Chapter 2

Pseudo-Differential Operators associated with modified Fractional Derivatives involving the Fractional Fourier Transform

2.1 Introduction

The theory of fractional Fourier transform encompasses the theory of Fourier transform and played a widespread role in mathematics, engineering, and other areas of sciences. Fractional Fourier transform is the generalization of the Fourier transform, which is the more flexible approach in the applications of ordinary differential equations, partial differential equations, quantum mechanics, signal processing, and other fields also. Many observations of this aforesaid theory have been done by authors and got important results. Ozaktas et al. [1] have discussed the fractional Fourier transform and its applications. The concept of the fractional Fourier transform was also investigated by De Bie et al. [2] and obtained many important results in the area of signal processing. A new concept of fractional Fourier transform was studied by Luchko et al. [3, 4] and discussed various properties on the Lizorkin space by taking the modified Liouville fractional derivatives and integrals.

Pseudo-differential operators considered interesting tools and were introduced by Kohn-Nirenberg [5], Hormander [6, 7], Wong [8] and others. By taking the Fourier transform theory, they studied pseudo-differential operators associated with different types of symbols on Schwartz space $S(\mathbb{R}^n)$. Cappiello [9, 10], Boutet de Monvel [11], Zanghirati [12] and Upadhyay et al. [13] considered the characterizations of pseudo-differential operators of the infinite order on Gevrey, Gelfand and Shilov types of spaces. Motivated from the results of [3, 4], Srivastava et al. [14] discussed pseudo-differential operators on Schwartz space $S(\mathbb{R})$ by exploiting the theory of the fractional Fourier transform.

Our main objective in this chapter is to study the pseudo-differential operators on $S(\mathbb{R}^n)$ and various properties by taking n-dimensional fractional Fourier transform. In the present chapter, author shall consider the following contents:

1. To introduce the n-dimensional fractional Fourier transform and its inversion formula.
2. To discuss the convolution property and Plancheral formula.
3. To find continuity properties on $S(\mathbb{R}^n)$ and $S'(\mathbb{R}^n)$.
4. To define pseudo-differential operators on $S(\mathbb{R}^n)$ space and their characterizations.
5. An integral representation and boundedness of pseudo-differential operators are given.

6. The boundedness of pseudo-differential operator on the Sobolev space discussed by exploiting the fractional Fourier transform theory.

7. Applications of pseudo-differential operators on the Lizorkin space $\Phi(\mathbb{R})$ are given and by using modified fractional derivative operator D^{α}_{β} and integral operator I^{α}_{β}, operational relations are obtained.

The entire chapter is organized in the following manner:

Section 2.1 is introductory, which contains brief history of the fractional Fourier transform and pseudo-differential operators. We define n-dimensional fractional Fourier transform and its inversion formula. In Section 2.2, we studied many properties of the n-dimensional fractional Fourier transform like convolution properties, continuity properties and Parseval formula. In Section 2.3, we established the continuity properties of pseudo-differential operators. An integral representation and boundedness of pseudo-differential operators are made in Section 2.4. The boundedness of pseudo-differential operator on the Sobolev space are discussed in Section 2.5. In Section 2.6, by using modified fractional derivative operator and integral operator, applications of pseudo-differential operators on the Lizorkin space are obtained.

Definition 2.1.1. (**Fractional Fourier Transform on** $\mathbb{R}^n$)
Let $\phi \in L^1(\mathbb{R}^n)$ and $\alpha \in (0,1]^n$, then the fractional Fourier transform of order α is defined by

$$(\mathcal{F}_\alpha \phi)(w) \equiv \hat{\phi}_\alpha(w) := (2\pi)^{\frac{-n}{2}} \int_{\mathbb{R}^n} e^{-i\sum_{j=1}^{n}(\operatorname{sign} w_j)|w_j|^{\frac{1}{\alpha_j}} x_j} \phi(x)dx, \quad w \in \mathbb{R}^n. \quad (2.1.1)$$

If we take $\alpha = (1, 1, \dots, 1)$, then (2.1.1) becomes the classical Fourier transform.

The inverse fractional Fourier transform is defined by

$$\mathcal{F}_\alpha^{-1}(\mathcal{F}_\alpha\phi)(x) := \frac{1}{(2\pi)^{\frac{n}{2}}[\alpha]} \int_{\mathbb{R}^n} e^{i\sum_{j=1}^n (\operatorname{sign} w_j)|w_j|^{\frac{1}{\alpha_j}} x_j} \left(\prod_{j=1}^n |w_j|^{\frac{1}{\alpha_j}-1}\right) \times (\mathcal{F}_\alpha\phi)(w)dw, \qquad x \in \mathbb{R}^n. \tag{2.1.2}$$

2.2 Properties of the fractional Fourier transform

In this section, we studied the properties of the fractional Fourier transform on $S(\mathbb{R}^n)$.

Proposition 2.2.1. *Let $\mathcal{F}_\alpha$ be the fractional Fourier transform, $\mathcal{F}$ be the Fourier transform and $\alpha \in (0,1]^n$. Then*

1. *$\mathcal{F}_\alpha : L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$ is a bounded linear operator.*
2. *If $\phi \in L^1(\mathbb{R}^n)$ and $\mathcal{F}(\phi) \in L^1(\mathbb{R}^n)$, then $\mathcal{F}_\alpha^{-1}(\mathcal{F}_\alpha\phi) = \phi$ a.e. on $\mathbb{R}^n$.*
3. *$(\mathcal{F}_\alpha\widetilde{\phi})(w) = \widetilde{\mathcal{F}_\alpha\phi}(w)$, $w \in \mathbb{R}^n$, where $\widetilde{\phi}(x) = \phi(-x)$, for all $x \in \mathbb{R}^n$.*
4. *Let $\phi \in L^1(\mathbb{R}^n)$, then*

$$(\mathcal{F}_\alpha\phi)(w) = (\mathcal{F}\phi)\left((\operatorname{sign} w_1)|w_1|^{\frac{1}{\alpha_1}}, (\operatorname{sign} w_2)|w_2|^{\frac{1}{\alpha_2}}, \dots, (\operatorname{sign} w_n)|w_n|^{\frac{1}{\alpha_n}}\right),$$

$$w = (w_1, w_2, \dots w_n) \in \mathbb{R}^n.$$

Proof. 1. Let $\phi \in L^1(\mathbb{R}^n)$, then by using (2.1.1), we have

$$|(\mathcal{F}_\alpha\phi)(w)| \le (2\pi)^{\frac{-n}{2}} \int_{\mathbb{R}^n} |\phi(x)|dx, \quad w \in \mathbb{R}^n,$$

this implies that

$$\sup_{x\in\mathbb{R}^n} |(\mathcal{F}_\alpha\phi)(w)| \leq (2\pi)^{\frac{-n}{2}} \|\phi\|_1.$$

2. Using (2.1.1) and (2.1.2), we have

$$\mathcal{F}_\alpha^{-1}(\mathcal{F}_\alpha\phi)(x) = \frac{1}{(2\pi)^n[\alpha]} \int_{\mathbb{R}^n} e^{i\sum_{j=1}^n (\operatorname{sign} w_j)|w_j|^{\frac{1}{\alpha_j}} x_j} \left(\prod_{j=1}^n |w_j|^{\frac{1}{\alpha_j}-1}\right)$$
$$\times \left(\int_{\mathbb{R}^n} e^{-i\sum_{j=1}^n (\operatorname{sign} w_j)|w_j|^{\frac{1}{\alpha_j}} t_j} \phi(t)dt\right) dw, \quad x \in \mathbb{R}^n.$$

Choosing $y_j = (\operatorname{sign} w_j)|w_j|^{\frac{1}{\alpha_j}}$, $j = 0, 1, \ldots, n$ and $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, then

$$\mathcal{F}_\alpha^{-1}(\mathcal{F}_\alpha\phi)(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix.y} \left(\int_{\mathbb{R}^n} e^{-it.y}\phi(t)dt\right) dy.$$

In view of [8, p. 25], the proof of Part 2 is completed.

The proof of Part 3 and Part 4 is obvious and can be obtained easily. □

Proposition 2.2.2. *Let $\phi \in L^1(\mathbb{R}^n)$, then*

1. $(\mathcal{F}_\alpha T_y\phi)(w) = e^{i\sum_{j=1}^n (\operatorname{sign} w_j)|w_j|^{\frac{1}{\alpha_j}} y_j} (\mathcal{F}_\alpha\phi)(w)$, $w \in \mathbb{R}^n$.
2. $(\mathcal{F}_\alpha D_a\phi)(w) = \frac{1}{a^n}(\mathcal{F}_\alpha\phi)(w^a_\alpha)$, *where* $w^a_\alpha = \left(\frac{w_1}{a^{\alpha_1}}, \frac{w_2}{a^{\alpha_2}}, \ldots, \frac{w_n}{a^{\alpha_n}}\right)$.

Proof. 1. Let $\phi \in L^1(\mathbb{R}^n)$, then by (1.1.5) and (2.1.1), we have

$$(\mathcal{F}_\alpha T_y\phi)(w) = \int_{\mathbb{R}^n} e^{-i\sum_{j=1}^n (\operatorname{sign} w_j)|w_j|^{\frac{1}{\alpha_j}} x_j} \phi(x+y)dx.$$

Let us assume $x + y = s = (s_1, s_2, \ldots, s_n) \in \mathbb{R}^n$, then

$$(\mathcal{F}_\alpha T_y\phi)(w) = (2\pi)^{\frac{-n}{2}} \int_{\mathbb{R}^n} e^{-i\sum_{j=1}^n (\operatorname{sign} w_j)|w_j|^{\frac{1}{\alpha_j}} (s_j-y_j)} \phi(s)ds$$
$$= (2\pi)^{\frac{-n}{2}} e^{i\sum_{j=1}^n (\operatorname{sign} w_j)|w_j|^{\frac{1}{\alpha_j}} y_j} \int_{\mathbb{R}^n} e^{-i\sum_{j=1}^n (\operatorname{sign} w_j)|w_j|^{\frac{1}{\alpha_j}} s_j} \phi(s)ds$$

$$= e^{i\sum_{j=1}^{n}(\text{sign}\, w_j)|w_j|^{\frac{1}{\alpha_j}} y_j}(\mathcal{F}_\alpha\phi)(w).$$

2. Let $\phi \in L^1(\mathbb{R}^n)$, then by (1.1.6) and (2.1.1), we have

$$(\mathcal{F}_\alpha D_a\phi)(w) = (2\pi)^{\frac{-n}{2}} \int_{\mathbb{R}^n} e^{-i\sum_{j=1}^{n}(\text{sign}\, w_j)|w_j|^{\frac{1}{\alpha_j}} x_j}\phi(ax)dx.$$

By taking $ax = s$, we get

$$\begin{aligned}(\mathcal{F}_\alpha D_a\phi)(w) &= (2\pi)^{\frac{-n}{2}} \int_{\mathbb{R}^n} e^{-i\sum_{j=1}^{n}(\text{sign}\, w_j)|w_j|^{\frac{1}{\alpha_j}} \frac{s_j}{a}}\phi(s)\frac{1}{a^n}ds \\ &= (2\pi)^{\frac{-n}{2}}\frac{1}{a^n} \int_{\mathbb{R}^n} e^{-i\sum_{j=1}^{n}\left(\text{sign}\, \frac{w_j}{a^{\alpha_j}}\right)\left|\frac{w_j}{a^{\alpha_j}}\right|^{\frac{1}{\alpha_j}} s_j}\phi(s)dx \\ &= \frac{1}{a^n}(\mathcal{F}_\alpha\phi)(w_\alpha^a).\end{aligned}$$

□

Proposition 2.2.3. *Let $\phi, \psi \in L^1(\mathbb{R}^n)$, then*

$$\mathcal{F}_\alpha(\phi * \psi) = (2\pi)^{\frac{n}{2}}(\mathcal{F}_\alpha\phi)(\mathcal{F}_\alpha\psi). \tag{2.2.1}$$

Proof. Using (1.1.4), (2.1.1) and the Fubini's theorem, we have

$$\begin{aligned}\mathcal{F}_\alpha(\phi * \psi)(w) &= (2\pi)^{\frac{-n}{2}} \int_{\mathbb{R}^n} e^{-i\sum_{j=1}^{n} x_j(\text{sign}\, w_j)|w_j|^{\frac{1}{\alpha_j}}}(\phi * \psi)(x)dx \\ &= (2\pi)^{\frac{-n}{2}} \int_{\mathbb{R}^n} e^{-i\sum_{j=1}^{n} x_j(\text{sign}\, w_j)|w_j|^{\frac{1}{\alpha_j}}}\left(\int_{\mathbb{R}^n}\phi(x-y)\psi(y)dy\right)dx \\ &= (2\pi)^{\frac{-n}{2}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\sum_{j=1}^{n} x_j(\text{sign}\, w_j)|w_j|^{\frac{1}{\alpha_j}}}\phi(x-y)\psi(y)dydx.\end{aligned}$$

On putting $x - y = s$, we get

$$\mathcal{F}_\alpha(\phi * \psi)(w) = (2\pi)^{\frac{-n}{2}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\sum_{j=1}^{n}(y_j+s_j)(\text{sign}\, w_j)|w_j|^{\frac{1}{\alpha_j}}}\phi(s)\psi(y)dyds$$

$$= (2\pi)^{\frac{-n}{2}} \int_{\mathbb{R}^n} e^{-i\sum_{j=1}^{n} y_j(\operatorname{sign} w_j)|w_j|^{\frac{1}{\alpha_j}}} \psi(y)$$
$$\times \left(\int_{\mathbb{R}^n} e^{-i\sum_{j=1}^{n} s_j(\operatorname{sign} w_j)|w_j|^{\frac{1}{\alpha_j}}} \phi(s)ds \right) dy.$$

By using the definition of fractional Fourier transform (2.1.1), we have

$$\mathcal{F}_\alpha(\phi * \psi)(w) = (2\pi)^{\frac{n}{2}}(\mathcal{F}_\alpha\phi)(w)(\mathcal{F}_\alpha\psi)(w), \quad \text{for all } w \in \mathbb{R}^n.$$

□

Proposition 2.2.4. *$\mathcal{F}_\alpha$ is a continuous linear isomorphism from $S(\mathbb{R}^n)$ onto $S(\mathbb{R}^n)$.*

Proof. Let $\phi \in S(\mathbb{R}^n)$, $\beta = (\beta_1, \beta_2, \dots \beta_n) \in \mathbb{N}_0^n$ and $w = (w_1, w_2, \dots w_n) \in \mathbb{R}^n$, then from (2.1.1), we get

$$D_w^\beta(\mathcal{F}_\alpha\phi)(w) = \int_{\mathbb{R}^n} D_w^\beta(e^{-i\sum_{j=1}^{n}(\operatorname{sign} w_j)|w_j|^{\frac{1}{\alpha_j}} x_j})\phi(x)dx, \quad w \in \mathbb{R}^n \tag{2.2.2}$$

From [14, p. 537], consider $\alpha_j\beta_j \leq 1$ for each j and on simplifying, (2.2.2) becomes

$$\left| \left(\prod_{j=1}^{n} w_j^{\frac{\gamma_j}{\alpha_j}}\right) D_w^\beta(\mathcal{F}_\alpha\phi)(w) \right| \leq \left(\prod_{j=1}^{n} \frac{\beta_j}{(\alpha_j)^{\beta_j}}\right)\left(|\prod_{j=1}^{n} w_j^{\frac{\beta_j}{\alpha_j}-\beta_j}|\right) \int_{\mathbb{R}^n} \sum_{v_1=0}^{\gamma_1}\binom{\gamma_1}{v_1}\sum_{v_2=0}^{\gamma_2}$$
$$\times \binom{\gamma_2}{v_2} \dots \sum_{v_n=0}^{\gamma_n}\binom{\gamma_n}{v_n} |(D_{x_1}^{\gamma_1-v_1} D_{x_2}^{\gamma_2-v_2} \dots D_{x_n}^{\gamma_n-v_n}\phi(x))$$
$$\times (x_1^{\beta_1-v_1} x_2^{\beta_2-v_2} \dots x_n^{\beta_n-v_n})|dx,$$

which implies that

$$\left| \left(\prod_{j=1}^{n} w_j^{\frac{\gamma_j-\beta_j}{\alpha_j}+\beta_j}\right) D_w^\beta(\mathcal{F}_\alpha\phi)(w) \right| \leq \left(\prod_{j=1}^{n} \frac{\beta_j}{(\alpha_j)^{\beta_j}}\right) \sum_{v\leq\gamma}\binom{\gamma}{v} \int_{\mathbb{R}^n} |x^{\beta-v}(D_x^{\gamma-v}\phi(x))|dx,$$

where $\gamma = (\gamma_1, \gamma_2 \ldots, \gamma_n)$ *and* $v = (v_1, v_2, \ldots, v_n)$ are multi-indices. Then

$$\left| \left(\prod_{j=1}^{n} w_j^{\frac{\gamma_j - \beta_j}{\alpha_j} + \beta_j} \right) D_w^{\beta} (\mathcal{F}_{\alpha}\phi)(w) \right| \leq \left(\prod_{j=1}^{n} \frac{\beta_j}{(\alpha_j)^{\beta_j}} \right) \sum_{v \leq \gamma} \binom{\gamma}{v} \int_{\mathbb{R}^n} \prod_{j=1}^{n} (1 + |x_j|^2)^{\frac{\beta_j - \gamma_j}{2} + 1} \times |(D_x^{\gamma - v} \phi(x))| \left(\prod_{j=1}^{n} (1 + |x_j|^2)^{-1} \right) dx.$$

Let m be an integer such that $\max\limits_{1 \leq j \leq n} \left| \frac{\beta_j - \gamma_j}{2} + 1 \right| \leq m$, $N = nm$ and notice that $|x_j|^2 \leq |x|^2 = \sum_{j=1}^{n} |x_j|^2$, we have

$$\left| \left(\prod_{j=1}^{n} w_j^{\frac{\gamma_j - \beta_j}{\alpha_j} + \beta_j} \right) D_w^{\beta} (\mathcal{F}_{\alpha}\phi)(w) \right| \leq \left(\prod_{j=1}^{n} \frac{\beta_j}{(\alpha_j)^{\beta_j}} \right) \sum_{v \leq \gamma} \binom{\gamma}{v} \int_{\mathbb{R}^n} (1 + |x|^2)^{N} \times |(D_x^{\gamma - v} \phi(x))| \left(\prod_{j=1}^{n} (1 + |x_j|^2)^{-1} \right) dx.$$

Since $\phi \in S(\mathbb{R}^n)$, then by (1.1.8), we obtain

$$\left| \left(\prod_{j=1}^{n} w_j^{\frac{\gamma_j - \beta_j}{\alpha_j} + \beta_j} \right) D_w^{\beta} (\mathcal{F}_{\alpha}\phi)(w) \right| \leq \left(\prod_{j=1}^{n} \frac{\beta_j}{(\alpha_j)^{\beta_j}} \right) \sum_{v \leq \gamma} B(v, \gamma, \beta) \|\phi\|_N. \tag{2.2.3}$$

The inequality (2.2.3) shows that $\mathcal{F}_{\alpha}\phi \in S(\mathbb{R}^n)$. Also the linearity of $\mathcal{F}_{\alpha}$ is easy to verify.

For continuity, let (ϕ_k) be a sequence of functions in $S(\mathbb{R}^n)$ such that $\phi_k \to 0$ in $S(\mathbb{R}^n)$ as $k \to \infty$, then we need to show that $\mathcal{F}_{\alpha}(\phi_k) \to 0$ in $S(\mathbb{R}^n)$ as $k \to \infty$. Now by using (2.2.3), we have

$$\left| \left(\prod_{j=1}^{n} w_j^{\frac{\gamma_j - \beta_j}{\alpha_j} + \beta_j} \right) D_w^{\beta} (\mathcal{F}_{\alpha}\phi_k)(w) \right| \leq \left(\prod_{j=1}^{n} \frac{\beta_j}{(\alpha_j)^{\beta_j}} \right) \sum_{v \leq \gamma} B(v, \gamma, \beta) \|\phi_k\|_N. \tag{2.2.4}$$

The right-hand side of the inequality (2.2.4) tends to 0 as $k \to \infty$, which proves that $\mathcal{F}_{\alpha}(\phi_k) \to 0$ as $k \to \infty$.

For bijection, let $\phi \in S(\mathbb{R}^n)$, then from (2.1.1) and (2.1.2), we have

$$\mathcal{F}_\alpha^{-1}(\mathcal{F}_\alpha\phi)(x) = \frac{1}{(2\pi)^n[\alpha]} \int_{\mathbb{R}^n} e^{i\sum_{j=1}^n (\operatorname{sign} w_j)|w_j|^{\frac{1}{\alpha_j}} x_j} \left(\prod_{j=1}^n |w_j|^{\frac{1}{\alpha}-1}\right) \times \left(\int_{\mathbb{R}^n} e^{-i\sum_{j=1}^n (\operatorname{sign} w_j)|w_j|^{\frac{1}{\alpha_j}} t_j} \phi(t)dt\right) dw, \quad \forall\, x \in \mathbb{R}^n.$$

Choosing $y_j = (\operatorname{sign} w_j)|w_j|^{\frac{1}{\alpha_j}},\ j = 0, 1, \ldots, n$ and $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, $t = (t_1, t_2, \ldots, t_n) \in \mathbb{R}^n$, then we have

$$\mathcal{F}_\alpha^{-1}(\mathcal{F}_\alpha\phi)(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix.y} \left(\int_{\mathbb{R}^n} e^{-it.y}\phi(t)dt\right) dy. \tag{2.2.5}$$

In view of ([8], p. 22), we get $\mathcal{F}_\alpha^{-1}(\mathcal{F}_\alpha\phi)(x) = \phi(x)$ for all $x \in \mathbb{R}^n$.

Similarly, we can show that $\mathcal{F}_\alpha(\mathcal{F}_\alpha^{-1}\phi)(x) = \phi(x),$ for all $x \in \mathbb{R}^n$. This proves that $\mathcal{F}_\alpha$ is a bijection from $S(\mathbb{R}^n)$ onto $S(\mathbb{R}^n)$. □

Proposition 2.2.5. *The fractional Fourier transform $\mathcal{F}_\alpha$ of order $\alpha \in (0, 1]^n$ is a bijective continuous linear mapping from $S'(\mathbb{R}^n)$ onto $S'(\mathbb{R}^n)$.*

Proof. The linearity of $\mathcal{F}_\alpha$ is easy to verify. Let T be a tempered distribution in $S'(\mathbb{R}^n)$, then firstly we prove that $\mathcal{F}_\alpha(T)$ is also a tempered distribution in $S'(\mathbb{R}^n)$.

Now, let (ϕ_k) be a sequence of functions in $S(\mathbb{R}^n)$ such that $\phi_k \to 0\ \ in\ \ S(\mathbb{R}^n)\ \ as\ \ k \to \infty$, then by Definition 1.1.4 and Proposition 2.2.4, we get

$$\langle \mathcal{F}_\alpha T, \phi_k\rangle = \langle T, \mathcal{F}_\alpha\phi_k\rangle \to 0, \quad \text{as} \quad k \to \infty.$$

Therefore, $\mathcal{F}_\alpha(T)$ is a tempered distribution.

Next, we have to prove that $\mathcal{F}_\alpha$ is continuous mapping. For this let (T_k) be a sequence of tempered distributions such that $T_k \to T$ in $S'(\mathbb{R}^n)$ as $k \to \infty$, then

$$\langle T_k, \phi\rangle \to \langle T, \phi\rangle, \quad \text{for all} \quad \phi \in S(\mathbb{R}^n).$$

Now for $\phi \in S(\mathbb{R}^n)$, we have

$$\langle \mathcal{F}_\alpha T_k, \phi\rangle = \langle T_k, \mathcal{F}_\alpha(\phi)\rangle \to \langle T, \mathcal{F}_\alpha(\phi)\rangle = \langle \mathcal{F}_\alpha T, \phi\rangle,$$

this implies that $\mathcal{F}_\alpha$ is continuous. Also

$$\langle (\mathcal{F}_\alpha^{-1}\mathcal{F}_\alpha)(T), \phi\rangle = \langle \mathcal{F}_\alpha T, \mathcal{F}_\alpha^{-1}\phi\rangle = \langle T, \mathcal{F}_\alpha\mathcal{F}_\alpha^{-1}\phi\rangle = \langle T, \phi\rangle, \text{ for all } \phi \in S(\mathbb{R}^n). \tag{2.2.6}$$

Similarly, we can show that

$$\langle (\mathcal{F}_\alpha\mathcal{F}_\alpha^{-1})(T), \phi\rangle = \langle T, \phi\rangle. \tag{2.2.7}$$

Thus from (2.2.6) and (2.2.7), we can say that $\mathcal{F}_\alpha^{-1}\mathcal{F}_\alpha = I = \mathcal{F}_\alpha\mathcal{F}_\alpha^{-1}$. Hence $\mathcal{F}_\alpha$ is a bijection. ☐

Proposition 2.2.6. *Let $\phi, \psi \in S(\mathbb{R}^n)$. Then the Parseval formula for the fractional Fourier transform is given by*

$$\int_{\mathbb{R}^n} \phi(x)\overline{\psi(x)}dx = \frac{1}{[\alpha]}\int_{\mathbb{R}^n} (\mathcal{F}_\alpha\phi)(w)\overline{(\mathcal{F}_\alpha\psi)(w)}\left(\prod_{j=1}^n |w_j|^{\frac{1}{\alpha_j}-1}\right) dw. \tag{2.2.8}$$

Proof. Let $\phi,\ \psi \in S(\mathbb{R}^n)$, then by using (2.1.2), Proposition 2.2.4, and the Fubini's theorem, we have

$$\int_{\mathbb{R}^n} \phi(x)\overline{\psi(x)}dx = \frac{1}{(2\pi)^{\frac{n}{2}}[\alpha]}\int_{\mathbb{R}^n} \overline{\psi(x)}\bigg(\int_{\mathbb{R}^n} e^{i\sum_{j=1}^n (\text{sign}\, w_j)|w_j|^{\frac{1}{\alpha_j}}x_j}\left(\prod_{j=1}^n |w_j|^{\frac{1}{\alpha_j}-1}\right)$$

$$\times (\mathcal{F}_\alpha \phi)(w)dw \Bigg) dx$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}[\alpha]} \int_{\mathbb{R}^n} (\mathcal{F}_\alpha \phi)(w) \left(\prod_{j=1}^{n} |w_j|^{\frac{1}{\alpha_j}-1} \right) \Bigg(\int_{\mathbb{R}^n} e^{i \sum_{j=1}^{n} (\text{sign}\, w_j)|w_j|^{\frac{1}{\alpha_j}} x_j}$$

$$\times \overline{\psi(x)} dx \Bigg) dw$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}[\alpha]} \int_{\mathbb{R}^n} (\mathcal{F}_\alpha \phi)(w) \left(\prod_{j=1}^{n} |w_j|^{\frac{1}{\alpha_j}-1} \right)$$

$$\times \left(\overline{\int_{\mathbb{R}^n} e^{-i \sum_{j=1}^{n} (\text{sign}\, w_j)|w_j|^{\frac{1}{\alpha_j}} x_j} \psi(x) dx} \right) dw.$$

Using (2.1.1), the above expression yields

$$\int_{\mathbb{R}^n} \phi(x)\overline{\psi(x)}dx = \frac{1}{[\alpha]} \int_{\mathbb{R}^n} (\mathcal{F}_\alpha \phi)(w)\overline{(\mathcal{F}_\alpha \psi)(w)} \left(\prod_{j=1}^{n} |w_j|^{\frac{1}{\alpha_j}-1} \right) dw.$$

□

For $\phi = \psi$, in (2.2.8), we get

$$\int_{\mathbb{R}^n} |\phi(x)|^2 dx = \frac{1}{[\alpha]} \int_{\mathbb{R}^n} |(\mathcal{F}_\alpha \phi)(w)|^2 \left(\prod_{j=1}^{n} |w_j|^{\frac{1}{\alpha_j}-1} \right) dw. \tag{2.2.9}$$

The above identity (2.2.9) known as the Plancherel identity.

2.3 Pseudo-differential operators

In this section, definition and properties of pseudo-differential operators are given by involving the n-dimensional fractional Fourier transform of order $\alpha \in (0, 1]^n$.

The linear partial differential operator $P_\alpha(x, D)$ of order m on $\mathbb{R}^n$ is given by

$$P_\alpha(x, D) = \sum_{|\beta| \leq m} a_\beta(x) D^\beta, \tag{2.3.1}$$

where $D^\beta = (-i)^{|\beta|} D_{x_1}^{\beta_1} D_{x_2}^{\beta_2} \dots D_{x_n}^{\beta_n}$ and coefficient $a_\beta(x)$ are functions defined on $\mathbb{R}^n$. If we replace D^β in (2.3.1) by monomials

$$\prod_{j=1}^{n} \Big((\text{sign } w_j) |w_j|^{\frac{1}{\alpha_j}} \Big)^{\beta_j} = ((\text{sign } w_1)|w_1|^{\frac{1}{\alpha_1}})^{\beta_1} \times ((\text{sign } w_2)|w_2|^{\frac{1}{\alpha_2}})^{\beta_2} \dots ((\text{sign } w_n)|w_n|^{\frac{1}{\alpha_n}})^{\beta_n} \tag{2.3.2}$$

in $\mathbb{R}^n$, then

$$P_\alpha(x, w) = \sum_{|\beta| \leq m} a_\beta(x) \Big(\prod_{j=1}^{n} ((\text{sign } w_j)|w_j|^{\frac{1}{\alpha_j}})^{\beta_j} \Big) \tag{2.3.3}$$

is called the symbol of the operator $P_\alpha(x, D)$.

Proposition 2.3.1. *If $\phi \in S(\mathbb{R}^n)$, then*

$$\mathcal{F}_\alpha(D^\beta \phi(x))(w) = \Big(\prod_{j=1}^{n} ((\text{sign } w_j)|w_j|^{\frac{1}{\alpha_j}})^{\beta_j} \Big) (\mathcal{F}_\alpha \phi)(w), \quad w \in \mathbb{R}^n. \tag{2.3.4}$$

Proof. From (2.1.1), we have

$$\begin{aligned} \mathcal{F}_\alpha(D^\beta \phi(x))(w) &= \int_{\mathbb{R}^n} e^{-i \sum_{j=1}^{n} (\text{sign } w_j)|w_j|^{\frac{1}{\alpha_j}} x_j} (D^\beta \phi(x)) dx \\ &= (-i)^{|\beta|} \int_{\mathbb{R}^n} e^{-i \sum_{j=1}^{n} (\text{sign } w_j)|w_j|^{\frac{1}{\alpha_j}} x_j} \big(D_{x_1}^{\beta_1} D_{x_2}^{\beta_2} \dots D_{x_n}^{\beta_n} \phi(x) \big) dx. \end{aligned} \tag{2.3.5}$$

Now using integration by parts in (2.3.5), we obtain

$$\mathcal{F}_\alpha(D^\beta \phi(x))(w) = (-i)^{|\beta|} (i)^{|\beta|} \big((\text{sign } w_1)|w_1|^{\frac{1}{\alpha_1}} \big)^{\beta_1} \big((\text{sign } w_2)|w_2|^{\frac{1}{\alpha_2}} \big)^{\beta_2} \dots$$

$$\times\left((\operatorname{sign} w_n)|w_n|^{\frac{1}{\alpha_n}}\right)^{\beta_n}\left(\int_{\mathbb{R}^n} e^{-i\sum_{j=1}^{n}(\operatorname{sign} w_j)|w_j|^{\frac{1}{\alpha_j}}x_j}\phi(x)dx\right)$$
$$=\left(\prod_{j=1}^{n}\left((\operatorname{sign} w_j)|w_j|^{\frac{1}{\alpha_j}}\right)^{\beta_j}\right)(\mathcal{F}_\alpha\phi)(w).$$

□

Theorem 2.3.2. *Integral representation of the partial differential operator $P_\alpha(x,D)$, $\alpha\in(0,1]^n$ is given by*

$$P_\alpha(x,D)\phi(x)=\frac{1}{(2\pi)^{\frac{n}{2}}[\alpha]}\int_{\mathbb{R}^n} e^{i\sum_{j=1}^{n}(\operatorname{sign} w_j)|w_j|^{\frac{1}{\alpha_j}}x_j}P_\alpha(x,w)\left(\prod_{j=1}^{n}|w_j|^{\frac{1}{\alpha_j}-1}\right)$$
$$\times(\mathcal{F}_\alpha\phi)(w)dw,\ \text{for all } \phi\in S(\mathbb{R}^n). \tag{2.3.6}$$

Proof. Let $\phi\in S(\mathbb{R}^n)$,

$$P_\alpha(x,D)\phi(x)=\sum_{|\beta|\le m}a_\beta(x)D^\beta\phi(x).$$

Then by Proposition 2.2.4, we get

$$P_\alpha(x,D)\phi(x)=\sum_{|\beta|\le m}a_\beta(x)\mathcal{F}_\alpha^{-1}(\mathcal{F}_\alpha D^\beta\phi)(x).$$

Now, using (2.3.4), we have

$$P_\alpha(x,D)\phi(x)=\sum_{|\beta|\le m}a_\beta(x)\mathcal{F}_\alpha^{-1}\left[\prod_{j=1}^{n}\left((\operatorname{sign} w_j)|w_j|^{\frac{1}{\alpha_j}}\right)^{\beta_j}(\mathcal{F}_\alpha\phi)\right](x)$$
$$=\sum_{|\beta|\le m}a_\beta(x)\frac{1}{(2\pi)^{\frac{n}{2}}[\alpha]}\int_{\mathbb{R}^n} e^{i\sum_{j=1}^{n}(\operatorname{sign} w_j)|w_j|^{\frac{1}{\alpha_j}}x_j}$$
$$\times\left(\prod_{j=1}^{n}|w_j|^{\frac{1}{\alpha_j}-1}\right)\left[\prod_{j=1}^{n}\left((\operatorname{sign} w_j)|w_j|^{\frac{1}{\alpha_j}}\right)^{\beta_j}\right](\mathcal{F}_\alpha\phi)(w)dw$$
$$=\frac{1}{(2\pi)^{\frac{n}{2}}[\alpha]}\int_{\mathbb{R}^n} e^{i\sum_{j=1}^{n}(\operatorname{sign} w_j)|w_j|^{\frac{1}{\alpha_j}}x_j}\left(\prod_{j=1}^{n}|w_j|^{\frac{1}{\alpha_j}-1}\right)$$

$$\times \Big[\sum_{|\beta|\le m} a_\beta(x) \prod_{j=1}^{n} \Big((\operatorname{sign} w_j)|w_j|^{\frac{1}{\alpha_j}} \Big)^{\beta_j} \Big] (\mathcal{F}_\alpha \phi)(w) dw$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}[\alpha]} \int_{\mathbb{R}^n} e^{i\sum_{j=1}^{n}(\operatorname{sign} w_j)|w_j|^{\frac{1}{\alpha_j}} x_j} P_\alpha(x,w) \Big(\prod_{j=1}^{n} |w_j|^{\frac{1}{\alpha_j}-1} \Big)$$

$$\times (\mathcal{F}_\alpha \phi)(w) dw.$$

□

Now, we define general symbol class and corresponding partial differential operator.

Definition 2.3.3. Let $m \in \mathbb{R}$ and $\alpha \in (0,1]^n$. We define $S^{\frac{m}{[\alpha]}}$ is the set of all functions $\sigma_\alpha(x,w) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that for all $\beta, \gamma \in \mathbb{N}_0^n$, there exists a constant $A_{\beta,\gamma} > 0$ depending on β and γ only, such that

$$|D_x^\beta D_w^\gamma \sigma_\alpha(x,w)| \le A_{\beta,\gamma}(1+|w|)^{\frac{m}{[\alpha]}-|\gamma|}. \tag{2.3.7}$$

We call any function $\sigma_\alpha \in \cup_m S^{\frac{m}{[\alpha]}}$ a symbol.

Definition 2.3.4. Let $\sigma_\alpha(x,w) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ be a symbol. Then the pseudo-differential operator $T_{\sigma,\alpha}$ associated to $\sigma_\alpha(x,w)$ is defined by

$$(T_{\sigma,\alpha}\phi)(x) = \frac{1}{(2\pi)^{\frac{n}{2}}[\alpha]} \int_{\mathbb{R}^n} e^{i\sum_{j=1}^{n}(\operatorname{sign} w_j)|w_j|^{\frac{1}{\alpha_j}} x_j} \Big(\prod_{j=1}^{n} |w_j|^{\frac{1}{\alpha_j}-1} \Big)$$

$$\times \sigma_\alpha(x,w)(\mathcal{F}_\alpha\phi)(w) dw, \tag{2.3.8}$$

for all $\phi \in S(\mathbb{R}^n)$.

Theorem 2.3.5. *Let $\sigma_\alpha(x,w) \in S^{\frac{m}{[\alpha]}}$, then the pseudo-differential operator $T_{\sigma,\alpha}$ is a continuous linear mapping from $S(\mathbb{R}^n)$ into $S(\mathbb{R}^n)$.*

Proof. We take $\phi \in S(\mathbb{R}^n)$, then for any $\beta,\ \gamma,\ \delta \in \mathbb{N}_0^n$ and using (2.3.8), we have

$$\begin{aligned}
(D^\gamma T_{\sigma,\alpha}\phi)(x) &= \frac{1}{(2\pi)^{\frac{n}{2}}[\alpha]} \int_{\mathbb{R}^n} D^\gamma \Big[e^{i\sum_{j=1}^n (\operatorname{sign} w_j)|w_j|^{\frac{1}{\alpha_j}} x_j} \sigma_\alpha(x,w) \Big] \Big(\prod_{j=1}^n |w_j|^{\frac{1}{\alpha}_j - 1} \Big) \\
&\quad \times (\mathcal{F}_\alpha \phi)(w) dw \\
&= \frac{(-i)^{|\gamma|}}{(2\pi)^{\frac{n}{2}}[\alpha]} \int_{\mathbb{R}^n} \sum_{\delta \le \gamma} \binom{\gamma}{\delta} \Big[D_x^{\gamma-\delta} \sigma_\alpha(x,w) \Big] \Big[D_x^\delta e^{i\sum_{j=1}^n (\operatorname{sign} w_j)|w_j|^{\frac{1}{\alpha_j}} x_j} \Big] \\
&\quad \times \Big(\prod_{j=1}^n |w_j|^{\frac{1}{\alpha}_j - 1} \Big) (\mathcal{F}_\alpha \phi)(w) dw \\
&= \frac{(-i)^{|\gamma|}}{(2\pi)^{\frac{n}{2}}[\alpha]} \int_{\mathbb{R}^n} \sum_{\delta \le \gamma} \binom{\gamma}{\delta} \big[D_x^{\gamma-\delta} \sigma_\alpha(x,w) \big] \Big[\prod_{j=1}^n \Big(i(\operatorname{sign} w_j)|w_j|^{\frac{1}{\alpha_j}} \Big)^{\delta_j} \Big] \\
&\quad \times e^{i\sum_{j=1}^n (\operatorname{sign} w_j)|w_j|^{\frac{1}{\alpha_j}} x_j} \Big(\prod_{j=1}^n |w_j|^{\frac{1}{\alpha}_j - 1} \Big) (\mathcal{F}_\alpha \phi)(w) dw. \qquad (2.3.9)
\end{aligned}$$

For $y = (y_1, y_2, \dots y_n) \in \mathbb{R}^n$, and $y_j = (\operatorname{sign} w_j)|w_j|^{\frac{1}{\alpha_j}}$, then for each j, $w_j = (\operatorname{sign} y_j)|y_j|^{\alpha_j}$, $\frac{1}{[\alpha]} \prod_{j=1}^n |w_j|^{\frac{1}{\alpha}_j - 1} dw = dy$ and

$$y_\alpha = \big((\operatorname{sign} y_1)|y_1|^{\alpha_1}, (\operatorname{sign} y_2)|y_2|^{\alpha_2}, \dots, (\operatorname{sign} y_n)|y_n|^{\alpha_n} \big).$$

Then (2.3.9), becomes

$$\begin{aligned}
x^\beta (D^\gamma T_{\sigma,\alpha}\phi)(x) &= \frac{(-i)^{|\gamma|}(i)^{|\delta|}}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \sum_{\delta \le \gamma} \binom{\gamma}{\delta} (D_y^\beta e^{ix.y}) [D_x^{\gamma-\delta} \sigma_\alpha(x, y_\alpha)] \\
&\quad \times (y^\delta \mathcal{F}_\alpha \phi(y_\alpha)) dy. \qquad (2.3.10)
\end{aligned}$$

Using integration by parts and Leibnitz's formula, (2.3.10) becomes

$$\begin{aligned}
x^\beta (D^\gamma T_{\sigma,\alpha}\phi)(x) &= \frac{(-1)^{|\gamma|+|\beta|}(i)^{|\gamma|+|\delta|}}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \sum_{\delta \le \gamma} \binom{\gamma}{\delta} e^{ix.y} D_y^\beta \Big(\big(D_x^{\gamma-\delta} \sigma_\alpha(x, y_\alpha) \big) \\
&\quad \times \big(y^\delta \mathcal{F}_\alpha \phi(y_\alpha) \big) \Big) dy
\end{aligned}$$

$$= \frac{(-1)^{|\gamma|+|\beta|}(i)^{|\gamma|+|\delta|}}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \sum_{\delta\leq\gamma} \sum_{\eta\leq\beta} \binom{\gamma}{\delta} \binom{\beta}{\eta} e^{ix.y} [D_y^{\beta-\eta} D_x^{\gamma-\delta}$$

$$\times \sigma_\alpha(x, y_\alpha)] \left[D_y^\eta \left(y^\delta \mathcal{F}_\alpha \phi(y_\alpha)\right)\right] dy. \tag{2.3.11}$$

Now using the fact that $\sigma_\alpha \in S^{\frac{m}{[\alpha]}}$ is a symbol, we can find a constant $A_\beta, \gamma, \delta, \eta > 0$ such that (2.3.11) takes the form

$$\sup_{x\in\mathbb{R}^n} |x^\beta (D^\gamma T_{\sigma,\alpha}\phi)(x)| \leq \frac{1}{(2\pi)^{\frac{n}{2}}} \sum_{\delta\leq\gamma} \sum_{\eta\leq\beta} \binom{\gamma}{\delta} \binom{\beta}{\eta} A_{\beta,\gamma,\delta,\eta} \int_{\mathbb{R}^n} (1 + |y_\alpha|)^{\frac{m}{[\alpha]}-|\beta|+|\eta|}$$

$$\times \left|D_y^\eta (y^\delta \mathcal{F}_\alpha \phi(y_\alpha))\right| dy. \tag{2.3.12}$$

Since the integral in (2.3.11) is absolutely integrable for $\phi \in S(\mathbb{R}^n)$, then we find that $\sup_{x\in\mathbb{R}^n} |x^\beta (D^\gamma T_{\sigma,\alpha}\phi)(x)| < \infty$, which proves that $T_{\sigma,\alpha}\phi \in S(\mathbb{R}^n)$. The linearity of $T_{\sigma,\alpha}$ is easy to verify. For continuity of $T_{\sigma,\alpha}$, let $\phi_k \to 0$ in $S(\mathbb{R}^n)$ as $k \to \infty$, then by using the relation (2.3.12), we have

$$\sup_{x\in\mathbb{R}^n} |x^\beta (D^\gamma T_{\sigma,\alpha}\phi_k)(x)| \leq \frac{1}{(2\pi)^{\frac{n}{2}}} \sum_{\delta\leq\gamma} \sum_{\eta\leq\beta} \binom{\gamma}{\delta} \binom{\beta}{\eta} A_{\beta,\gamma,\delta,\eta} \int_{\mathbb{R}^n} (1 + |y_\alpha|)^{\frac{m}{[\alpha]}-|\beta|+|\eta|}$$

$$\times \left|D_y^\eta (y^\delta \mathcal{F}_\alpha \phi_k(y_\alpha))\right| dy. \tag{2.3.13}$$

By using Proposition 2.2.4, we can say that
$(1 + |y_\alpha|^2)^{\frac{1}{2}\left(\frac{m}{[\alpha]}-|\beta|-|\eta|\right)} \left|D_y^\eta (y^\delta \mathcal{F}_\alpha \phi_k(y_\alpha))\right| \to 0$ in $S(\mathbb{R}^n)$ as $k \to \infty$, which shows that the right side of (2.3.13) tends to 0 as $k \to \infty$.

This proves that $T_{\sigma,\alpha}\phi_k \to 0$ in $S(\mathbb{R}^n)$ as $k \to \infty$. Thus $T_{\sigma,\alpha}$ is continuous. □

Definition 2.3.6. Let $T_{\sigma,\alpha}$ be a pseudo differential operator on $S(\mathbb{R}^n)$ and $u \in S'(\mathbb{R}^n)$, then $T_{\sigma,\alpha}$ is defined on $S'(\mathbb{R}^n)$ by

$$(T_{\sigma,\alpha}u)(\phi) = u(\overline{T^*_{\sigma,\alpha}\overline{\phi}}), \qquad \phi \in S(\mathbb{R}^n), \tag{2.3.14}$$

where $T^*_{\sigma,\alpha}$ is the formal adjoint of $T_{\sigma,\alpha}$, which is given by

$$(T_{\sigma,\alpha}\phi, \psi) = (\phi, T^*_{\sigma,\alpha}\psi), \quad \phi, \psi \in S(\mathbb{R}^n). \tag{2.3.15}$$

Theorem 2.3.7. *$T_{\sigma,\alpha} : S'(\mathbb{R}^n) \to S'(\mathbb{R}^n)$ be a continuous linear mapping.*

Proof. The linearity of $T_{\sigma,\alpha}$ is obvious. Let $\phi_k \to 0$ in $S(\mathbb{R}^n)$ as $k \to \infty$, then by Theorem 2.3.5, $T^*_{\sigma,\alpha}\overline{\phi}_k \to 0$ in $S(\mathbb{R}^n)$, as $k \to \infty$. Now for $u \in S'(\mathbb{R}^n)$ and using (2.3.14), we have $(T_{\sigma,\alpha}u)(\phi_k) \to 0$ in $S(\mathbb{R}^n)$ as $k \to \infty$. This gives that $T_{\sigma,\alpha}u \in S'(\mathbb{R}^n)$. For continuity, consider $\phi \in S(\mathbb{R}^n)$ and let $u_k \to 0$ in $S'(\mathbb{R}^n)$ as $k \to \infty$, therefore from (2.3.14), we have $(T_{\sigma,\alpha}(u_k))(\phi) = u_k(\overline{T^*_{\sigma,\alpha}\overline{\phi}}) \to 0$ in $S'(\mathbb{R}^n)$ as $k \to \infty$. This yields that $T_{\sigma,\alpha}u_k \to 0$ in $S'(\mathbb{R}^n)$ as $k \to \infty$. Hence $T_{\sigma,\alpha}$ is a continuous mapping. □

2.4 Integral representation and boundedness of the pseudo-differential operator $T_{\sigma,\alpha}$

In this section, the integral representation, boundedness of pseudo-differential operator $T_{\sigma,\alpha}$ and its various properties are discussed by exploiting the theory of the fractional Fourier transform.

Theorem 2.4.1. *Let $\sigma_\alpha(x, w) \in S^{\frac{m}{[\alpha]}}$, then the pseudo-differential operator $T_{\sigma,\alpha}$ can be represented by*

$$(T_{\sigma,\alpha}\phi)(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \phi(y)(\mathcal{F}^{-1}_\alpha \sigma_\alpha)(x, x - y)dy, \tag{2.4.1}$$

where

$$(\mathcal{F}_\alpha^{-1}\sigma_\alpha)(x,x-y) = \frac{1}{(2\pi)^{\frac{n}{2}}[\alpha]}\int_{\mathbb{R}^n} e^{i\sum_{j=1}^n(\operatorname{sign} w_j)|w_j|^{\frac{1}{\alpha_j}}(x_j-y_j)}\Big(\prod_{j=1}^n |w_j|^{\frac{1}{\alpha_j}-1}\Big) \times \sigma_\alpha(x,w)dw. \tag{2.4.2}$$

Moreover, if $\phi \in L^1(\mathbb{R}^n)$ *then for*

$$n + \sum_{j=1}^n \frac{1}{\alpha_j} + \frac{m}{[\alpha]} < 0,$$

we have

$$\|T_{\sigma,\alpha}\phi\|_\infty \leq \mathcal{C}''\|\phi\|_1, \tag{2.4.3}$$

for some positive constant $\mathcal{C}''$.

Proof. Using (2.3.8) and the Fubini's theorem, we have

$$\begin{aligned}(T_{\sigma,\alpha}\phi)(x) &= \frac{1}{(2\pi)^{\frac{n}{2}}[\alpha]}\int_{\mathbb{R}^n} e^{i\sum_{j=1}^n(\operatorname{sign} w_j)|w_j|^{\frac{1}{\alpha_j}}x_j}\Big(\prod_{j=1}^n |w_j|^{\frac{1}{\alpha_j}-1}\Big)\sigma_\alpha(x,w) \\ &\quad\times (\mathcal{F}_\alpha\phi)(w)dw \\ &= \frac{1}{(2\pi)^n[\alpha]}\int_{\mathbb{R}^n} e^{i\sum_{j=1}^n(\operatorname{sign} w_j)|w_j|^{\frac{1}{\alpha_j}}x_j}\Big(\prod_{j=1}^n |w_j|^{\frac{1}{\alpha_j}-1}\Big)\sigma_\alpha(x,w) \\ &\quad\times \Big(\int_{\mathbb{R}^n} e^{-i\sum_{j=1}^n(\operatorname{sign} w_j)|w_j|^{\frac{1}{\alpha_j}}y_j}\phi(y)dy\Big)dw \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\phi(y)\Big(\frac{1}{(2\pi)^{\frac{n}{2}}[\alpha]}\int_{\mathbb{R}^n} e^{i\sum_{j=1}^n(\operatorname{sign} w_j)|w_j|^{\frac{1}{\alpha_j}}(x_j-y_j)} \\ &\quad\times \Big(\prod_{j=1}^n |w_j|^{\frac{1}{\alpha_j}-1}\Big)\sigma_\alpha(x,w)dw\Big)dy \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\phi(y)(\mathcal{F}_\alpha^{-1}\sigma_\alpha)(x,x-y)dy.\end{aligned}$$

Now, in view of (2.3.7) and the following inequalities,

$$\prod_{j=1}^{n} |w_j|^{\frac{1}{\alpha_j}-1} \leq (1+|w|)^{\sum_{j=1}^{n}\frac{1}{\alpha_j}-n} \text{ and } (1+|w|^2)^n \leq (1+|w|)^{2n},$$

there exist $C>0$ such that (2.4.2) becomes

$$\begin{aligned}|(\mathcal{F}_\alpha^{-1}\sigma_\alpha)(x,x-y)| &\leq \frac{1}{(2\pi)^{\frac{n}{2}}[\alpha]}C\int_{\mathbb{R}^n}\left(\prod_{j=1}^{n}|w_j|^{\frac{1}{\alpha_j}-1}\right)(1+|w|)^{\frac{m}{[\alpha]}}\,dw \\ &\leq \frac{1}{(2\pi)^{\frac{n}{2}}[\alpha]}C\int_{\mathbb{R}^n}(1+|w|)^{\sum_{j=1}^{n}\frac{1}{\alpha_j}-n}(1+|w|)^{\frac{m}{[\alpha]}}\,dw \\ &\leq \frac{1}{(2\pi)^{\frac{n}{2}}[\alpha]}C\int_{\mathbb{R}^n}(1+|w|)^{\sum_{j=1}^{n}\frac{1}{\alpha_j}+n+\frac{m}{[\alpha]}}(1+|w|^2)^{-n}dw.\end{aligned}$$

Assuming

$$n+\sum_{j=1}^{n}\frac{1}{\alpha_j}+\frac{m}{[\alpha]}<0, \text{ we have}$$

$$|(\mathcal{F}_\alpha^{-1}\sigma_\alpha)(x,x-y)| \leq \frac{1}{(2\pi)^{\frac{n}{2}}[\alpha]}C\int_{\mathbb{R}^n}(1+|w|^2)^{-n}dw. \tag{2.4.4}$$

Since the integral on the right hand-side of (2.4.4) is integrable therefore there exists a positive constant $\mathcal{C}'$ such that,

$$|(\mathcal{F}_\alpha^{-1}\sigma_\alpha)(x,x-y)| \leq \mathcal{C}'. \tag{2.4.5}$$

In view of (2.4.1) and (2.4.5), we obtain

$$|(T_{\sigma,\alpha}\phi)(x)| \leq \mathcal{C}'\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}|\phi(y)|dy, \quad x\in\mathbb{R}^n.$$

This implies that there exist a positive constant $\mathcal{C}''>0$, such that

$$\|(T_{\sigma,\alpha}\phi)\|_\infty \leq \mathcal{C}''\|\phi\|_1.$$

This proves (2.4.3) and hence the above proof is completed. □

2.5 The Sobolev space

In this section, the properties of Sobolev space associated with the pseudo-differential operator $J^s_{\sigma,\alpha}$ with a symbol $\sigma_{s,\alpha}$ are discussed by exploiting the theory of fractional Fourier transform of order $\alpha \in (0,1]^n$.

For $s \in \mathbb{R}$, let $\sigma_{s,\alpha}$ be a symbol, which is given by

$$\sigma_{s,\alpha}(w) = \Big(1 + \sum_{k=1}^{n} |w_k|^{\frac{2}{\alpha_k}}\Big)^{\frac{-s}{[\alpha]}}, \quad w \in \mathbb{R}^n. \tag{2.5.1}$$

Then the pseudo-differential operator $J^s_{\sigma,\alpha}$ associated to the given symbol (2.5.1) is defined by

$$\begin{aligned}(J^s_{\sigma,\alpha}\phi)(x) &= \frac{1}{(2\pi)^{\frac{n}{2}}[\alpha]} \int_{\mathbb{R}^n} e^{i\sum_{j=1}^{n}(\operatorname{sign} w_j)|w_j|^{\frac{1}{\alpha_j}} x_j} \Big(\prod_{j=1}^{n} |w_j|^{\frac{1}{\alpha_j}-1}\Big) \\ &\quad \times \Big(1 + \sum_{k=1}^{n} |w_k|^{\frac{2}{\alpha_k}}\Big)^{\frac{-s}{[\alpha]}} (\mathcal{F}_\alpha \phi)(w) dw \\ &= \mathcal{F}_\alpha^{-1}\Big[\Big(1 + \sum_{k=1}^{n} |w_k|^{\frac{2}{\alpha_k}}\Big)^{\frac{-s}{[\alpha]}} (\mathcal{F}_\alpha\phi)(w)\Big](x), \text{ for } \phi \in S(\mathbb{R}^n).\end{aligned}$$

It can be easily shown that for $u \in S'(\mathbb{R}^n)$, the product $\sigma_{s,\alpha}u$ of $\sigma_{s,\alpha}$ and u defined by

$$(\sigma_{s,\alpha}u)(\phi) = u(\sigma_{s,\alpha}\phi), \quad \phi \in S(\mathbb{R}^n),$$

is also in $S'(\mathbb{R}^n)$. Moreover, for $u \in S'(\mathbb{R}^n)$,

$$(J^s_{\sigma,\alpha}u) = \mathcal{F}_\alpha^{-1}\Big[\Big(1 + \sum_{k=1}^{n} |w_k|^{\frac{2}{\alpha_k}}\Big)^{\frac{-s}{[\alpha]}} (\mathcal{F}_\alpha u)\Big]. \tag{2.5.2}$$

From (2.5.2), it is clear that $J^s_{\sigma,\alpha}$ is an element of $S'(\mathbb{R}^n)$.

Theorem 2.5.1. *Let $u \in S'(\mathbb{R}^n)$, then*

1. $J^s_{\sigma,\alpha}(J^t_{\sigma,\alpha}u) = J^{s+t}_{\sigma,\alpha}u.$

2. $J^0_{\sigma,\alpha}u = u.$

Proof. Now, we find

$$\begin{aligned}
J^s_{\sigma,\alpha}(J^t_{\sigma,\alpha}u) &= J^s_{\sigma,\alpha}\Big[\mathcal{F}_\alpha^{-1}\Big[\Big(1+\sum_{k=1}^n |w_k|^{\frac{2}{\alpha_k}}\Big)^{\frac{-t}{[\alpha]}}(\mathcal{F}_\alpha u)\ big]\Big] \\
&= \mathcal{F}_\alpha^{-1}\Big[\Big(1+\sum_{k=1}^n |w_k|^{\frac{2}{\alpha_k}}\Big)^{\frac{-s}{[\alpha]}}\Big(1+\sum_{k=1}^n |w_k|^{\frac{2}{\alpha_k}}\Big)^{\frac{-t}{[\alpha]}}(\mathcal{F}_\alpha u)\Big] \\
&= \mathcal{F}_\alpha^{-1}\Big[\Big(1+\sum_{k=1}^n |w_k|^{\frac{2}{\alpha_k}}\Big)^{\frac{-s-t}{[\alpha]}}(\mathcal{F}_\alpha u)\Big] \\
&= J^{s+t}_{\sigma,\alpha}u.
\end{aligned}$$

Next, by using Proposition 2.2.5, we have

$$\begin{aligned}
J^0_{\sigma,\alpha}u &= \mathcal{F}_\alpha^{-1}\Big[\Big(1+\sum_{k=1}^n |w_k|^{\frac{2}{\alpha_k}}\Big)^{\frac{0}{[\alpha]}}(\mathcal{F}_\alpha u)\Big] \\
&= \mathcal{F}_\alpha^{-1}(\mathcal{F}_\alpha u) \\
&= u.
\end{aligned}$$

□

Definition 2.5.2. Let $s \in \mathbb{R}$ and $1 \le p < \infty$, then $H^{s,p}_\alpha$ is defined by

$$H^{s,p}_\alpha = \{u \in S'(\mathbb{R}^n) : J^{-s}_{\sigma,\alpha}u \in L^p(\mathbb{R}^n)\}. \tag{2.5.3}$$

It can easily show that $H_\alpha^{s,p}$ is a vector space.

Let $\|.\|_{s,p} : H_\alpha^{s,p} \to \mathbb{R}$, is defined by

$$\|u\|_{s,p} = \|J_{\sigma,\alpha}^{-s} u\|_p, \; u \in H_\alpha^{s,p}. \tag{2.5.4}$$

Then $H_\alpha^{s,p}$ is a normed linear space with respect to the norm $\|.\|_{s,p}$ given by (2.5.4). Moreover, from the concept of Wong ([8], p. 88) it is a Banach space. We call $H_\alpha^{s,p}$, the L^p- Sobolev space of order $\frac{s}{[\alpha]}$.

Theorem 2.5.3. *$J_{\sigma,\alpha}^{t} : H_\alpha^{s,p} \to H_\alpha^{s+t,p}$ is an onto isometry.*

Proof. Let $u \in H_\alpha^{s,p}$, then

$$\|J_{\sigma,\alpha}^{t} u\|_{s+t,p} = \|J_{\sigma,\alpha}^{-s-t} J_{\sigma,\alpha}^{t} u\|_p = \|J_{\sigma,\alpha}^{-s} u\|_p = \|u\|_{s,p}.$$

This proves that $J_{\sigma,\alpha}^{t}$ is an isometry.

For onto, let $x \in H_\alpha^{s+t,p}$, then $J_{\sigma,\alpha}^{-s-t} x \in L^p(\mathbb{R}^n)$. Now from Theorem 2.5.1

$$J_{\sigma,\alpha}^{-s-t} x = J_{\sigma,\alpha}^{-s}(J_{\sigma,\alpha}^{-t} x) \in L^p(\mathbb{R}^n),$$

which implies that $J_{\sigma,\alpha}^{-t} x \in H_\alpha^{s,p}$, so $J_{\sigma,\alpha}^{t}(J_{\sigma,\alpha}^{-t} x) = J_{\sigma,\alpha}^{0} x = x$. This proves that $J_{\sigma,\alpha}^{t}$ is an onto mapping. □

From Wong ([8], p. 88), we give the following example:

Example 1. Let $s > 0$. We define the function G_s on $\mathbb{R}^n$ by

$$G_s(x) = \frac{1}{2^{\frac{s}{2}}\Gamma(\frac{s}{2})} \int_0^\infty e^{-\frac{r}{2}} e^{-\frac{|x|^2}{2r}} r^{-\frac{(n-s)}{2}} \frac{dr}{r}, \quad x \in \mathbb{R}^n.$$

Then, we have

$G_s \in L^1(\mathbb{R}^n)$ with $\|G_s\|_1 = (2\pi)^{\frac{n}{2}}$ and $(\mathcal{F}G_s)(w) = (1+|w|^2)^{\frac{-s}{2}}$, $w \in \mathbb{R}^n$.

Now, using Proposition 2.2.1 and Example 1, the fractional Fourier transform of G_s is given by

$$\begin{aligned}(\mathcal{F}_\alpha G_s)(w) &= (\mathcal{F}G_s)\big((\operatorname{sign} w_1)|w_1|^{\frac{1}{\alpha_1}}, (\operatorname{sign} w_2)|w_2|^{\frac{1}{\alpha_2}}, \ldots, (\operatorname{sign} w_n)|w_n|^{\frac{1}{\alpha_n}}\big) \\ &= \left(1+\sum_{k=1}^{n}|w_k|^{\frac{2}{\alpha_k}}\right)^{\frac{-s}{2}}, \text{ for all } w=(w_1,w_2,\ldots,w_n)\in\mathbb{R}^n. \qquad (2.5.5)\end{aligned}$$

Theorem 2.5.4. *Let $s > 0$ and $1 \le p < \infty$. Then $J^s_{\sigma,\alpha}$ is a bounded linear operator from $L^p(\mathbb{R}^n)$ into itself.*

Proof. Let $\phi \in S(\mathbb{R}^n)$, then

$$\mathcal{F}_\alpha(J^s_{\sigma,\alpha}\phi)(w) = \left[\left(1+\sum_{k=1}^{n}|w_k|^{\frac{2}{\alpha_k}}\right)^{\frac{-s}{[\alpha]}}(\mathcal{F}_\alpha\phi)\right](w).$$

Next, by using Proposition 2.2.3 and (2.5.5), we have

$$\mathcal{F}_\alpha(G_s*\phi)(w) = (2\pi)^{\frac{n}{2}}\left(1+\sum_{k=1}^{n}|w_k|^{\frac{2}{\alpha_k}}\right)^{\frac{-s}{2}}(\mathcal{F}_\alpha\phi)(w),$$

so that

$$G_s*\phi = (2\pi)^{\frac{n}{2}}\mathcal{F}_\alpha{}^{-1}\left[\left(1+\sum_{k=1}^{n}|w_k|^{\frac{2}{\alpha_k}}\right)^{\frac{-s}{2}}(\mathcal{F}_\alpha\phi)\right]. \qquad (2.5.6)$$

Since (2.5.6) is true for all $s > 0$, so we can replace s by $\frac{2s}{[\alpha]} > 0$, in (2.5.6) thus, we have

$$G_{\frac{2s}{[\alpha]}}*\phi = (2\pi)^{\frac{n}{2}}\mathcal{F}_\alpha{}^{-1}\left[\left(1+\sum_{k=1}^{n}|w_k|^{\frac{2}{\alpha_k}}\right)^{\frac{-s}{[\alpha]}}(\mathcal{F}_\alpha\phi)\right].$$

Therefore we get,

$$J^s_{\sigma,\alpha}\phi = (2\pi)^{\frac{-n}{2}}\left(G_{\frac{2s}{[\alpha]}}*\phi\right)$$

Now, using the properties of convolution ([8], p. 9), we have

$$\|J^{s}_{\sigma,\alpha}\phi\|_p = (2\pi)^{\frac{-n}{2}}\|G_{\frac{2s}{[\alpha]}} * \phi\|_p \leq (2\pi)^{\frac{-n}{2}}\|G_{\frac{2s}{[\alpha]}}\|_1\|\phi\|_p = \|\phi\|_p. \tag{2.5.7}$$

Relation (2.5.7) proves that $J^{s}_{\sigma,\alpha}$ is a bounded linear map from $S(\mathbb{R}^n)$ into itself. From ([8], p. 14), $S(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)(1 \leq p < \infty)$, therefore $J^{s}_{\sigma,\alpha}$ can be extended to a bounded linear operator on $L^p(\mathbb{R}^n)$ satisfying (2.5.7). □

Theorem 2.5.5. *Let $1 < p < \infty$ and $s < t$, then*

$$H^{t,p}_{\alpha} \subseteq H^{s,p}_{\alpha} \quad \textit{and} \quad \|u\|_{s,p} \leq \|u\|_{t,p},$$

for all $u \in H^{t,p}_{\alpha}$.

Proof. Let $u \in H^{t,p}_{\alpha}$ then, $J^{-t}_{\sigma,\alpha}u \in L^p(\mathbb{R}^n)$.

Now, we can write

$$J^{-s}_{\sigma,\alpha}u = J^{t-s}_{\sigma,\alpha}(J^{-t}_{\sigma,\alpha}u).$$

Exploiting (2.5.7), we obtain

$$\|u\|_{s,p} = \|J^{-s}_{\sigma,\alpha}u\|_p = \|J^{t-s}_{\sigma,\alpha}(J^{-t}_{\sigma,\alpha}u)\|_p \leq \|J^{-t}_{\sigma,\alpha}u\|_p \leq \|u\|_p, \quad u \in H^{t,p}_{\alpha}.$$

□

Theorem 2.5.6. *For $s \in \mathbb{R},\ 1 < p < \infty$, $S(\mathbb{R}^n)$ is dense in $H^{s,p}_{\alpha}$.*

Proof. Let $u \in H^{s,p}_{\alpha}$, then $J^{-s}_{\sigma,\alpha}u \in L^p(\mathbb{R}^n)$. Since $S(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, so there exists a sequence (ϕ_j) in $S(\mathbb{R}^n)$ such that $\phi_j \to J^{-s}_{\sigma,\alpha}u$ in $L^p(\mathbb{R}^n)$ as $j \to \infty$.

Assume $\chi_j = J^{s}_{\sigma,\alpha}\phi_j \in S(\mathbb{R}^n)$.

From (2.5.4), we find that

$$\|\chi_j - u\|_{s,p} = \|J^{-s}_{\sigma,\alpha}(\chi_j - u)\|_p = \|J^{-s}_{\sigma,\alpha}\chi_j - J^{-s}_{\sigma,\alpha}u\|_p = \|\phi_j - J^{-s}_{\sigma,\alpha}u\|_p \to 0,$$

as $j \to \infty$. This shows that $S(\mathbb{R}^n)$ is dense in $H^{s,p}_{\alpha}$. □

Theorem 2.5.7. *For $s > 0$, $1 < p < \infty$, $J^s_{\sigma,\alpha}$ is a bounded linear operator from $H^{s,p}_{\alpha}$ into itself.*

Proof. From inequality (2.5.7), $J^s_{\sigma,\alpha}$ is a bounded linear operator from $S(\mathbb{R}^n)$ into itself. From Theorem 2.5.6, we have seen that $S(\mathbb{R}^n)$ is dense in $H^{s,p}_{\alpha}$. So that, $J^s_{\sigma,\alpha}$ can be extended to a bounded linear operator on $H^{s,p}_{\alpha}$. □

2.6 Applications of pseudo-differential operators in fractional calculus

In this section, applications of pseudo-differential operators involving fractional derivatives and fractional integrals on Lizorkin space $\Phi(\mathbb{R})$ are discussed by exploiting the theory of the fractional Fourier transform.

Lemma 2.6.1. *If $T_{\sigma,\alpha}$ is the pseudo-differential operator corresponding to the symbol $\sigma_\alpha(x,w) = \sigma_\alpha(w) \in S^{\frac{m}{\alpha}}$, then for $\phi \in S(\mathbb{R})$, we have to prove the following expression*

$$(\mathcal{F}_\alpha T_{\sigma,\alpha}\phi)(w) = \sigma_\alpha(w)(\mathcal{F}_\alpha\phi)(w). \tag{2.6.1}$$

Proof. For $\phi \in S(\mathbb{R})$, using (2.3.8) for $n = 1$ and taking $\sigma_\alpha(x, w) = \sigma_\alpha(w) \in S^{\frac{m}{\alpha}}$, we get

$$(T_{\sigma,\alpha}\phi)(x) = \frac{1}{(2\pi)^{\frac{1}{2}}\alpha} \int_{\mathbb{R}} e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}} x} |w|^{\frac{1}{\alpha}-1} \sigma_\alpha(w)(\mathcal{F}_\alpha\phi)(w)dw. \tag{2.6.2}$$

From (2.1.2), (2.6.2) becomes

$$(T_{\sigma,\alpha}\phi)(x) = \mathcal{F}_\alpha^{-1}[\sigma_\alpha(w)(\mathcal{F}_\alpha\phi)(w)](x). \tag{2.6.3}$$

Since $\phi \in S(\mathbb{R})$, therefore by using Proposition 2.2.4, we get $\mathcal{F}_\alpha\phi \in S(\mathbb{R})$. This implies that R.H.S of (2.6.3) be an element of $S(\mathbb{R})$.
Now, by the property of the fractional Fourier transform, we can write (2.6.3) by the following way:

$$(\mathcal{F}_\alpha T_{\sigma,\alpha}\phi)(w) = \sigma_\alpha(w)(\mathcal{F}_\alpha\phi)(w),$$

which is an element of $S(\mathbb{R})$. □

Lemma 2.6.2. *If $\mathcal{F}_\alpha\phi \in V(\mathbb{R})$, which is defined in (1.3.1), then the pseudo differential operator $T_{\sigma,\alpha}\phi$ is given in (2.6.2) is an element of the Lizorkin space $\Phi(\mathbb{R})$.*

Proof. From (2.6.1) and Leibnitz formula for differentiation, we have

$$D_w^m\left((\mathcal{F}_\alpha T_{\sigma,\alpha}\phi)(w)\right) = \sum_{r=0}^{m} \binom{m}{r} \left(D_w^{m-r}\sigma_\alpha(w)\right)\left(D_w^r(\mathcal{F}_\alpha\phi)(w)\right), \text{ for } m \in \mathbb{N}. \tag{2.6.4}$$

Since $\mathcal{F}_\alpha\phi \in V(\mathbb{R})$, and at $w = 0$, the right hand side of (2.6.4) will be zero. For making $T_{\sigma,\alpha}\phi$ is in the Lizorkin space, put $w = 0$ in (2.6.1), we get $(\mathcal{F}_\alpha T_{\sigma,\alpha}\phi)(0) = 0$. Finally $T_{\sigma,\alpha}\phi$ belongs to the Lizorkin space $\Phi(\mathbb{R})$. □

Theorem 2.6.3. *If $\mathcal{F}_\alpha\phi \in V(\mathbb{R})$, then following operational relation holds for $0 < \alpha \leq 1$,*

$$(\mathcal{F}_\alpha D^\alpha_\beta T_{\sigma,\alpha}\phi)(w) = (ic_\alpha w)(\sigma_\alpha(w)(\mathcal{F}_\alpha\phi)(w)), \quad w \in \mathbb{R}, \tag{2.6.5}$$

where β is any parameter and c_α is

$$c_\alpha = sin(\alpha\pi/2) - i(\operatorname{sign} w)(1-2\beta)cos(\alpha\pi/2), \tag{2.6.6}$$

and $T_{\sigma,\alpha}\phi$ is defined by (2.6.2).

Proof. By using Lemma 2.6.2, we prove the above theorem by the following way:

Case 1: If $\alpha = 1$, then (2.6.5) holds easily.

Case 2: If $w = 0, \quad \alpha \neq 1$, then we need to show that $(\mathcal{F}_\alpha D^\alpha_\beta T_{\sigma,\alpha}\phi)(0) = 0$.

Now, by using the definition of the fractional Fourier transform, D^α_β and Theorem 4.1 from [3], we get

$$\begin{aligned}
(\mathcal{F}_\alpha D^\alpha_\beta T_{\sigma,\alpha}\phi)(0) &= \int_{-\infty}^{+\infty} (\mathcal{F}_\alpha D^\alpha_\beta T_{\sigma,\alpha}\phi)(x)dx \\
&= \Big[(1-\beta)\big(\frac{d}{dx}\big)(I_+^{1-\alpha}T_{\sigma,\alpha}\phi)(x) + \beta\big(\frac{d}{dx}\big)(I_-^{1-\alpha}T_{\sigma,\alpha}\phi)(x)\Big]_{-\infty}^{+\infty} \\
&= 0,
\end{aligned}$$

by using the fact that $\Phi(\mathbb{R})$ is a proper subset of $S(\mathbb{R})$.

Case 3: $\alpha \neq 1, \ w > 0$, then from (1.3.4), (1.2.9), (1.2.10) and [3, p. 465], we have

$$\begin{aligned}
(\mathcal{F}_\alpha D^\alpha_\beta T_{\sigma,\alpha}\phi)(w) &= \int_{-\infty}^{+\infty} e^{-i|w|^{\frac{1}{\alpha}}x}(D^\alpha_\beta T_{\sigma,\alpha}\phi)(x)dx \\
&= (1-\beta)\int_{-\infty}^{+\infty} e^{-i|w|^{\frac{1}{\alpha}}x}(D^\alpha_+ T_{\sigma,\alpha}\phi)(x)dx \\
&\quad - \beta\int_{-\infty}^{+\infty} e^{-i|w|^{\frac{1}{\alpha}}x}(D^\alpha_- T_{\sigma,\alpha}\phi)(x)dx \\
&= (1-\beta)\int_{-\infty}^{+\infty} (D^\alpha_- e^{-i|w|^{\frac{1}{\alpha}}t})(x)(T_{\sigma,\alpha}\phi)(x)dx
\end{aligned}$$

$$
\begin{aligned}
&- \beta \int_{-\infty}^{+\infty} (D_+^{\alpha} e^{-i|w|^{\frac{1}{\alpha}} t})(x)(T_{\sigma,\alpha}\phi)(x)dx \\
&= \big((1-2\beta)cos(\alpha\pi/2) + isin(\alpha\pi/2)\big) w \int_{-\infty}^{+\infty} e^{-i|w|^{\frac{1}{\alpha}} x} T_{\sigma,\alpha}\phi(x)dx \\
&= \big((1-2\beta)cos(\alpha\pi/2) + isin(\alpha\pi/2)\big) w (\mathcal{F}_\alpha T_{\sigma,\alpha}\phi(x))(w).
\end{aligned}
$$

Now, by using Lemma 2.6.1, we have

$$
(\mathcal{F}_\alpha D_\beta^\alpha T_{\sigma,\alpha}\phi)(w) = (iw)\big(sin(\alpha\pi/2) - i(1-2\beta)cos(\alpha\pi/2)\big)(\sigma_\alpha(w)(\mathcal{F}_\alpha\phi)(w)).
$$

Case 4: For $\alpha \neq 1,\ w < 0$, then we get

$$
(\mathcal{F}_\alpha D_\beta^\alpha T_{\sigma,\alpha}\phi)(w) = \int_{-\infty}^{+\infty} e^{i|w|^{\frac{1}{\alpha}} x} (\mathcal{F}_\alpha D_\beta^\alpha T_{\sigma,\alpha}\phi)(x)dx.
$$

Taking the concept of [3, p.465] and case 3, we have

$$
(\mathcal{F}_\alpha D_\beta^\alpha T_{\sigma,\alpha}\phi)(w) = (iw)\big(sin(\alpha\pi/2) + i(1-2\beta)cos(\alpha\pi/2)\big)(\sigma_\alpha(w)(\mathcal{F}_\alpha\phi)(w)).
$$

Hence by combining cases 1 to 4, we find the proof of the Theorem. □

Theorem 2.6.4. *If $\mathcal{F}_\alpha\phi \in V(\mathbb{R})$, then following operational relation holds for $0 < \alpha \leq 1$,*

$$
(\mathcal{F}_\alpha I_\beta^\alpha T_{\sigma,\alpha}\phi)(w) = \frac{d_\alpha(\beta)i}{w}(\sigma_\alpha(w)(\mathcal{F}_\alpha\phi)(w)), \quad w \in \mathbb{R};\ w \neq 0, \tag{2.6.7}
$$

where

$$
d_\alpha(\beta) = sin(\frac{\alpha\pi}{2}) - i(\operatorname{sign} w)(1-2\beta)cos(\frac{\alpha\pi}{2}), \tag{2.6.8}
$$

and $T_{\sigma,\alpha}\phi$ is given in (2.6.2).

Proof. By using Lemma 2.6.2, the proof of aforesaid theorem can be easily obtained from [4, p. 789-790]. □

2.7 Conclusions

Fractional Fourier transform is a useful tool for solving the problems of Riemann-Liouville fractional derivatives and other problems of fractional calculus. The importance of this theory can be seen in the papers of Luchko et al [3, 4]. Properties of the fractional Fourier transform, pseudo-differential operators on Schwartz space $S(\mathbb{R}^n)$ and applications of pseudo-differential operators on the Sobolev space of type $H_{\alpha}^{s,p}$ associated with a certain class of symbols are discussed in this paper. The authors also studied applications of pseudo-differential operators on Lizorkin space $\Phi(\mathbb{R})$ by utilizing the theory of the fractional Fourier transform, modified fractional derivative operator D_{β}^{α} and modified fractional integral operator I_{β}^{α}. This paper is of great importance in mathematical sciences, physical sciences, engineering, and other areas. This paper is useful for those researchers, who are interested to work in the area of pseudo-differential operators, fractional calculus, image processing, signal processing, and others. The authors cited all the possible recent developments regarding the fractional Fourier transform, pseudo-differential operators, and others in the form of references.

***