Chapter 3

Abelian Theorems involving the Fractional Wavelet Transform

3.1 Introduction

The fractional Fourier transform, which was studied by Luchko et al. [3] in the year 2008, plays a significant role for finding the fractional derivatives. Later, in the year 2010, Kilbas et al. [4] discussed the composition of the fractional Fourier transform with some modified fractional integral and derivatives. More recently, the calculus of pseudo-differential operators, which are associated with the fractional Fourier transform on the Schwartz space, were considered in [14, 15]. Motivated by these and other related developments, Srivastava, Khatterwani and Upadhyay [16] investigated various potentially useful properties of the continuous fractional wavelet transform.

Abelian theorems are useful for finding the initial value with the help of the final value and with help of the intitial value, the final value is found by exploiting different integral transforms. By using different integral transforms, Abelian theorems were investigated by many authors in classical and distributional sense both and found different observations.

In 1955, J.L. Griffith [19] gave a theorem concerning the asymptotic behavior of Hankel transforms. In 1966, Zemanian [21] considered some Abelian theorems for the distributional Hankel and K transformations. Hayek and Gonzalez [25] established Abelian theorems for the generalized index ${}_2F_1$ -transform in 1992. Pathak [22, 49] investigated the Abelian theorems for the wavelet transform by exploiting the theory of the Fourier transforms in 2001. After that, Upadhyay et al. [35] found the abelian theorems for the Bessel wavelet transform in 2020. Abelian theorems for the Laplace and the Mehler- Fock transforms of general order over distributions of compact support and over certain spaces of generalized functions are proved by Gonzalez and Negrın [26] in 2020. In 2022, Prasad et al. [24] discussed the Abelian theorems for the quadratic-phase Fourier wavelet transform and got different results.

Motivated from the above results, our main objective in the present chapter is to discuss Abelian theorems for the fractional wavelet transform in classical and distributions sense by using the technique of the fractional Fourier transform. An application of Abelian theorems associated with the continuous fractional wavelet transform are discussed by using the Mexican hat wavelet function.

The entire chapter is organized in the following manner:

Section 3.1 is introductory, which contains definitions and properties of the continuous fractional wavelet transform. In Section 3.2, the initial-value and final-value theorems for the fractional wavelet transform of functions are given. In Section 3.3, Abelian theorems for the fractional wavelet transform of distributions are investigated and their properties are obtained by exploiting the theory of the fractional

Fourier transform. An application and justification of Abelian theorems regarding the fractional wavelet transform is presented by using the Mexican hat wavelet function in Section 3.4.

Definition 3.1.1. Let $\psi \in L^2(\mathbb{R})$ and $0 < \alpha \leq 1$. Then the fractional wavelet $\psi_{\alpha,a,b}(t)$ is defined by

$$\psi_{\alpha,a,b}(t) = \frac{1}{|a|^{\frac{1}{\alpha}}}\psi\left(\frac{t-b}{|a|^{\frac{1}{\alpha}}}\right), \quad a \neq 0,\ b \in \mathbb{R}. \tag{3.1.1}$$

Definition 3.1.2. Let $\psi \in L^2(\mathbb{R})$. Then the continuous fractional wavelet transform of a given signal $\phi \in L^2(\mathbb{R})$ for $0 < \alpha \leq 1$ is defined by

$$\begin{aligned}\big(W_{\psi_\alpha}\phi\big)(b,a) &= \langle \phi, \psi_{\alpha,a,b}\rangle \\ &= \int_{-\infty}^{+\infty} \phi(t)\frac{1}{|a|^{\frac{1}{\alpha}}}\overline{\psi\left(\frac{t-b}{|a|^{\frac{1}{\alpha}}}\right)}dt.\end{aligned} \tag{3.1.2}$$

From (1.4.1) and (3.1.1), we have

$$F_\alpha(\psi_{\alpha,a,b}(t))(w) = e^{-i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}}b}\hat{\psi}_\alpha(aw). \tag{3.1.3}$$

Let $\phi, \psi \in L^2(\mathbb{R})$. Then the Parseval formula (see [16]) for the fractional Fourier transform is given by

$$\langle \phi, \psi\rangle = \frac{1}{2\pi\alpha}\big\langle |w|^{\frac{1}{\alpha}-1}\hat{\phi}_\alpha(w), \hat{\psi}_\alpha(w)\big\rangle. \tag{3.1.4}$$

In view of (3.1.2), (3.1.3) and (3.1.4), we get the following relation

$$\big(W_{\psi_\alpha}\phi\big)(b,a) = \frac{1}{2\pi\alpha}\int_{-\infty}^{+\infty} e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}}b}|w|^{\frac{1}{\alpha}-1}\hat{\phi}_\alpha(w)\overline{\hat{\psi}_\alpha(aw)}dw. \tag{3.1.5}$$

3.2 Abelian theorems for the fractional wavelet transform of functions

In this section, we present the initial-value and final-value theorems for the fractional wavelet transform of functions.

Let us suppose that

$$\hat{\psi}_\alpha(w) = O(|w|^\mu), \quad |w| \to 0 \tag{3.2.1}$$

and that

$$1 + \frac{1}{\alpha} < \eta < \mu + 1 + \frac{1}{\alpha}, \quad \text{for} \quad 0 < \alpha \leq 1.$$

Then the following integral:

$$\int_{-\infty}^{+\infty} \overline{\hat{\psi}_\alpha(w)} |w|^{\frac{1}{\alpha}-\eta} dw,$$

is convergent.

Now, we set

$$\int_{-\infty}^{+\infty} \overline{\hat{\psi}_\alpha(w)} |w|^{\frac{1}{\alpha}-\eta} dw = F_1(\alpha, \eta). \tag{3.2.2}$$

Theorem 3.2.1. ***(Initial-Value Theorem)***

Let

$$1 + \frac{1}{\alpha} < \eta < \mu + 1 + \frac{1}{\alpha} \quad \textit{and} \quad \mu > 0.$$

Assume also that

$$|w|^{\frac{1}{\alpha}-\eta} \hat{\psi}_\alpha(w) \in L^1(\mathbb{R}),$$

$$|\hat{\psi}_\alpha(w)| \leq M, \quad M > 0$$

and

$$|w|^{\frac{1}{\alpha}-1}\hat{\phi}_\alpha(w) \in L^1(\delta,\infty),\ \forall\ \delta > 0.$$

If

$$\lim_{|w|\to 0}(2\pi\alpha)^{-1}|w|^{-1+\eta}\hat{\phi}_\alpha(w) = F_2(\alpha,\eta), \tag{3.2.3}$$

then

$$\lim_{a\to\infty} a^{1-\eta+\frac{1}{\alpha}}\big(W_{\psi_\alpha}\phi\big)(b,a) = F_1(\alpha,\eta)F_2(\alpha,\eta). \tag{3.2.4}$$

Proof. From (3.1.5) and (3.2.2), we have

$$\begin{aligned}
&\Big|a^{1-\eta+\frac{1}{\alpha}}\big(W_{\psi_\alpha}\phi\big)(b,a) - F_1(\alpha,\eta)F_2(\alpha,\eta)\Big| \\
&= \Big|a^{1-\eta+\frac{1}{\alpha}}\frac{1}{2\pi\alpha}\int_{-\infty}^{+\infty} e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}}b}|w|^{\frac{1}{\alpha}-1}\hat{\phi}_\alpha(w)\overline{\hat{\psi}_\alpha(aw)}dw \\
&\quad - F_2(\alpha,\eta)\int_{-\infty}^{+\infty}\overline{\hat{\psi}_\alpha(aw)}|aw|^{\frac{1}{\alpha}-\eta}adw\Big| \\
&= a\Big|\int_{-\infty}^{+\infty}(2\pi\alpha)^{-1}e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}}b}|aw|^{\frac{1}{\alpha}-1}|aw|^{1-\eta}|w|^{-1+\eta}\hat{\phi}_\alpha(w)\overline{\hat{\psi}_\alpha(aw)}dw \\
&\quad - F_2(\alpha,\eta)\int_{-\infty}^{+\infty}\overline{\hat{\psi}_\alpha(aw)}|aw|^{1-\eta}|aw|^{\frac{1}{\alpha}-1}dw\Big| \\
&= a\Big|\int_{-\infty}^{+\infty}\big[(2\pi\alpha)^{-1}e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}}b}|w|^{-1+\eta}\hat{\phi}_\alpha(w) - F_2(\alpha,\eta)\big] \\
&\quad\times|aw|^{1-\eta}|aw|^{\frac{1}{\alpha}-1}\overline{\hat{\psi}_\alpha(aw)}dw\Big| \\
&\le a\sup_{|w|<\delta}\Big|(2\pi\alpha)^{-1}e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}}b}|w|^{-1+\eta}\hat{\phi}_\alpha(w) - F_2(\alpha,\eta)\Big| \\
&\quad\times\int_{|w|<\delta}|aw|^{1-\eta}|aw|^{\frac{1}{\alpha}-1}|\hat{\psi}_\alpha(aw)|dw \\
&\quad + a\int_{|w|>\delta}\Big|(2\pi\alpha)^{-1}e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}}b}|w|^{-1+\eta}\hat{\phi}_\alpha(w) - F_2(\alpha,\eta)\Big| \\
&\quad\times|aw|^{1-\eta}|aw|^{\frac{1}{\alpha}-1}|\hat{\psi}_\alpha(aw)|dw \\
&\le \sup_{|w|<\delta}\Big|(2\pi\alpha)^{-1}e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}}b}|w|^{-1+\eta}\hat{\phi}_\alpha(w) - F_2(\alpha,\eta)\Big| \\
&\quad\times\int_{-\infty}^{+\infty}|w|^{\frac{1}{\alpha}-\eta}|\hat{\psi}_\alpha(w)|dw
\end{aligned}$$

$$+ Ma^{1+\frac{1}{\alpha}-\eta} \int_{|w|>\delta} \left| (2\pi\alpha)^{-1} e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}} b} |w|^{-1+\eta} \hat{\phi}_\alpha(w) - F_2(\alpha,\eta) \right|$$
$$\times |w|^{\frac{1}{\alpha}-\eta} dw. \tag{3.2.5}$$

Since $|w|^{\frac{1}{\alpha}-\eta}\hat{\psi}_\alpha(w) \in L^1(\mathbb{R})$, there exist a positive real number M_1 such that

$$\int_{-\infty}^{+\infty} |w|^{\frac{1}{\alpha}-\eta} |\hat{\psi}_\alpha(w)| dw = M_1 < \infty.$$

Also, for any $\epsilon > 0$, we have

$$\sup_{|w|<\delta} \left| (2\pi\alpha)^{-1} e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}} b} |w|^{-1+\eta} \hat{\phi}_\alpha(w) - F_2(\alpha,\eta) \right| < \frac{\epsilon}{2M_1},$$

by choosing δ small enough.

Next, since $\eta > 1 + \frac{1}{\alpha}$, we find that

$$a^{1+\frac{1}{\alpha}-\eta} \to 0, \quad \text{as} \quad a \to \infty$$

and the integral in second term of (3.2.5) is convergent. So, for any $\epsilon > 0$, we can made the second term in (3.2.5) less than $\frac{\epsilon}{2}$. Hence for any $\epsilon > 0$, the following consequence:

$$\left| a^{1-\eta+\frac{1}{2\alpha}} \left(W_{\psi_\alpha}\phi \right)(b,a) - F_1(\alpha,\eta) F_2(\alpha,\eta) \right| < \epsilon,$$

for sufficiently large a. □

Theorem 3.2.2. ***(Final-Value Theorem)***

Let

$$1 + \frac{1}{\alpha} < \eta < \mu + 1 + \frac{1}{\alpha}, \ \textit{and} \ \ \mu > 0.$$

Suppose also that

$$|w|^{\frac{1}{\alpha}-\eta}\hat{\psi}_\alpha(w) \in L^1(\mathbb{R})$$

and that

$$|w|^{\mu+\frac{1}{\alpha}-1}\hat{\phi}_\alpha(w) \in L^1(-X, X), \quad \forall\ X > 0.$$

If

$$\lim_{|w|\to\infty} (2\pi\alpha)^{-1} e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}} b} |w|^{-1+\eta}\hat{\phi}_\alpha(w) = F_3(\alpha, b, \eta), \tag{3.2.6}$$

then

$$\lim_{a\to 0} a^{1-\eta+\frac{1}{\alpha}} \big(W_{\psi_\alpha}\phi\big)(b, a) = F_1(\alpha, \eta)F_3(\alpha, b, \eta). \tag{3.2.7}$$

Proof. In view of the proof of the previous theorem, we find $\forall\ X > 0$ that

$$\begin{aligned}
&\left|a^{1-\eta+\frac{1}{\alpha}} \big(W_{\psi_\alpha}\phi\big)(b, a) - F_1(\alpha, \eta)F_3(\alpha, b, \eta)\right| \\
&\leq a \int_{|w|<X} \left|(2\pi\alpha)^{-1} e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}} b}|w|^{-1+\eta}\hat{\phi}_\alpha(w) - F_3(\alpha, b, \eta)\right| \\
&\quad \times |aw|^{1-\eta}|aw|^{\frac{1}{\alpha}-1}|\hat{\psi}_\alpha(aw)|dw \\
&\quad + a \int_{|w|>X} \left|(2\pi\alpha)^{-1} e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}} b}|w|^{-1+\eta}\hat{\phi}_\alpha(w) - F_3(\alpha, b, \eta)\right| \\
&\quad \times |aw|^{1-\eta}|aw|^{\frac{1}{\alpha}-1}|\hat{\psi}_\alpha(aw)|dw \\
&\leq a^{1-\eta+\frac{1}{\alpha}} \int_{-X}^{+X} \left|(2\pi\alpha)^{-1} e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}} b}|w|^{-1+\eta}\hat{\phi}_\alpha(w) - F_3(\alpha, b, \eta)\right| \\
&\quad \times |w|^{1-\eta}|w|^{\frac{1}{\alpha}-1}|\hat{\psi}_\alpha(aw)|dw \\
&\quad + \sup_{|w|>X} \left|(2\pi\alpha)^{-1} e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}} b}|w|^{-1+\eta}\hat{\phi}_\alpha(w) - F_3(\alpha, b, \eta)\right| \\
&\quad \times \int_{-\infty}^{+\infty} |w|^{1-\eta}|w|^{\frac{1}{\alpha}-1}|\hat{\psi}_\alpha(w)|dw.
\end{aligned}$$

Thus, by using (3.2.1), there exists a positive constant $M_2 > 0$ such that

$$|\hat{\psi}_\alpha(aw)| \ \leq\ M_2(a|w|)^\mu.$$

Hence we have

$$\begin{aligned}
&\left|a^{1-\eta+\frac{1}{\alpha}}\left(W_{\psi_\alpha}\phi\right)(b,a) - F_1(\alpha,\eta)F_3(\alpha,b,\eta)\right| \\
&\leq M_2 a^{\mu+1-\eta+\frac{1}{\alpha}} \int_{-X}^{+X} \left|(2\pi\alpha)^{-1} e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}}b} \hat{\phi}_\alpha(w) - F_3(\alpha,b,\eta)|w|^{1-\eta}\right| \\
&\quad \times |w|^{\frac{1}{\alpha}-1+\mu} dw \\
&\quad + \sup_{|w|>X} \left|(2\pi\alpha)^{-1} e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}}b} |w|^{-1+\eta} \hat{\phi}_\alpha(w) - F_3(\alpha,b,\eta)\right| \\
&\quad \times \int_{-\infty}^{+\infty} |w|^{\frac{1}{\alpha}-\eta} |\hat{\psi}_\alpha(w)| dw. \qquad (3.2.8)
\end{aligned}$$

Now, since both of the integrals on the right hand side of (3.2.8) are convergent and

$$\eta < 1 + \frac{1}{\alpha} + \mu,$$

therefore, as $a \to 0$, the first term can be made less than $\frac{\epsilon}{2}$. Also, for sufficiently large X, the second term, which is independent of a, can be made less than $\frac{\epsilon}{2}$. Hence we obtain

$$\left|a^{1-\eta+\frac{1}{\alpha}}\left(W_{\psi_\alpha}\phi\right)(b,a) - F_1(\alpha,\eta)F_3(\alpha,b,\eta)\right| < \epsilon,$$

for sufficiently small a. □

3.3 Abelian theorems for the fractional wavelet transform of distributions

In this section, Abelian theorems for the fractional wavelet transform of distributions are investigated and their properties are obtained by exploiting the theory of the fractional Fourier transform.

If $\phi \in S^{'}(\mathbb{R})$, then $|w|^{\frac{1}{\alpha}-1}\hat{\phi}_\alpha(w) \in S^{'}(\mathbb{R})$. Hence we can define the fractional wavelet transform of distributions.

Definition 3.3.1. The fractional wavelet transform of $|w|^{\frac{1}{\alpha}-1}\hat{\phi}_\alpha(w) \in S^{'}(\mathbb{R})$ is defined by

$$\big(W_{\psi_\alpha}\phi\big)(b,a) = \frac{1}{2\pi\alpha}\big\langle |w|^{\frac{1}{\alpha}-1}\hat{\phi}_\alpha(w), e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}}b}\overline{\hat{\psi}_\alpha(aw)}\big\rangle. \tag{3.3.1}$$

Theorem 3.3.2. *If $\phi \in S^{'}(\mathbb{R})$, then the differentiability of the fractional wavelet transform:*

$$\big(W_{\psi_\alpha}\phi\big)(b,a)$$

is exhibited by

$$\begin{aligned}
&\left(\frac{\partial}{\partial a}\right)^m\left(\frac{\partial}{\partial b}\right)^n\big(W_{\psi_\alpha}\phi\big)(b,a)\\
&\quad = \frac{1}{2\pi\alpha}\Big\langle |w|^{\frac{1}{\alpha}-1}\hat{\phi}_\alpha(w),\ \big(i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}}\big)^n e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}}b}\left(\frac{\partial}{\partial a}\right)^m\overline{\hat{\psi}_\alpha(aw)}\Big\rangle,
\end{aligned} \tag{3.3.2}$$

for $a > 0$ and for all $m, n \in \mathbb{N}_0$.

Proof. For $h > 0$, we have

$$\begin{aligned}
&\frac{1}{h}\Big(\big(W_{\psi_\alpha}\phi\big)(b,a+h) - \big(W_{\psi_\alpha}\phi\big)(b,a)\Big)\\
&\quad - \frac{1}{2\pi\alpha}\Big\langle |w|^{\frac{1}{\alpha}-1}\hat{\phi}_\alpha(w),\ e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}}b}\frac{\partial}{\partial a}\overline{\hat{\psi}_\alpha(aw)}\Big\rangle\\
&\quad = \frac{1}{2\pi\alpha}\Big\langle |w|^{\frac{1}{\alpha}-1}\hat{\phi}_\alpha(w),\ e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}}b}\\
&\qquad \times\Big\{\frac{1}{h}\Big(\overline{\hat{\psi}_\alpha((a+h)w)} - \overline{\hat{\psi}_\alpha(aw)}\Big) - \frac{\partial}{\partial a}\overline{\hat{\psi}_\alpha(aw)}\Big\}\Big\rangle.
\end{aligned}$$

Thus, clearly we have to show that

$$e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}}b}\Big\{\frac{1}{h}\Big(\overline{\hat{\psi}_\alpha((a+h)w)} - \overline{\hat{\psi}_\alpha(aw)}\Big) - \frac{\partial}{\partial a}\overline{\hat{\psi}_\alpha(aw)}\Big\} \to 0,$$

in $S(\mathbb{R})$, as $h \to 0$. For this, we have to take for the following way:

$$\begin{aligned}
&\left|w^k\left(\frac{\partial}{\partial w}\right)^m\left(e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}}b}\left\{\frac{1}{h}\left(\overline{\hat{\psi}_\alpha((a+h)w)}-\overline{\hat{\psi}_\alpha(aw)}\right)-\frac{\partial}{\partial a}\overline{\hat{\psi}_\alpha(aw)}\right\}\right)\right|\\
&=\left|w^k\sum_{r=0}^{m}\binom{m}{r}\left(\left(\frac{\partial}{\partial w}\right)^{m-r}e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}}b}\right)\right.\\
&\quad\left.\times\left(\left(\frac{\partial}{\partial w}\right)^{r}\left\{\frac{1}{h}\left(\overline{\hat{\psi}_\alpha((a+h)w)}-\overline{\hat{\psi}_\alpha(aw)}\right)-\frac{\partial}{\partial a}\overline{\hat{\psi}_\alpha(aw)}\right\}\right)\right|\\
&=\left|w^k\sum_{r=0}^{m}\binom{m}{r}\left(\left(\frac{\partial}{\partial w}\right)^{m-r}e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}}b}\right)\right.\\
&\quad\left.\times\left\{\frac{1}{h}\left(\left(\frac{\partial}{\partial w}\right)^{r}\overline{\hat{\psi}_\alpha((a+h)w)}-\left(\frac{\partial}{\partial w}\right)^{r}\overline{\hat{\psi}_\alpha(aw)}\right)-\frac{\partial}{\partial a}\left(\frac{\partial}{\partial w}\right)^{r}\overline{\hat{\psi}_\alpha(aw)}\right\}\right|\\
&=\left|w^k\sum_{r=0}^{m}\binom{m}{r}\left(\left(\frac{\partial}{\partial w}\right)^{m-r}e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}}b}\right)\right.\\
&\quad\left.\times\frac{1}{h}\left(\int_a^{a+h}\left\{\left(\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial w}\right)^{r}\overline{\hat{\psi}_\alpha(tw)}-\left(\frac{\partial}{\partial a}\right)\left(\frac{\partial}{\partial w}\right)^{r}\overline{\hat{\psi}_\alpha(aw)}\right\}dt\right)\right|\\
&=\left|w^k\sum_{r=0}^{m}\binom{m}{r}\left(\left(\frac{\partial}{\partial w}\right)^{m-r}e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}}b}\right)\right.\\
&\quad\left.\times\frac{1}{h}\int_a^{a+h}\left(\int_a^{t}\left(\frac{\partial}{\partial u}\right)^{2}\left(\frac{\partial}{\partial w}\right)^{r}\overline{\hat{\psi}_\alpha(uw)}du\right)dt\right|\\
&\leq\left|w^k\sum_{r=0}^{m}\binom{m}{r}\left(\left(\frac{\partial}{\partial w}\right)^{m-r}e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}}b}\right)\right|\\
&\quad\times\frac{h}{2}\sup_{a\leq u\leq a+h}\left|\left(\frac{\partial}{\partial u}\right)^{2}\left(\frac{\partial}{\partial w}\right)^{r}\overline{\hat{\psi}_\alpha(uw)}\right|. \qquad (3.3.3)
\end{aligned}$$

From [14, p.535], we can write

$$\begin{aligned}
&\left(\frac{\partial}{\partial w}\right)^{m-r}e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}}b}\\
&=(m-r)!\left(\frac{(ib)e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}}b}\left(\frac{1}{\alpha}\right)\left(\frac{1}{\alpha}-1\right)\left(\frac{1}{\alpha}-2\right)\ldots\left(\frac{1}{\alpha}-(m-r)+1\right)}{1!(m-r)!}\right.\\
&\qquad\times|w|^{\frac{1}{\alpha}-(m-r)}(\operatorname{sign} w)^{m-r+1}+\ldots\\
&\qquad\left.+\frac{(ib)^{m-r}e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}}b}\frac{1}{\alpha^{m-r}}}{(1!)^{m-r}(m-r)!}|w|^{\frac{m-r}{\alpha}-(m-r)}(\operatorname{sign} w)^{2(m-r)}\right)\\
&=(m-r)!\Big(A_1(ib)e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}}b}(\operatorname{sign} w)^{m-r+1}|w|^{\frac{1}{\alpha}-(m-r)}+\ldots\\
&\qquad+A_{m-r}(ib)^{m-r}e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}}b}(\operatorname{sign} w)^{2(m-r)}|w|^{\frac{m-r}{\alpha}-(m-r)}\Big), \qquad (3.3.4)
\end{aligned}$$

where $A_1, A_2, ...A_{m-r}$ are constants.

Next, from (3.3.4), we have

$$\left|\left(\frac{\partial}{\partial w}\right)^{m-r} e^{i(\text{sign}\, w)|w|^{\frac{1}{\alpha}}b}\right| \leq (m-r)!\Big(|A_1||b||w|^{\frac{1}{\alpha}-(m-r)} + \dots + |A_{m-r}||b|^{m-r}|w|^{(m-r)(\frac{1}{\alpha}-1)}\Big). \tag{3.3.5}$$

Using (3.3.3) and (3.3.5), we obtain

$$\begin{aligned}&\left|w^k\left(\frac{\partial}{\partial w}\right)^m\left(e^{i(\text{sign}\, w)|w|^{\frac{1}{\alpha}}b}\left(\frac{1}{h}\big(\overline{\hat{\psi}_\alpha((a+h)w)} - \overline{\hat{\psi}_\alpha(aw)}\big) - \frac{\partial}{\partial a}\overline{\hat{\psi}_\alpha(aw)}\right)\right)\right|\\ &\leq \sum_{r=0}^{m}\binom{m}{r}(m-r)!\Big(|A_1||b||w|^{\frac{1}{\alpha}-(m-r)} + \dots + |A_{m-r}||b|^{m-r}|w|^{(m-r)(\frac{1}{\alpha}-1)}\Big)\\ &\quad\times \frac{h}{2}\sup_{a\leq u\leq a+h}\left|w^k\left(\frac{\partial}{\partial u}\right)^2\left(\frac{\partial}{\partial w}\right)^r\overline{\hat{\psi}_\alpha(uw)}\right|. \end{aligned}\tag{3.3.6}$$

By substituting $uw = z$ in (3.3.6), we get

$$\begin{aligned}&\left|w^k\Big(\frac{\partial}{\partial w}\Big)^m e^{i(\text{sign}\, w)|w|^{\frac{1}{\alpha}}b}\left(\frac{1}{h}\Big(\overline{\hat{\psi}_\alpha((a+h)w)} - \overline{\hat{\psi}_\alpha(aw)}\Big) - \frac{\partial}{\partial a}\overline{\hat{\psi}_\alpha(aw)}\right)\right|\\ &\leq \sum_{r=0}^{m}\binom{m}{r}(m-r)!\left(|A_1||b|\left|\frac{z}{u}\right|^{\frac{1}{\alpha}-(m-r)} + \dots + |A_{m-r}||b|^{m-r}\left|\frac{z}{u}\right|^{(m-r)(\frac{1}{\alpha}-1)}\right)\\ &\quad\times \frac{h}{2}\sup_{a\leq u\leq a+h}\left|z^{k+2}u^{r-2-k}\left(\frac{\partial}{\partial z}\right)^{r+2}\overline{\hat{\psi}_\alpha(z)}\right|\\ &\leq \sum_{r=0}^{m}\binom{m}{r}(m-r)!\frac{h}{2}\left(|A_1||b|\sup_{z\in\mathbb{R}}\left|z^{k+2+\frac{1}{\alpha}-(m-r)}\left(\frac{\partial}{\partial z}\right)^{r+2}\overline{\hat{\psi}_\alpha(z)}\right|\right.\\ &\quad\times \sup_{a\leq u\leq a+h}|u|^{r-2-k-\frac{1}{\alpha}+(m-r)} + \dots + |A_{m-r}||b|^{m-r}\\ &\quad\times \sup_{z\in\mathbb{R}}\left|z^{k+2+(m-r)(\frac{1}{\alpha}-1)}\left(\frac{\partial}{\partial z}\right)^{r+2}\overline{\hat{\psi}_\alpha(z)}\right|\\ &\quad\left.\times \sup_{a\leq u\leq a+h}|u|^{r-2-k-(m-r)(\frac{1}{\alpha}-1)}\right)\\ &\leq \frac{h}{2}\sum_{r=0}^{m}\binom{m}{r}(m-r)!\Big[|A_1||b|\sup_{a\leq u\leq a+h}|u|^{m-2-k-\frac{1}{\alpha}}\\ &\quad\times \gamma_{k+2+\frac{1}{\alpha}-(m-r),r+2}(\hat{\psi}_\alpha) + \dots + |A_{m-r}||b|^{m-r}\end{aligned}$$

$$\times \sup_{a\leq u\leq a+h} |u|^{r-2-k-(m-r)(\frac{1}{\alpha}-1)}$$

$$\times\, \gamma_{k+2+(m-r)(\frac{1}{\alpha}-1),r+2}(\hat{\psi}_\alpha)\Big] \to 0, \quad \text{as} \quad h \to 0.$$

Hence, finally we find that

$$\lim_{h\to 0} \frac{\big(W_{\psi_\alpha}\phi\big)(b, a+h) - \big(W_{\psi_\alpha}\phi\big)(b,a)}{h}$$

$$= \frac{1}{2\pi\alpha}\Big\langle |w|^{\frac{1}{\alpha}-1}\hat{\phi}_\alpha(w),\ e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}}b}\frac{\partial}{\partial a}\overline{\hat{\psi}_\alpha(aw)}\Big\rangle.$$

Similarly, we can prove the differentiability with respect to the variable b and, in general, we can find (3.3.2). □

Theorem 3.3.3. *Let* $\big(W_{\psi_\alpha}\phi\big)(b,a)$ *be the fractional wavelet transform of*

$$|w|^{\frac{1}{\alpha}-1}\hat{\phi}_\alpha(w) \in S'(\mathbb{R}).$$

defined by (3.3.1). Then, for large k *and* $a > 0$*, it is asserted that*

$$\big(W_{\psi_\alpha}\phi\big)(b,a) = O(a^{-k-\frac{k}{\alpha}}|b|^k), \quad a \to 0; \tag{3.3.7}$$

$$= O(a^{2k-\frac{k}{\alpha}}), \quad a \to \infty; \tag{3.3.8}$$

$$= O\big(a^{-\frac{k}{\alpha}}(1+a^2)^k\big), \quad |b| \to 0; \tag{3.3.9}$$

$$= O\big(a^{-k-\frac{k}{\alpha}}(1+a^2)^k|b|^k\big), \quad |b| \to \infty. \tag{3.3.10}$$

Proof. In view of the boundedness property of generalized functions ([27], p.111) there exist a constant $C > 0$ and a non-negative integer k depending on $|w|^{\frac{1}{\alpha}-1}\hat{\phi}_\alpha(w)$ such that

$$|\big(W_{\psi_\alpha}\phi\big)(b,a)| \leq C \sup_w \Big|(1+w^2)^k\Big(\frac{\partial}{\partial w}\Big)^k\{e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}}b}\hat{\psi}_\alpha(aw)\}\Big|$$

$$= C\sup_w \Big|(1+w^2)^k \sum_{s=0}^{k}\binom{k}{s}\Big(\Big(\frac{\partial}{\partial w}\Big)^s e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}} b}\Big)\Big(\Big(\frac{\partial}{\partial w}\Big)^{k-s}\hat{\psi}_\alpha(aw)\Big)\Big|$$

$$= C\sup_w \Big|\sum_{s=0}^{k}\sum_{r=0}^{k}\binom{k}{s}\binom{k}{r} w^{2r}\Big(\Big(\frac{\partial}{\partial w}\Big)^s e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}} b}\Big)\Big(\Big(\frac{\partial}{\partial w}\Big)^{k-s}\hat{\psi}_\alpha(aw)\Big)\Big|.$$

On the other hand, in view of (3.3.5), there exist positive constants $A_1, A_2, \ldots, A_s$ such that

$$|(W_{\psi_\alpha}\phi)(b,a)| \le C\sup_w \Big|\sum_{s=0}^{k}\sum_{r=0}^{k}\binom{k}{s}\binom{k}{r} w^{2r}(s)!\Big(A_1|b||w|^{\frac{1}{\alpha}-s} + \ldots$$
$$+ A_s|b|^s|w|^{\frac{s}{\alpha}-s}\Big)\Big(\Big(\frac{\partial}{\partial w}\Big)^{k-s}\hat{\psi}_\alpha(aw)\Big)\Big|.$$

Let $z = aw$, then we can find

$$|(W_{\psi_\alpha}\phi)(b,a)| \le C\sup_z \Big|\sum_{s=0}^{k}\sum_{r=0}^{k}\binom{k}{s}\binom{k}{r} a^{k-s-2r} z^{2r}(s)!\Big(A_1|b|\Big|\frac{z}{a}\Big|^{\frac{1}{\alpha}-s} + \ldots$$
$$+ A_s|b|^s\Big|\frac{z}{a}\Big|^{\frac{s}{\alpha}-s}\Big)\Big(\Big(\frac{\partial}{\partial w}\Big)^{k-s}\hat{\psi}_\alpha(z)\Big)\Big|$$
$$\le C\sup_z \Big|\sum_{s=0}^{k}\sum_{r=0}^{k}\binom{k}{s}\binom{k}{r}(s)!\Big(A_1|b|a^{k-\frac{1}{\alpha}-2r}|z|^{2r+\frac{1}{\alpha}-s} + \ldots$$
$$+ A_s|b|^s a^{k-\frac{s}{\alpha}-2r}|z|^{2r+\frac{s}{\alpha}-s}\Big)\Big(\Big(\frac{d}{dz}\Big)^{k-s}\hat{\psi}_\alpha(z)\Big)\Big|$$
$$\le C\sum_{s=0}^{k}\sum_{r=0}^{k}\binom{k}{s}\binom{k}{r}(s)!\Big(A_1|b|a^{k-\frac{1}{\alpha}-2r}\sup_z\Big|z^{2r+\frac{1}{\alpha}-s}\Big(\frac{d}{dz}\Big)^{k-s}$$
$$\times\hat{\psi}_\alpha(z)\Big| + \cdots + A_s|b|^s a^{k-\frac{s}{\alpha}-2r}\sup_z\Big|z^{2r+\frac{s}{\alpha}-s}\Big(\frac{d}{dz}\Big)^{k-s}\hat{\psi}_\alpha(z)\Big|\Big)$$
$$\le C\sum_{s=0}^{k}\sum_{r=0}^{k}\binom{k}{s}\binom{k}{r}(s)!\Big(A_1|b|a^{k-\frac{1}{\alpha}-2r}\gamma_{2r+\frac{1}{\alpha}-s,k-s}\big(\hat{\psi}_\alpha(z)\big) +$$
$$\cdots + A_s|b|^s a^{k-\frac{s}{\alpha}-2r}\gamma_{2r+\frac{s}{\alpha}-s,k-s}\big(\hat{\psi}_\alpha(z)\big)\Big)$$
$$= C\sum_{s=0}^{k}\sum_{r=0}^{k}\sum_{l=1}^{s}\binom{k}{s}\binom{k}{r}(s)!A_l|b|^l a^{k-\frac{l}{\alpha}-2r}\gamma_{2r+\frac{l}{\alpha}-s,k-s}\big(\hat{\psi}_\alpha(z)\big)$$
$$\le C'\sum_{s=0}^{k}\sum_{r=0}^{k}\binom{k}{s}\binom{k}{r}(s)!A_s|b|^s a^{k-\frac{s}{\alpha}-2r}\gamma_{2r+\frac{s}{\alpha}-s,k-s}\big(\hat{\psi}_\alpha(z)\big)$$

$$\leq C'' \sum_{r=0}^{k} \binom{k}{r} a^{-2r} a^{k-\frac{k}{\alpha}} (a+|b|)^k$$
$$= C''(1+a^{-2})^k a^{k-\frac{k}{\alpha}} (a+|b|)^k. \tag{3.3.11}$$

Thus, from (3.3.11) we are led to the assertions (3.3.7), (3.3.8), (3.3.9) and (3.3.10).

□

For proving Theorem 3.3.4, we assume that

$$D^s \hat{\psi}_\alpha(w) = O(|w|^\mu), \quad |w| \to 0, \quad \forall s \in \mathbb{N}_0, \tag{3.3.12}$$

for some real number μ.

Theorem 3.3.4. *Let $\psi \in S(\mathbb{R})$ and $\phi \in S^{'}(\mathbb{R})$ be a distribution of compact support in $\mathbb{R}$. Then*

$$\big(W_{\psi_\alpha}\phi\big)(b,a) = \frac{1}{2\pi\alpha}\big\langle |w|^{\frac{1}{\alpha}-1}\hat{\phi}_\alpha(w), e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}}b}\overline{\hat{\psi}_\alpha(aw)}\big\rangle,$$

is a smooth function on $\mathbb{R}\times\mathbb{R}_+$ and satisfies the following condition:

$$\big(W_{\psi_\alpha}\phi\big)(b,a) = O\big(a^\mu(1+a+|b|)^k\big), \quad |a| \to 0, \;\; k \in \mathbb{N}. \tag{3.3.13}$$

Proof. Let $\phi \in S^{'}(\mathbb{R})$. Then, from Theorem 2.3 of [14], we have $\hat{\phi}_\alpha \in S^{'}(\mathbb{R})$. We assume that $\hat{\phi}_\alpha$ is of compact support $K \subset \mathbb{R}$. We also let $\lambda(w) \in \mathcal{D}(\mathbb{R})$, the space of all C^∞- functions of compact support such that $\lambda(w) = 1$ in a neighborhood of K. Therefore, we get

$$\big(W_{\psi_\alpha}\phi\big)(b,a) = \frac{1}{2\pi\alpha}\big\langle |w|^{\frac{1}{\alpha}-1}\hat{\phi}_\alpha(w), e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}}b}\overline{\hat{\psi}_\alpha(aw)}\big\rangle$$
$$= \frac{1}{2\pi\alpha}\big\langle |w|^{\frac{1}{\alpha}-1}\hat{\phi}_\alpha(w), \lambda(w)e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}}b}\overline{\hat{\psi}_\alpha(aw)}\big\rangle.$$

So, by Theorem 3.3.2, $\big(W_{\psi_\alpha}\phi\big)(b,a)$ is infinitely differentiable with respect to the variables b and a. Thus, by the boundedness property of generalized functions as used in Theorem 3.3.3, we have

$$\begin{aligned}
\left|\big(W_{\psi_\alpha}\phi\big)(b,a)\right| &= \frac{1}{2\pi\alpha}\left|\left\langle |w|^{\frac{1}{\alpha}-1}\hat{\phi}_\alpha(w), e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}}b}\overline{\hat{\psi}_\alpha(aw)}\right\rangle\right| \\
&\le C\max_r \sup_{w\in K}\left|D_w^r\left[\lambda(w)e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}}b}\overline{\hat{\psi}_\alpha(aw)}\right]\right| \\
&\le C\max_r \sup_{w\in K}\sum_{n=0}^{r}\binom{r}{n}\left|\big(D_w^{r-n}\lambda(w)\big)D_w^n\Big(e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}}b}\overline{\hat{\psi}_\alpha(aw)}\Big)\right| \\
&\le C\max_r \sup_{w\in K}\sum_{n=0}^{r}\binom{r}{n}\Big|\big(D_w^{r-n}\lambda(w)\big) \\
&\quad\times\sum_{s=0}^{n}\binom{n}{s}\Big(D_w^{n-s}e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}}b}\Big)\Big(D_w^s\overline{\hat{\psi}_\alpha(aw)}\Big)\Big|.
\end{aligned}$$

In view of (3.3.5), there exists positive constants $A_1,\dots A_{n-s}$ such that

$$\begin{aligned}
\left|\big(W_{\psi_\alpha}\phi\big)(b,a)\right| &\le C\max_r \sup_{w\in K}\sum_{n=0}^{r}\sum_{s=0}^{n}\binom{r}{n}\binom{n}{s}\left|\big(D_w^{r-n}\lambda(w)\big)\right| \\
&\quad\times(n-s)!\left(A_1|b||w|^{\frac{1}{\alpha}-(n-s)}+\cdots+A_{n-s}|b|^{n-s}|w|^{(n-s)\left(\frac{1}{\alpha}-1\right)}\right) \\
&\quad\times\left|\Big(D_w^s\overline{\hat{\psi}_\alpha(aw)}\Big)\right| \\
&\le C'\max_r \sup_{w\in K}\sum_{n=0}^{r}\sum_{s=0}^{n}\binom{r}{n}\binom{n}{s}\left|\big(D_w^{r-n}\lambda(w)\big)\right| \\
&\quad\times\left(\sum_{l=1}^{n-s}A_l|b|^l|w|^{\frac{l}{\alpha}-(n-s)}\right)\left|\Big(D_w^s\overline{\hat{\psi}_\alpha(aw)}\Big)\right| \\
&\le C''\max_r \sup_{w\in K}\sum_{n=0}^{r}\sum_{s=0}^{n}\binom{r}{n}\binom{n}{s}|b|^{n-s}|w|^{(n-s)\left(\frac{1}{\alpha}-1\right)}a^{s+\mu}|w|^{\mu} \\
&\le C''\max_r\sum_{n=0}^{r}\binom{r}{n}\left(\sum_{s=0}^{n}\binom{n}{s}|b|^{n-s}\right)a^{s+\mu} \\
&\le C''\max_r\sum_{n=0}^{r}\binom{r}{n}\big(a+|b|\big)^n a^{\mu}
\end{aligned}$$

$$= C'' \max_r \left(1 + a + |b|\right)^r a^\mu,$$

where C'' is a positive constants. Hence we have

$$\left|\left(W_{\psi_\alpha}\phi\right)(b,a)\right| \leq C'' \max_r \left(1 + a + |b|\right)^r a^\mu.$$

□

The aforesaid theorem is useful for finding Abelian theorems, which are given below:

Theorem 3.3.5. ***(Initial-Value Theorem)***
Let $\hat{\phi}_\alpha \in S'(\mathbb{R})$ which can be decomposed into

$$\hat{\phi}_\alpha = \phi_1 + \phi_2,$$

where ϕ_1 is an ordinary function and $\phi_2 \in \mathscr{E}'(\mathbb{R} - \{0\})$ is of order k. Also let real numbers μ and η be such that

$$1 + \frac{1}{\alpha} + 2k - \frac{k}{\alpha} < \eta < \mu + 1 + \frac{1}{\alpha}.$$

Suppose also that

$$|w|^{\frac{1}{\alpha}-\eta}\hat{\psi}_\alpha(w) \in L^1(\mathbb{R})$$

and

$$|w|^{\frac{1}{\alpha}-1}\phi_1(w) \in L^1(\delta,\infty), \ \forall \ \ \delta > 0,$$

and assume that

$$\left(W_{\psi_\alpha}\phi\right)(b,a)$$

is the distributional wavelet transform of $|w|^{\frac{1}{\alpha}-1}\hat{\phi}_\alpha$, *which is defined by (3.3.1). Then*

$$\lim_{a\to\infty} a^{1-\eta+\frac{1}{\alpha}}\big(W_{\psi_\alpha}\phi\big)(b,a) = F_1(\alpha,\eta)\lim_{|w|\to 0}(2\pi\alpha)^{-1}|w|^{-1+\eta}\hat{\phi}_\alpha(w). \tag{3.3.14}$$

Proof. By Theorem 3.3.2, we see that

$$\big(W_{\psi_\alpha}\phi_2\big)(b,a) = \frac{1}{2\pi\alpha}\Big\langle |w|^{\frac{1}{\alpha}-1}\phi_2(w),\ e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}}b}\overline{\hat{\psi}_\alpha(aw)}\Big\rangle,$$

is an infinitely differentiable function on $\mathbb{R}\times\mathbb{R}_+$. Furthermore, by Theorem 3.3.3,

$$\big(W_{\psi_\alpha}\phi_2\big)(b,a) = O(a^{2k-\frac{k}{\alpha}}),\quad a\to\infty.$$

Hence there exists a constant $C>0$ such that

$$\Big|a^{1-\eta+\frac{1}{\alpha}}\big(W_{\psi_\alpha}\phi_2\big)(b,a)\Big| \le Ca^{1-\eta+\frac{1}{\alpha}+2k-\frac{k}{\alpha}}. \tag{3.3.15}$$

Since

$$1-\eta+\frac{1}{\alpha}+2k-\frac{k}{\alpha}<0,$$

the right hand side of (3.3.15) tends to 0, as $a\to\infty$. Also, since the support of $\phi_2\in\mathscr{E}'(\mathbb{R}-\{0\})$ is a compact subset of $\mathbb{R}-\{0\}$, we get

$$\lim_{w\to 0} e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}}b}|w|^{-1+\eta}\phi_2(w) = 0.$$

The result asserted by Theorem 3.3.5 follows by an application of Theorem 3.2.1 with $\hat{\phi}_\alpha(w)$ replaced by $\phi_1(w)$. □

Theorem 3.3.6. ***(Final-Value Theorem)***

Let

$$1+\frac{1}{\alpha}<\eta<\mu+1+\frac{1}{\alpha},\quad \mu>0.$$

Assume thet $\hat{\phi}_\alpha \in S'(\mathbb{R})$ can be decomposed into $\hat{\phi}_\alpha = \phi_1 + \phi_2$, where ϕ_1 is an ordinary function satisfying the following condition:

$$|w|^{\mu+\frac{1}{\alpha}-1}\phi_1(w) \in L^1(-X, X), \quad \forall\ X > 0$$

and $\phi_2 \in \mathscr{E}'(\mathbb{R} - \{0\})$. If $\big(W_{\psi_\alpha}\phi\big)(b, a)$ is the distributional wavelet transform of $|w|^{\frac{1}{\alpha}-1}\hat{\phi}_\alpha$ defined by (3.3.1), then

$$\lim_{a\to 0} a^{1-\eta+\frac{1}{\alpha}}\big(W_{\psi_\alpha}\phi\big)(b, a) = F_1(\alpha, \eta)(2\pi\alpha)^{-1} \lim_{w\to\infty} e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}}b}|w|^{-1+\eta}\hat{\phi}_\alpha(w). \tag{3.3.16}$$

Proof. By Theorem 3.3.2 and Theorem 3.3.4, we observe that

$$\big(W_{\psi_\alpha}\phi_2\big)(b, a) = \frac{1}{2\pi\alpha}\big\langle|w|^{\frac{1}{\alpha}-1}\phi_2(w), e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}}b}\overline{\hat{\psi}_\alpha(aw)}\big\rangle,$$

is an infinitely differentiable function on $\mathbb{R} \times \mathbb{R}_+$ and

$$\big(W_{\psi_\alpha}\phi_2\big)(b, a) = Ca^\mu(1 + |b|)^k, \quad \text{as} \quad a \to 0,$$

C being a large constant.

$$\text{Since} \quad 1 - \eta + \frac{1}{\alpha} + \mu > 0, \quad \text{we have}$$

$$a^{1-\eta+\frac{1}{\alpha}}\big|\big(W_{\psi_\alpha}\phi_2\big)(b, a)\big| \le Ca^{1-\eta+\frac{1}{\alpha}+\mu}(1 + |b|)^k, \to 0 \quad \text{as} \quad a \to 0.$$

By taking $\hat{\phi}_\alpha(w)$ to be $\phi_2(w)$, the final result follows from Theorem 3.2.2. □

3.4 Application

In this section, an application and justification of Abelian theorems regarding the fractional wavelet transform is presented by using the Mexican hat wavelet function.

The Mexican hat wavelet function is given by

$$\psi(x) = (1 - x^2)e^{-\frac{1}{2}x^2}. \tag{3.4.1}$$

Also, from Example 1.6.4 of [23, p.11], the Fourier transform of (3.4.1) is given by

$$\hat{\psi}(w) = \sqrt{2\pi}|w|^2 e^{-\frac{|w|^2}{2}},$$

where $\hat{\psi}(w)$ is the Fourier transform of $\psi(x)$.

Now, by Remark 5 of [4, p. 787], the fractional Fourier transform of (3.4.1) is given by

$$\begin{aligned}\hat{\psi}_\alpha(w) &= \hat{\psi}((\operatorname{sign} w)|w|^{\frac{1}{\alpha}}) \\ &= \sqrt{2\pi}|w|^{\frac{2}{\alpha}} e^{-\frac{1}{2}|w|^{\frac{2}{\alpha}}}.\end{aligned} \tag{3.4.2}$$

Furthermore, the following asymptotic order of $\hat{\psi}_\alpha(w)$ holds true:

$$\hat{\psi}_\alpha(w) = O(w^{\frac{2}{\alpha}}), \quad |w| \to 0. \tag{3.4.3}$$

Hence, in view of (3.1.5) and (3.4.2), we have the following fractional wavelet transform:

$$\big(W_{\psi_\alpha}\phi\big)(b, a) = \frac{1}{\sqrt{2\pi}\alpha} \int_{-\infty}^{+\infty} e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}}b} |w|^{\frac{1}{\alpha}-1} \hat{\phi}_\alpha(w)$$

$$\times |aw|^{\frac{2}{\alpha}} e^{-\frac{1}{2}|aw|^{\frac{2}{\alpha}}} dw. \tag{3.4.4}$$

Thus, from (3.2.2) and (3.4.2), we find the following expression of $F_1(\alpha, \eta)$:

$$\begin{aligned} F_1(\alpha,\eta) &= \int_{-\infty}^{+\infty} \sqrt{2\pi}|w|^{\frac{2}{\alpha}} e^{-\frac{1}{2}|w|^{\frac{2}{\alpha}}} |w|^{\frac{1}{\alpha}-\eta} dw \\ &= \int_{-\infty}^{+\infty} \sqrt{2\pi}|w|^{\frac{3}{\alpha}-\eta} e^{-\frac{1}{2}|w|^{\frac{2}{\alpha}}} dw, \end{aligned}$$

which, in view of the following familiar Gamma-function result:

$$\int_0^{\infty} x^m e^{-ax^n} dx = \frac{1}{na^{\left(\frac{m+1}{n}\right)}} \Gamma\left(\frac{m+1}{n}\right), \quad \text{for } Re(m) > -1;\ min\{Re(n), Re(a)\} > 0,$$

can be rewritten as follows:

$$F_1(\alpha,\eta) = \alpha\pi^{\frac{1}{2}} 2^{\frac{1}{2}(4+\alpha-\alpha\eta)} \Gamma\left(\frac{3-\alpha\eta+\alpha}{2}\right), \quad \text{for } \eta < \frac{3}{\alpha}+1. \tag{3.4.5}$$

Therefore, by a modification of proof of Theorem 3.2.1, for

$$\eta < \frac{3}{\alpha} + 1$$

and

$$e^{-\frac{1}{2}|w|^{\frac{2}{\alpha}}} |w|^{\frac{1}{\alpha}-1} \hat{\phi}_\alpha(w) \in L^1(\delta,\infty), \quad \forall\ \delta > 0$$

and, by using (3.4.5), we find that

$$\begin{aligned} \lim_{a\to\infty} a^{1-\eta+\frac{1}{\alpha}} \big(W_{\psi_\alpha}\phi\big)(b,a) &= \pi^{-\frac{1}{2}} 2^{\frac{1}{2}(2+\alpha-\alpha\eta)} \Gamma\left(\frac{3-\alpha\eta+\alpha}{2}\right) \\ &\quad \times \lim_{|w|\to 0} |w|^{-1+\eta} \hat{\phi}_\alpha(w). \end{aligned} \tag{3.4.6}$$

Furthermore, by applying Theorem 3.2.2, for

$$\eta < \frac{3}{\alpha} + 1$$

and

$$|w|^{\frac{3}{\alpha}-1}\hat{\phi}_\alpha(w) \in L^1(-X, X), \quad \forall\ X > 0$$

and, by using (3.4.5), we get

$$\lim_{a\to 0} a^{1-\eta+\frac{1}{\alpha}}\left(W_{\psi_\alpha}\phi\right)(b,a) = \pi^{-\frac{1}{2}}2^{\frac{1}{2}(2+\alpha-\alpha\eta)}\Gamma\left(\frac{3-\alpha\eta+\alpha}{2}\right) \times \lim_{|w|\to\infty} e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}}b}|w|^{-1+\eta}\hat{\phi}_\alpha(w). \tag{3.4.7}$$

Finally, upon taking into account the fact that the kernel $\hat{\psi}_\alpha(w)$ is exponentially decreasing, the conditions of validity of the initial-value and final-value results are relaxed in this example. By using (3.4.6) and (3.4.7), we can obtain the corresponding results derivable from Theorem 3.3.5 and Theorem 3.3.6 respectively.

3.5 Conclusions

In several earlier developments (see, for example, [3, 4, 14–16]), one can find that the fractional wavelet transform has rich theory and extensive mathematical background. This theory presents a study of Abelian theorems in classical as well as distributional sense both. In our investigation herein, we have established several Abelian theorems of the fractional wavelet transform. Moreover, with the help of an example, we have shown that the Mexican hat function is a fractional wavelet function, which contains adequate time and frequency localization and justifies Abelian theorems involving the fractional wavelet transform. Upon a systematic survey of the existing literature on Abelian theorems for many different integral transforms, we conclude that Abelian theorems associated with the fractional wavelet transform provide potentially useful information about the initial and final values of the fractional wavelet transform which we have investigated herein.

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