w). \end{aligned}$$

Using the tool of Riemann sum, we get

$$\begin{aligned} B_\psi f(t,a) &\approx \sum_k (h_{\mu,\alpha}f)(w_k)L(j(w_k^{\frac{1}{\alpha}}t))\frac{w_k^{\frac{1}{\alpha}-1}}{\alpha}\Delta w \\ &\approx L\Big(\sum_k (h_{\mu,\alpha}f)(w_k)j(w_k^{\frac{1}{\alpha}}t)\frac{w_k^{\frac{1}{\alpha}-1}}{\alpha}\Delta w\Big) \end{aligned}$$

$$= L\Big(\int_0^\infty (h_{\mu,\alpha}f)(w)j(w^{\frac{1}{\alpha}}t)\frac{w^{\frac{1}{\alpha}-1}}{\alpha}d\sigma(w)\Big)$$

By using the definition of inverse fractional Hankel transform (4.2.10), above expression becomes

$$B_\psi f(t,a) = (Lf)(t),$$

which evidently completes the proof of Theorem 5.4.4. □

Theorem 5.4.5. *The Fredholm integral equation is given by*

$$\int_0^\infty f(t)g\Big(\frac{t}{a^{\frac{1}{\alpha}}},\frac{b}{a^{\frac{1}{\alpha}}}\Big)d\sigma(t) + \Lambda f(b) = v(b), \tag{5.4.14}$$

can be written in the form of linear time-invariant filter

$$(Lf)(b) + \Lambda f(b) = v(b). \tag{5.4.15}$$

Proof. Let

$$g\Big(\frac{t}{a^{\frac{1}{\alpha}}},\frac{b}{a^{\frac{1}{\alpha}}}\Big) = a^{-2\mu-\frac{1}{\alpha}}\overline{\phi_\alpha\left(\frac{t}{a^{\frac{1}{\alpha}}},\frac{b}{a^{\frac{1}{\alpha}}}\right)},$$

then (5.4.14) becomes

$$a^{-2\mu-\frac{1}{\alpha}}\int_0^\infty f(t)\overline{\phi_\alpha\left(\frac{t}{a^{\frac{1}{\alpha}}},\frac{b}{a^{\frac{1}{\alpha}}}\right)}d\sigma(t) + \Lambda f(b) = v(b).$$

By using (5.1.4), we have

$$(B_\psi f)(b,a) + \Lambda f(b) = v(b).$$

From Theorem 5.4.4, we get

$$(Lf)(b) + \Lambda f(b) = v(b).$$

□

Theorem 5.4.6. *Let $f \in L^1_\sigma(I)$, then the solution of (5.4.15) can be obtained by*

$$f(x) = \int_0^\infty j(w^{\frac{1}{\alpha}}x)\left(\frac{(h_{\mu,\alpha}v)(w)}{(h_{\mu,\alpha}\bar{\psi})(aw) + \Lambda}\right)\frac{w^{\frac{1}{\alpha}-1}}{\alpha}d\sigma(w). \tag{5.4.16}$$

Proof. Taking the fractional Hankel transform of (5.4.15), we have

$$h_{\mu,\alpha}((Lf)(b))(w) + \Lambda h_{\mu,\alpha}(f(b))(w) = h_{\mu,\alpha}(v(b))(w).$$

By using Theorem 5.4.4 and Theorem 5.2.1, the above yields

$$(h_{\mu,\alpha}f)(w)(h_{\mu,\alpha}\bar{\psi})(aw) + \Lambda(h_{\mu,\alpha}f)(w) = (h_{\mu,\alpha}v)(w).$$

Therefore

$$(h_{\mu,\alpha}f)(w) = \left(\frac{(h_{\mu,\alpha}v)(w)}{(h_{\mu,\alpha}\bar{\psi})(aw) + \Lambda}\right). \tag{5.4.17}$$

Taking the inverse fractional Hankel transform on both sides of (5.4.17), we get

$$f(x) = \int_0^\infty j(w^{\frac{1}{\alpha}}x)\left(\frac{(h_{\mu,\alpha}v)(w)}{(h_{\mu,\alpha}\bar{\psi})(aw) + \Lambda}\right)\frac{w^{\frac{1}{\alpha}-1}}{\alpha}d\sigma(w).$$

□

5.5 Conclusions

In the present chapter, the theory of fractional Bessel wavelet transform is introduced, and discussed its boundedness properties by taking the theory of the fractional Hankel transform. Time-invariant linear filter and its various properties are discussed in the present chapter by using the concepts of the fractional Hankel transform. Author also showed that time-invariant linear filter can be written in the form of the fractional Bessel wavelet transform. An application of the fractional Bessel wavelet transform related to linear time-invariant filter, in integral equation is given. Various properties of the fractional Bessel wavelet transform associated with certain weighted Sobolev-type space are done by using by using the technique of the fractional Hankel transform. This theory is useful to study the various properties of Riemann-Liouville fractional derivatives and integral operators and other problems of fractional calculus.

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