# Chapter 4

# The Fractional Hankel Transform and its Properties

## 4.1 Introduction

Recently, the fractional Fourier transform of the real order $\alpha$ was introduced and studied by Luchko *et al.* [3]. This transform plays the same role for the fractional derivatives as the Fourier transform does for the ordinary derivatives. Moreover, in the case when $\alpha = 1$, the fractional Fourier transform reduces to the Fourier transform in the usual sense. Several important properties of the fractional Fourier transform including the inversion formula and the operational relations for the fractional derivatives, together with its applications in solving some partial differential equations of fractional order were also given by Luchko *et al.* [3, 4].

Motivated by these theoretical developments, Upadhyay and Khatterwani [29] considered the conventional fractional Hankel transform and presented the relation between a two-dimensional fractional Fourier transform and the fractional Hankel transform in terms of radial functions. They also derived other operational properties of the Hankel transform and the fractional Hankel transform.

The Hankel transform played an important role for solving the problem in mathematics, physics and engineering and also a useful technique in the solution of infinite cylindrical boundary value problems. Many researchers exploited the theory of Hankel transformation and found many important observations in different research works. The theory of Hankel transform was introduced by Haimo [17], Hirshman [30], and Cholewinski [31]. Considering this transform, these researchers investigated integral equations and approximations associated with Hankel convolution.

In this sequel to the above-mentioned developments, our contributions in this chapter are given below:

1. Taking the concepts of [17, 30, 31], with the help of two-dimensional fractional Fourier transform, the fractional Hankel transform and its inversion formula are introduced. The fractional Hankel translation and the fractional Hankel convolution are defined by exploiting the theory of fractional Hankel transform.
2. Parseval formula and convolution property for the fractional Hankel transform are found.
3. Boundedness properties of the fractional Hankel translation and of the fractional Hankel convolution are proved.

Our plan in this chapter is as follows:

In Section 4.2, the fractional Hankel transform and its inversion formula are introduced, with the help of two-dimensional fractional Fourier transform. The fractional Hankel translation and the fractional Hankel convolution are defined. In Section 4.3, the Parseval formula and the properties of fractional Hankel convolution are derived. By using the fractional Hankel convolution theory, boundedness properties for the fractional Hankel translation and the fractional Hankel convolution have been given.

## 4.2 Foundation of the fractional Hankel transform

In this section, from [17, 30, 31], the fractional Hankel transform and its inversion formula are obtained, with the help of two-dimensional fractional Fourier transform. Later on, the fractional Hankel translation and the fractional Hankel convolution are defined.

**Theorem 4.2.1.** *The two-dimensional fractional Fourier transform can be expressed in terms of the fractional Hankel transform as*

$$(\mathcal{F}_\alpha f)(w_1, w_2) = \frac{\rho^{-\frac{1}{2\alpha}}}{\alpha} h_0^\alpha(F(r))(\rho), \text{ for all } w_1, w_2 \in \mathbb{R}, \tag{4.2.1}$$

*where* $F(r) = r^{\frac{1}{2}} f(r)$, $r$ *and* $\rho$ *are radial functions and*

$$h_0^\alpha(F(r))(\rho) = \int_0^\infty (r\rho^{\frac{1}{\alpha}})^{\frac{1}{2}} J_0(\rho^{\frac{1}{\alpha}} r) r^{\frac{1}{2}} f(r) dr, \tag{4.2.2}$$

*for* $r = \sqrt{x_1^2 + x_2^2}$ *and* $\rho = \sqrt{w_1^2 + w_2^2}$.

***Proof.*** Two dimensional fractional Fourier transform of radial function $f$ is given by

$$(\mathcal{F}_\alpha f)(w_1, w_2) = \frac{1}{2\pi\alpha} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{i(x_1 w_1 |w_1|^{\frac{1}{\alpha}-1} + x_2 w_2 |w_2|^{\frac{1}{\alpha}-1})} f\left(\sqrt{x_1^2 + x_2^2}\right) dx_1 dx_2.$$

By taking the substitutions,

$x_1 = rcos\theta,\ x_2 = rsin\theta, w_1 = \rho cos\psi,\ w_2 = \rho sin\psi$, we get

$$(\mathcal{F}_\alpha f)(w_1, w_2) = \frac{1}{2\pi\alpha}\int_0^{2\pi}\int_0^{+\infty} e^{ir\rho^{\frac{1}{\alpha}}\left(cos\theta cos\psi|cos\psi|^{\frac{1}{\alpha}-1}+sin\theta sin\psi|sin\psi|^{\frac{1}{\alpha}-1}\right)} f(r)rd\theta dr.$$

If we take $\psi = 2n\pi, \quad n = \mathbb{N}\cup\{0\}$, then $(w_1, w_2)$ becomes $(\rho, 0)$ with $0 \leq \rho < \infty$ and we get

$$(\mathcal{F}_\alpha f)(w_1, w_2) = \frac{1}{2\pi\alpha}\int_0^{2\pi}\int_0^{+\infty} e^{ircos\theta(\rho^{\frac{1}{\alpha}})} f(r)rd\theta dr.$$

Changing the order of integration by Fubini's theorem and using the formula $\int_0^{2\pi} e^{ircos\theta(\rho^{\frac{1}{\alpha}})}d\theta = 2\pi J_0(r\rho^{\frac{1}{\alpha}})$ from Sneddon [33, p. 62], we get

$$\begin{aligned}(\mathcal{F}_\alpha f)(w_1, w_2) &= \frac{1}{2\pi\alpha}\int_0^{+\infty}\Big(\int_0^{2\pi} e^{ircos\theta(\rho^{\frac{1}{\alpha}})}d\theta\Big)rf(r)dr\\ &= \frac{1}{\alpha}\int_0^{+\infty} J_0(r\rho^{\frac{1}{\alpha}})rf(r)dr\\ &= \frac{\rho^{\frac{-1}{2\alpha}}}{\alpha}\int_0^{+\infty}(r\rho^{\frac{1}{\alpha}})^{\frac{1}{2}}J_0(r\rho^{\frac{1}{\alpha}})r^{\frac{1}{2}}f(r)dr\\ &= \frac{\rho^{-\frac{1}{2\alpha}}}{\alpha}h_0^\alpha(F(r))(\rho).\end{aligned}$$

□

**Corollary 4.2.2.** *Let* $\phi(\rho) = \rho^{-(\frac{n-1}{2\alpha})}(\mathcal{F}_\alpha f)(w_1, w_2, ..., w_n), \quad \frac{n}{2} - 1 = \nu$ *and*

$$r^{\frac{n-1}{2}}f(r) = \phi(r), \quad then$$

$$\phi(\rho) = \frac{1}{\alpha^{n/2}}\int_0^\infty(\rho^{\frac{1}{\alpha}}r)^{\frac{1}{2}}J_\nu(\rho^{\frac{1}{\alpha}}r)\phi(r)dr,\ for\ 0 < \alpha \leq 1. \tag{4.2.3}$$

*Proof.* From Theorem 4.2.1 and [33, pp. 62-65], we can find

$$(\mathcal{F}_\alpha f)(w_1, w_2, ..., w_n) = \frac{\rho^{-\frac{1}{\alpha}(\frac{n}{2}-1)}}{\alpha^{n/2}} \int_0^\infty r(r^{\frac{n}{2}-1} f(r)) J_{\frac{n}{2}-1}(\rho^{\frac{1}{\alpha}} r) dr. \tag{4.2.4}$$

Therefore

$$\rho^{-(\frac{n-1}{2\alpha})}(\mathcal{F}_\alpha f)(w_1, w_2, ..., w_n) = \frac{1}{\alpha^{n/2}} \int_0^\infty (\rho^{\frac{1}{\alpha}} r)^{\frac{1}{2}} J_{\frac{n}{2}-1}(\rho^{\frac{1}{\alpha}} r)(r^{\frac{n-1}{2}} f(r)) dr.$$

If we take $\phi(\rho) = \rho^{-(\frac{n-1}{2\alpha})}(\mathcal{F}_\alpha f)(w_1, w_2, ..., w_n)$, then

$$\phi(\rho) = \frac{1}{\alpha^{n/2}} \int_0^\infty (\rho^{\frac{1}{\alpha}} r)^{\frac{1}{2}} J_{\frac{n}{2}-1}(\rho^{\frac{1}{\alpha}} r)(r^{\frac{n-2}{2}} f(r)) dr. \tag{4.2.5}$$

By considering $\frac{n}{2} - 1 = \nu$ and $r^{\frac{n-1}{2}} f(r) = \phi(r)$, (4.2.5) becomes

$$\phi(\rho) = \frac{1}{\alpha^{n/2}} \int_0^\infty (\rho^{\frac{1}{\alpha}} r)^{\frac{1}{2}} J_\nu(\rho^{\frac{1}{\alpha}} r) \phi(r) dr.$$

□

Now, from (4.2.1) and (4.2.3), we can define the conventional fractional Hankel transform of a function $\phi$ as

$$(h_\mu^\alpha \phi)(w) := \int_0^\infty (w^{\frac{1}{\alpha}} x)^{\frac{1}{2}} J_\mu(w^{\frac{1}{\alpha}} x) \phi(x) dx, \quad \mu \geq -\frac{1}{2}, \quad 0 < \alpha \leq 1. \tag{4.2.6}$$

To convert (4.2.6) in Haimo type fractional Hankel transform, we take $x^\gamma \phi(x) \in L_\sigma^1(I)$ and $\mu = \gamma - \frac{1}{2}$, then from (4.2.6), we have

$$(h_{\gamma-\frac{1}{2}}^\alpha x^\gamma \phi)(w) = \int_0^\infty (w^{\frac{1}{\alpha}} x)^{\frac{1}{2}} J_{\gamma-\frac{1}{2}}(w^{\frac{1}{\alpha}} x) x^\gamma \phi(x) dx.$$

By multiplying $w^{-\frac{\gamma}{\alpha}}$ in the above equation, we get

$$\begin{aligned} w^{-\frac{\gamma}{\alpha}}(h^{\alpha}_{\gamma-\frac{1}{2}}x^{\gamma}\phi)(w) &= \int_0^{\infty} 2^{\gamma-\frac{1}{2}}\Gamma\left(\gamma+\frac{1}{2}\right)(w^{\frac{1}{\alpha}}x)^{\frac{1}{2}-\gamma}J_{\gamma-\frac{1}{2}}(w^{\frac{1}{\alpha}}x)\phi(x) \\ &\quad\times \frac{x^{2\gamma}}{2^{\gamma-\frac{1}{2}}\Gamma\left(\gamma+\frac{1}{2}\right)}dx \\ &= \int_0^{\infty} C_{\gamma}(w^{\frac{1}{\alpha}}x)^{\frac{1}{2}-\gamma}J_{\gamma-\frac{1}{2}}(w^{\frac{1}{\alpha}}x)\phi(x)d\sigma(x), \end{aligned}$$

where

$$C_{\gamma} = 2^{\gamma-\frac{1}{2}}\Gamma\left(\gamma+\frac{1}{2}\right), \quad d\sigma(x) = \frac{x^{2\gamma}}{2^{\gamma-\frac{1}{2}}\Gamma\left(\gamma+\frac{1}{2}\right)}dx$$

and $J_{\mu-\frac{1}{2}}$ denotes the Bessel function of order $\mu-\frac{1}{2}$. Now, by taking

$$j(w^{\frac{1}{\alpha}}x) = C_{\gamma}(w^{\frac{1}{\alpha}}x)^{\frac{1}{2}-\gamma}J_{\gamma-\frac{1}{2}}(w^{\frac{1}{\alpha}}x),$$

we find the following expression

$$w^{-\frac{\gamma}{\alpha}}(h^{\alpha}_{\gamma-\frac{1}{2}}x^{\gamma}\phi)(w) = \int_0^{\infty} j(w^{\frac{1}{\alpha}}x)\phi(x)d\sigma(x).$$

By setting $\Phi(w) = w^{-\frac{\gamma}{\alpha}}(h^{\alpha}_{\mu}x^{\gamma}\phi)(w)$, we get

$$\Phi(w) = \int_0^{\infty} j(w^{\frac{1}{\alpha}}x)\phi(x)d\sigma(x). \tag{4.2.7}$$

Now, we are able to define the fractional Hankel transform in the Haimo setting by the following way:

**Definition 4.2.3.** Let $\phi \in L^1_{\sigma}(I)$, the fractional Hankel transform of $\phi$ is defined by

$$(h_{\mu,\alpha}\phi)(w) \equiv \hat{\phi}_{\alpha}(w) := \int_0^{\infty} j(w^{\frac{1}{\alpha}}x)\phi(x)d\sigma(x), \quad 0 \le x < \infty, \quad 0 < \alpha \le 1. \tag{4.2.8}$$

If $\phi \in L^1_\sigma(I)$, then $\hat{\phi}_\alpha$ is continuous and bounded on $[0, \infty)$ and

$$||\hat{\phi}_\alpha||_{\infty,\sigma} \leq ||\phi||_{1,\sigma}. \tag{4.2.9}$$

**Definition 4.2.4.** If $\phi \in L^1_\sigma(I)$ and $h_{\mu,\alpha}\phi \in L^1_\sigma(I)$, then for $0 < \alpha \leq 1$ the inverse fractional Hankel transform is defined by

$$\phi(x) := \int_0^\infty j(w^{\frac{1}{\alpha}}x)(h_{\mu,\alpha}\phi)(w)\frac{w^{\frac{1}{\alpha}-1}}{\alpha}d\sigma(w), \quad 0 < x < \infty. \tag{4.2.10}$$

**Definition 4.2.5.** Let $\phi \in L^1_\sigma(I)$, then the fractional Hankel translation is defined by

$$(\tau_y\phi)(x) \equiv \phi_\alpha(x, y) := \int_0^\infty \phi(z)D_\alpha(x, y, z)d\sigma(z), \quad 0 < x, y < \infty, \tag{4.2.11}$$

where

$$D_\alpha(x, y, z) = \int_0^\infty j(w^{\frac{1}{\alpha}}x)j(w^{\frac{1}{\alpha}}y)j(w^{\frac{1}{\alpha}}z)\frac{w^{\frac{1}{\alpha}-1}}{\alpha}d\sigma(w). \tag{4.2.12}$$

**Definition 4.2.6.** Let $\phi \in L^1_\sigma(I)$ and $\psi \in L^1_\sigma(I)$, then the fractional Hankel convolution is defined by

$$(\phi\#\psi)(x) := \int_0^\infty \phi_\alpha(x, y)\psi(y)d\sigma(y), \quad 0 < \alpha \leq 1, \tag{4.2.13}$$

where $\phi_\alpha(x, y)$ is given by (4.2.11).

Using (4.2.8) and (4.2.10), we get

$$\int_0^\infty j(w^{\frac{1}{\alpha}}x)D_\alpha(x, y, z)d\sigma(x) = j(w^{\frac{1}{\alpha}}y)j(w^{\frac{1}{\alpha}}z), \; 0 < x, y < \infty, \; 0 \leq w < \infty. \tag{4.2.14}$$

Setting $w = 0$, we obtain

$$\int_0^\infty D_\alpha(x, y, z) d\sigma(z) = 1. \tag{4.2.15}$$

## 4.3 Properties of the fractional Hankel transform

In this section, the Parseval formula and the properties of fractional Hankel convolution are derived. By using the fractional Hankel convolution theory, boundedness properties for the fractional Hankel translation and the fractional Hankel convolution have been given.

**Theorem 4.3.1.** *If $\phi,\ \psi \in L^1_\sigma(I)$ and $\hat{\phi}_\alpha,\ \hat{\psi}_\alpha \in L^1_\sigma(I)$, then the Parseval formula is given by*

$$\int_0^\infty \phi(x)\psi(x) d\sigma(x) = \int_0^\infty \hat{\phi}_\alpha(w)\hat{\psi}_\alpha(w) \frac{w^{\frac{1}{\alpha}-1}}{\alpha} d\sigma(x). \tag{4.3.1}$$

*Proof.* From (4.2.8) and (4.2.10), we have

$$\begin{aligned}
\int_0^\infty \phi(x)\psi(x) d\sigma(x) &= \int_0^\infty \phi(x) d\sigma(x) \int_0^\infty j(w^{\frac{1}{\alpha}} x)\hat{\psi}_\alpha(w) \frac{w^{\frac{1}{\alpha}-1}}{\alpha} d\sigma(w) \\
&= \int_0^\infty \hat{\psi}_\alpha(w) \frac{w^{\frac{1}{\alpha}-1}}{\alpha} d\sigma(w) \int_0^\infty j(w^{\frac{1}{\alpha}} x)\phi(x) d\sigma(x) \\
&= \int_0^\infty \hat{\phi}_\alpha(w)\hat{\psi}_\alpha(w) \frac{w^{\frac{1}{\alpha}-1}}{\alpha} d\sigma(w).
\end{aligned}$$

□

**Theorem 4.3.2.** *Let $\phi \in L^p_\sigma(I)$, then*

$$||\phi_\alpha(x, .)||_{p,\sigma} \leq ||\phi||_{p,\sigma}, \tag{4.3.2}$$

*for any fixed $x$.*

*Proof.* From (4.2.11), we have

$$\begin{aligned}\phi_\alpha(x,y) &= \int_0^\infty \phi(z)D_\alpha(x,y,z)d\sigma(z)\\ &= \int_0^\infty \phi(z)D_\alpha^{\frac{1}{p}}(x,y,z)D_\alpha^{\frac{1}{q}}(x,y,z)d\sigma(z),\quad \frac{1}{p}+\frac{1}{q}=1,\end{aligned}$$

which yields that

$$|\phi_\alpha(x,y)| \le \int_0^\infty |\phi(z)|D_\alpha^{\frac{1}{p}}(x,y,z)D_\alpha^{\frac{1}{q}}(x,y,z)d\sigma(z).$$

Applying Holder inequality, we get

$$|\phi_\alpha(x,y)| \le \left(\int_0^\infty |\phi(z)|^p D_\alpha(x,y,z)d\sigma(z)\right)^{\frac{1}{p}} \left(\int_0^\infty D_\alpha(x,y,z)d\sigma(z)\right)^{\frac{1}{q}}.$$

Using (4.2.15), we obtain

$$|\phi_\alpha(x,y)| \le \left(\int_0^\infty |\phi(z)|^p D_\alpha(x,y,z)d\sigma(z)\right)^{\frac{1}{p}}.$$

Thus, we have

$$\begin{aligned}\int_0^\infty |\phi_\alpha(x,y)|^p d\sigma(y) &\le \int_0^\infty \left(\int_0^\infty |\phi(z)|^p D_\alpha(x,y,z)d\sigma(z)\right) d\sigma(y)\\ &= \int_0^\infty |\phi(z)|^p d\sigma(z) \left(\int_0^\infty D_\alpha(x,y,z)d\sigma(y)\right).\end{aligned}$$

Again, by using (4.2.15), we yield

$$\int_0^\infty |\phi_\alpha(x,y)|^p d\sigma(y) \le \int_0^\infty |\phi(z)|^p d\sigma(z).$$

Thus for any fixed $x$, we get

$$||\phi_\alpha(x,.)||_{p,\sigma} \leq ||\phi||_{p,\sigma}.$$

□

**Theorem 4.3.3.** *Let $\phi \in L^1_\sigma(I)$ and $\psi \in L^p_\sigma(I)$ then,*

$$||\phi\#\psi||_{1,\sigma} \leq ||\phi||_{1,\sigma}||\psi||_{p,\sigma}, \qquad 0 < \alpha \leq 1. \tag{4.3.3}$$

*Proof.* From (4.2.13), we have

$$\begin{aligned}(\phi\#\psi)(x) &= \int_0^\infty \phi_\alpha(x,y)\psi(y)d\sigma(y)\\ &= \int_0^\infty |\phi_\alpha^{\frac{1}{p}}(x,y)\psi(y)||\phi_\alpha^{\frac{1}{q}}(x,y)|d\sigma(y), \quad \frac{1}{p}+\frac{1}{q}=1,\end{aligned}$$

which implies that

$$|(\phi\#\psi)(x)| \leq \int_0^\infty |\phi_\alpha^{\frac{1}{p}}(x,y)\psi(y)||\phi_\alpha^{\frac{1}{q}}(x,y)|d\sigma(y).$$

Applying Holder inequality, we get

$$\begin{aligned}|(\phi\#\psi)(x)| &\leq \left(\int_0^\infty |\phi_\alpha(x,y)||\psi(y)|^p d\sigma(y)\right)^{\frac{1}{p}} \left(\int_0^\infty |\phi_\alpha(x,y)|d\sigma(y)\right)^{\frac{1}{q}}\\ &= (||\phi_\alpha(x,.)||_{1,\sigma})^{\frac{1}{q}} \left(\int_0^\infty |\phi_\alpha(x,y)||\psi(y)|^p d\sigma(y)\right)^{\frac{1}{p}}.\end{aligned}$$

Now, by using (4.3.2), we get

$$|(\phi\#\psi)(x)|^p \leq (||\phi||_{1,\sigma})^{\frac{p}{q}} \left(\int_0^\infty |\phi_\alpha(x,y)||\psi(y)|^p d\sigma(y)\right).$$

On integrating both sides and using Fubini's theorem, we get

$$\begin{aligned}\int_0^\infty |(\phi\#\psi)(x)|^p d\sigma(x) &\leq (||\phi||_{1,\sigma})^{\frac{p}{q}} \left(\int_0^\infty \int_0^\infty |\phi_\alpha(x,y)||\psi(y)|^p d\sigma(y) d\sigma(x)\right)\\ &\leq (||\phi||_{1,\sigma})^{\frac{p}{q}} \int_0^\infty |\psi(y)|^p d\sigma(y) \left(\int_0^\infty |\phi_\alpha(x,y)| d\sigma(x)\right)\\ &\leq (||\phi||_{1,\sigma})^{\frac{p}{q}+1} \left(\int_0^\infty |\psi(y)|^p d\sigma(y)\right).\end{aligned}$$

Now, using the relation $\frac{1}{p}+\frac{1}{q}=1$, we get

$$\begin{aligned}||\phi\#\psi||_{p,\sigma} &\leq (||\phi||_{1,\sigma})^{\frac{1}{p}(\frac{p}{q}+1)} \left(\int_0^\infty |\psi(y)|^p d\sigma(y)\right)^{\frac{1}{p}}\\ &= ||\phi||_{1,\sigma}||\psi||_{p,\sigma}.\end{aligned}$$

□

**Theorem 4.3.4.** *Let $\phi \in L^p_\sigma(I)$ and $\psi \in L^q_\sigma(I)$ then,*

$$||\phi\#\psi||_{r,\sigma} \leq ||\phi||_{p,\sigma}||\psi||_{q,\sigma}, \quad 0<\alpha\leq 1 \quad \textit{and} \quad \frac{1}{r}=\frac{1}{p}+\frac{1}{q}-1. \tag{4.3.4}$$

*Proof.* From (4.2.13), we have

$$(\phi\#\psi)(x) = \int_0^\infty \phi_\alpha(x,y)\psi(y) d\sigma(y).$$

Now, we can write

$$|(\phi\#\psi)(x)| \leq \int_0^\infty |\phi_\alpha(x,y)||\psi(y)| d\sigma(y).$$

Applying Young's inequality, we get

$$|(\phi\#\psi)(x)| \leq \left(\int_0^\infty |\phi_\alpha(x,y)|^p|\psi(y)|^q d\sigma(y)\right)^{\frac{1}{r}} \left(\int_0^\infty |\phi_\alpha(x,y)|^p d\sigma(y)\right)^{1-\frac{1}{q}}$$

$$\times \Big(\int_0^\infty |\psi(y)|^q d\sigma(y)\Big)^{1-\frac{1}{q}}.$$

Above implies that

$$\int_0^\infty |(\phi\#\psi)(x)|^r d\sigma(x) \leq \int_0^\infty \Big(\Big(\int_0^\infty |\phi_\alpha(x,y)|^p|\psi(y)|^q d\sigma(y)\Big) \times \Big(\int_0^\infty |\phi_\alpha(x,y)|^p d\sigma(y)\Big)^{r-\frac{r}{q}} \times \Big(\int_0^\infty |\psi(y)|^q d\sigma(y)\Big)^{r-\frac{r}{q}}\Big) d\sigma(x).$$

Using Fubini's theorem, we get

$$\int_0^\infty |(\phi\#\psi)(x)|^r d\sigma(x) \leq \Big(\int_0^\infty |\phi_\alpha(x,y)|^p d\sigma(x)\Big)\Big(\int_0^\infty |\psi(y)|^q d\sigma(y)\Big) \times \Big(\int_0^\infty |\phi_\alpha(x,y)|^p d\sigma(y)\Big)^{r-\frac{r}{q}}\Big(\int_0^\infty |\psi(y)|^q d\sigma(y)\Big)^{r-\frac{r}{q}}.$$

Now, by using translation symmetry in (4.2.11) and the relation,

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1,$$

we can write the above expression as

$$\begin{aligned}\Big(\int_0^\infty |(\phi\#\psi)(x)|^r d\sigma(x)\Big)^{\frac{1}{r}} &\leq \Big(\int_0^\infty |\phi_\alpha(x,y)|^p d\sigma(y)\Big)^{\frac{1}{r}(1+r-\frac{r}{q})} \\ &\quad \times \Big(\int_0^\infty |\psi(y)|^q d\sigma(y)\Big)^{\frac{1}{r}(1+r-\frac{r}{q})} \\ &= \Big(\int_0^\infty |\phi_\alpha(x,y)|^p d\sigma(y)\Big)^{\frac{1}{p}}\Big(\int_0^\infty |\psi(y)|^q d\sigma(y)\Big)^{\frac{1}{q}} \\ &= ||\phi_\alpha(x,.)||_{p,\sigma}||\psi_\alpha||_{q,\sigma}.\end{aligned}$$

Thus, by using (4.3.2), we get

$$||\phi\#\psi||_{r,\sigma} \leq ||\phi||_{p,\sigma}||\psi||_{q,\sigma}, \quad 0 < \alpha \leq 1.$$

□

**Theorem 4.3.5.** *Let $\phi \in L^p_\sigma(I)$ and $\psi \in L^q_\sigma(I)$ then,*

$$||\phi\#\psi||_{\infty,\sigma} \leq ||\phi||_{p,\sigma}||\psi||_{q,\sigma}, \quad 0 < \alpha \leq 1. \tag{4.3.5}$$

*Proof.* From the definition of the fractional Hankel convolution (4.2.13), we have

$$(\phi\#\psi)(x) = \int_0^\infty \phi_\alpha(x,y)\psi(y)d\sigma(y),$$

which implies that

$$|(\phi\#\psi)(x)| \leq \int_0^\infty |\phi_\alpha(x,y)||\psi(y)|d\sigma(y).$$

By using Holder inequality, the above expression becomes

$$|(\phi\#\psi)(x)| \leq ||\phi_\alpha(x,.)||_{p,\sigma}||\psi||_{q,\sigma}.$$

Now, in view of (4.3.2), we get

$$|(\phi\#\psi)(x)| \leq ||\phi||_{p,\sigma}||\psi||_{q,\sigma}, \quad \forall\ x \in I.$$

□

**Theorem 4.3.6.** *Let $\phi \in L^1_\sigma(I)$ and $\psi \in L^1_\sigma(I)$ then,*

$$h_{\mu,\alpha}(\phi\#\psi) = (h_{\mu,\alpha}\phi)(h_{\mu,\alpha}\psi), \quad 0 < \alpha \leq 1. \tag{4.3.6}$$

*Proof.* By the definition of the fractional Hankel transform (4.2.8) and the fractional Hankel convolution (4.2.13), we have

$$h_{\mu,\alpha}((\phi\#\psi)(x))(w) = \int_0^\infty j(w^{\frac{1}{\alpha}}x)(\phi\#\psi)(x)d\sigma(x)$$

From (4.2.13), we can write

$$h_{\mu,\alpha}((\phi\#\psi)(x))(w) = \int_0^\infty j(w^{\frac{1}{\alpha}}x)\Big(\int_0^\infty \phi_\alpha(x,u)\psi(u)d\sigma(u)\Big)d\sigma(x)$$

From (4.2.11), then we get

$$h_{\mu,\alpha}((\phi\#\psi)(x))(w) = \int_0^\infty j(w^{\frac{1}{\alpha}}x)\int_0^\infty \psi(u)\Big(\int_0^\infty \phi(v)D_\alpha(x,u,v)d\sigma(v)d\sigma(u)\Big)d\sigma(x)$$

Using Fubini's theorem, we get

$$h_{\mu,\alpha}((\phi\#\psi)(x))(w) = \int_0^\infty \int_0^\infty \psi(u)\phi(v)\Big(\int_0^\infty j(w^{\frac{1}{\alpha}}x)D_\alpha(x,u,v)d\sigma(x)\Big)d\sigma(u)d\sigma(v)$$

From (4.2.14), we find

$$\begin{aligned} h_{\mu,\alpha}((\phi\#\psi)(x))(w) &= \int_0^\infty \int_0^\infty \psi(u)\phi(v)j(w^{\frac{1}{\alpha}}u)j(w^{\frac{1}{\alpha}}v)d\sigma(u)d\sigma(v) \\ &= \Big(\int_0^\infty \psi(u)j(w^{\frac{1}{\alpha}}u)d\sigma(u)\Big)\Big(\int_0^\infty \phi(v)j(w^{\frac{1}{\alpha}}v)d\sigma(v)\Big) \end{aligned}$$

By the definition of the fractional Hankel transform (4.2.8), the above expression becomes

$$h_{\mu,\alpha}((\phi\#\psi)(x))(w) = (h_{\mu,\alpha}\phi)(w)(h_{\mu,\alpha}\psi)(w).$$

□

## 4.4 Conclusions

Like Fourier transform, Hankel transform contains rich calculus and deep mathematical background. It is useful tool to find the solution of cylindrical boundary value problem. Many authors used this theory and found many applications in differential equations and other areas of mathematics. The fractional Hankel transform is modification and generalization of the Hankel transform for $0 < \alpha \leq 1$. From [17, 30, 31], in the present chapter, the properties of the fractional Hankel convolution and other associated results are investigated by exploiting the theory of the fractional Hankel transform. Using the aforesaid theory, the author is able to study the theory of the fractional Bessel wavelet transform and its various properties which will be given in Chapter 5.

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