Chapter 1

Introduction

The theory of Fourier transform of fractional order was initially introduced by Wiener (1929). The fractional Fourier transform played an important role in finding problems of quantum mechanics, signal processing, image processing, stochastic processes, optics, wavelets and other areas of mathematical sciences. The importance of the fractional Fourier transform is that it is a generalization of the Fourier transform. Many definitions and properties regarding the fractional Fourier transform are given, which have different applications in different areas. Condon (1937) constructed a continuous group of functional transforms which contained ordinary Fourier transforms as a subgroup. The Fourier transform of fractional order was observed by Namias (1980) and found the integral representation and generalized operational calculus. A. McBride and F. Kerr (1987) found more specific and modified results of Namias's fractional operators. Zayed (1997) gave the relation between the Fourier transform and the fractional Fourier transform. Using analytical techniques and algebraic techniques, Zayed (1998) extended the results of Namias for the fractional Fourier transform on different spaces of generalized functions. He had also observed convolution operation for the fractional Fourier transform. Ozaktas

(2001) introduced six different definitions regarding the fractional Fourier transform and found different applications in the voice, signal processing and image processing. Luchko et al. (2008) introduced the fractional Fourier transform, which is useful to find properties of the fractional derivative and fractional integral operators in the Lizorkin space. With the help of the aforesaid technique, the partial diffusion-type differential equation of fractional order was solved by Kilbas et al. [4]. The solution of model partial differential equations of fractional order was discussed by Luchko et al. [3]. The compositions of the fractional Fourier transform with modified fractional integrals and derivatives were considered by Kilbas et al.(2010).

The pseudo-differential operator is an important tool, which is the generalization of the partial differential operator and useful to find the solution of the partial differential equation. By exploiting the theory of many integral transforms, many authors had discussed properties of the pseudo-differential operator and found applications in the Sobolev type space. Kohn-Nirenberg [5], Hormander [6, 7], Wong [8] and others discussed properties of the pseudo-differential operator by exploiting the theory of the Fourier transform. For the Hankel transform concern, the pseudo-differential operator were found by Pathak and Pandey [38]. Several results of the pseudo-differential operator associated with the fractional Fourier transform, the fractional Hankel transform and other integral transforms were investigated by Prasad et al. [39, 40] and Upadhyay et al. [29]. Recently, Srivastava et al. [14], studied properties of the pseudo-differential operator by exploiting the Luchko theory of the fractional Fourier transform on the Schwartz space $S(\mathbb{R})$.

The wavelet transform associated to the Fourier transform is an integral transform, whose kernel contains the translation and dilation in the time domain. A wavelet transform gives local as well as global information of a signal. The wavelet transform by exploiting the Fourier transform is given in [49]. Pathak and Dixit [32] developed continuous and discrete Bessel wavelet transform in 2003. Recently, Upadhyay, Srivastava and Khatterwani [16], discussed the continuous fractional wavelet transform by taking the fractional Fourier transform theory.

From [3, 4, 8, 14, 16, 16, 17, 28, 32, 49], several definitions, formulae, and properties are given in this chapter that will be used in the subsequent chapters.

1.1 Basic definitions

From [8, 28, 49], some basic definitions which are useful for the present thesis are given below:

Definition 1.1.1. (L^p**-spaces**)

Let $1 \leq p < \infty$. A function $f : \mathbb{R}^n \to \mathbb{C}$ is said to be in $L^p(\mathbb{R}^n)$ if it is measurable and its norm

$$||f||_p := \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{\frac{1}{p}} \tag{1.1.1}$$

is finite. In the case $p = \infty$, f is said to be in $L^\infty(\mathbb{R}^n)$ if it is measurable and essentially bounded, i.e., if

$$||f||_\infty := ess\, sup_{x\in\mathbb{R}^n}|f(x)| < \infty. \tag{1.1.2}$$

Here $ess\, sup_{x\in\mathbb{R}^n}|f(x)|$ is defined as the smallest M such that $|f(x)| \leq M$ for almost all $x \in \mathbb{R}^n$.

If $\phi, \psi \in L^2(\mathbb{R}^n)$, then the inner product of ϕ and ψ is defined by

$$\langle \phi, \psi \rangle := \int_{\mathbb{R}^n} \phi(x)\overline{\psi(x)}dx. \tag{1.1.3}$$

If $\phi, \psi \in L^1(\mathbb{R}^n)$, then the convolution of ϕ and ψ is defined as

$$(\phi * \psi)(x) = \int_{\mathbb{R}^n} \phi(x-y)\psi(y)dy, \quad x \in \mathbb{R}^n. \tag{1.1.4}$$

Let ϕ be a measurable function, which is defined on $\mathbb{R}^n$. For any fixed $y \in \mathbb{R}^n$ and a be a positive real number, we define translation function $T_y\phi$ and dilation function $D_a\phi$ by

$$(T_y\phi)(x) = \phi(x+y), \tag{1.1.5}$$

$$(D_a\phi)(x) = \phi(ax). \tag{1.1.6}$$

Definition 1.1.2. (**Schwartz space**)

The Schwartz space $S(\mathbb{R}^n)$ is the vector space of all complex valued infinitely differentiable functions ϕ on $\mathbb{R}^n$ such that for all multi-indices β,$\gamma \in \mathbb{N}_0^n$, we have

$$\|\phi\|_{\beta,\gamma} = \sup_{x\in\mathbb{R}^n} |x^\beta (D^\gamma \phi)(x)| < \infty. \tag{1.1.7}$$

Moreover,

$$\|\phi\|_{\beta,\gamma} < \infty \text{ iff } \|\phi\|_N = \sup_{x\in\mathbb{R}^n} (1+|x|^2)^N |(D^\beta\phi)(x)| < \infty, \quad N \in \mathbb{N}_0. \tag{1.1.8}$$

Definition 1.1.3. (**Convergence in the Schwartz space**)

A sequence (ϕ_j) of functions in the Schwartz space $S(\mathbb{R}^n)$ is said to converge to zero in $S(\mathbb{R}^n)$ (denoted by $\phi_j \to 0$ in $S(\mathbb{R}^n)$) if for all multi-indices $\beta, \gamma \in \mathbb{N}_0^n$, we have, $\sup_{x\in\mathbb{R}^n} |x^\beta (D^\gamma \phi_j)(x)| \to 0$, as $j \to \infty$.

Definition 1.1.4. (**Tempered distributions**)

A linear functional T on $S(\mathbb{R}^n)$ is called continuous if for any sequence (ϕ_j) of functions in $S(\mathbb{R}^n)$ converging to zero in $S(\mathbb{R}^n)$, we have $T(\phi_j) \to 0$ as $j \to \infty$. Continuous linear functionals on $S(\mathbb{R}^n)$ are called tempered distributions and the set of continuous linear functionals is denoted by $S'(\mathbb{R}^n)$. Let $T \in S'(\mathbb{R}^n)$, then $T : S(\mathbb{R}^n) \to \mathbb{C}$ is a continuous linear map such that $T(\phi) = \langle T, \phi \rangle$, for all $\phi \in S(\mathbb{R}^n)$.

Definition 1.1.5. (**Convergence in** $S'(\mathbb{R}^n)$)
A sequence (T_j) of functions in $S'(\mathbb{R}^n)$ is said to converge to zero in $S'(\mathbb{R}^n)$, if for any $\phi \in S(\mathbb{R}^n)$, the sequence $\langle T_j, \phi \rangle$ converges to zero in $\mathbb{C}$, as $j \to \infty$.

Definition 1.1.6. (**Distributions of compact support**)
We denote $\mathcal{D}(\mathbb{R})$ be the set of all complex-valued infinitely differentiable functions on $\mathbb{R}$ having compact support. $\mathcal{D}'(\mathbb{R})$ is the dual of the space $\mathcal{D}(\mathbb{R})$ and its elements are called Schwartz distributions. The space of all those distributions in $\mathcal{D}'(\mathbb{R})$ which have compact support is denoted by $\mathscr{E}'(\mathbb{R})$.

1.2 Riemann-Liouville fractional derivatives and integrals

In this section, from [3, 4] various properties and formulae of Riemann-Liouville fractional derivatives and integrals are given below:
The modified fractional derivative D^α_β is defined by

$$D^\alpha_\beta u := (1-\beta) D^\alpha_+ u - \beta D^\alpha_- u, \quad 0 < \alpha \le 1, \quad \beta \in \mathbb{R}, \tag{1.2.1}$$

where $D^\alpha_+ u$ and $D^\alpha_- u$ are the left-hand and right-hand side Riemann-Liouville fractional derivatives on the real axis, which are defined by

$$(D^\alpha_+ u)(x) := \left(\frac{d}{dx}\right)(I^{1-\alpha}_+ u)(x), \tag{1.2.2}$$

where I_+^α is the Riemann-Liouville fractional integral operator

$$(I_+^\alpha u)(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} (x-t)^{\alpha-1} u(t) dt, \tag{1.2.3}$$

and

$$(D_-^\alpha u)(x) := \Big(-\frac{d}{dx}\Big)(I_-^{1-\alpha} u)(x), \tag{1.2.4}$$

where I_-^α is the Riemann-Liouville fractional integral operator

$$(I_-^\alpha u)(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{+\infty} (t-x)^{\alpha-1} u(t) dt. \tag{1.2.5}$$

For $\alpha > 0$ and real $\beta \in \mathbb{R}$, the modified fractional integral operator I_β^α is defined by

$$I_\beta^\alpha u := (1-\beta) I_+^\alpha u - \beta I_-^\alpha u. \tag{1.2.6}$$

Properties of the fractional derivative operator and fractional integral operator

Let $w \in \mathbb{R}$, $w \neq 0$ and $0 < \alpha < 1$, then

$$\Big(I_+^\alpha e^{iwt}\Big)(x) = e^{iwx}|w|^{-\alpha}\Big(cos(\alpha\pi/2) - i(\operatorname{sign} w) sin(\alpha\pi/2)\Big), \tag{1.2.7}$$

$$\Big(I_-^\alpha e^{iwt}\Big)(x) = e^{iwx}|w|^{-\alpha}\Big(cos(\alpha\pi/2) + i(\operatorname{sign} w) sin(\alpha\pi/2)\Big), \tag{1.2.8}$$

$$\Big(D_+^\alpha e^{iwt}\Big)(x) = e^{iwx}|w|^{\alpha}\Big(cos(\alpha\pi/2) + i(\operatorname{sign} w) sin(\alpha\pi/2)\Big), \tag{1.2.9}$$

$$\Big(D_-^\alpha e^{iwt}\Big)(x) = e^{iwx}|w|^{\alpha}\Big(cos(\alpha\pi/2) - i(\operatorname{sign} w) sin(\alpha\pi/2)\Big). \tag{1.2.10}$$

1.3 Lizorkin space

This section presents the definition and properties of the Lizorkin space as described by Luchko et al. (2008):

Let $V(\mathbb{R})$ be the set of functions $\phi \in S(\mathbb{R})$ satisfying the conditions:

$$\left.\frac{d^n\phi(x)}{dx^n}\right|_{x=0} = 0, \quad n = 0, 1, 2, ... \tag{1.3.1}$$

Definition 1.3.1. The Lizorkin space $\Phi(\mathbb{R})$ is introduced as the Fourier pre-image of the space $V(\mathbb{R})$ in the space $S(\mathbb{R})$, i.e.,

$$\Phi(\mathbb{R}) = \{\phi \in S(\mathbb{R}) : \quad \mathcal{F}\phi \in V(\mathbb{R})\}, \tag{1.3.2}$$

where $\mathcal{F}$ denotes the Fourier transform.

Properties of the Lizorkin space

1. The following orthogonality conditions hold:

$$\int_{-\infty}^{+\infty} x^n\phi(x)dx = 0, \quad \phi \in \Phi(\mathbb{R}), \quad n = 0, 1, 2, ... \tag{1.3.3}$$

2. The Lizorkin space is invariant with respect to the fractional integration and differentiation operators.

3. If $\phi, \psi \in \Phi(\mathbb{R})$, the following formula for integration by parts for the fractional Riemann-Liouville derivatives holds true:

$$\int_{-\infty}^{+\infty} \phi(x)(D^\alpha_+\psi)(x)dx = \int_{-\infty}^{+\infty} (D^\alpha_-\phi)(x)\psi(x)dx. \tag{1.3.4}$$

4. If $\phi, \psi \in \Phi(\mathbb{R})$, the following formula for integration by parts for the fractional Riemann-Liouville integral holds true:

$$\int_{-\infty}^{+\infty} \phi(x)(I^\alpha_+\psi)(x)dx = \int_{-\infty}^{+\infty} (I^\alpha_-\phi)(x)\psi(x)dx. \tag{1.3.5}$$

1.4 Fractional Fourier transform

From [3, 4, 14, 16], in this section various definitions and properties of the fractional Fourier transform are discussed.

Definition 1.4.1. The fractional Fourier transform for a given function ϕ of order α $(0 < \alpha \leq 1)$ is defined by

$$(\mathcal{F}_\alpha \phi)(w) \equiv \hat{\phi}_\alpha(w) := \int_{\mathbb{R}} e^{-i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}} x} \phi(x)dx, \quad \forall\ w \in \mathbb{R}, \tag{1.4.1}$$

provided the integral on R.H.S. of (1.4.1) is convergent.

The inverse fractional Fourier transform of $\mathcal{F}_\alpha \phi$ of order α $(0 < \alpha \leq 1)$ is defined by

$$\mathcal{F}_\alpha^{-1}(\mathcal{F}_\alpha \phi)(x) := \frac{1}{2\pi\alpha} \int_{\mathbb{R}} e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}} x} |w|^{\frac{1}{\alpha}-1} (\mathcal{F}_\alpha \phi)(w)dw, \quad \forall\ x \in \mathbb{R}, \tag{1.4.2}$$

provided the integral on R.H.S. of (1.4.2) is convergent.

Properties of the fractional Fourier transform

1. $\mathcal{F}_\alpha : L^1(\mathbb{R}) \to L^\infty(\mathbb{R})$ is a bounded linear operator for $0 < \alpha \leq 1$.

2. If $\phi \in L^1(\mathbb{R})$, then

$$(\mathcal{F}_\alpha \phi)(w) = (\mathcal{F}\phi)((\operatorname{sign} w)|w|^{\frac{1}{\alpha}}), \quad w \in \mathbb{R};\ \alpha > 0, \tag{1.4.3}$$

   where $\mathcal{F}$ denotes the Fourier transform operator.

3. Let $\phi, \psi \in L^1(\mathbb{R})$, then

$$\mathcal{F}_\alpha(\phi * \psi) = (\mathcal{F}_\alpha \phi)(\mathcal{F}_\alpha \psi). \tag{1.4.4}$$

4. Let $\phi \in L^1(\mathbb{R})$, then

$$(\mathcal{F}_\alpha T_y \phi)(w) = e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}} y}(\mathcal{F}_\alpha \phi)(w), \quad w \in \mathbb{R}. \tag{1.4.5}$$

5. Let $\phi \in L^1(\mathbb{R})$, then for any positive real number a, we have

$$(\mathcal{F}_\alpha D_a \phi)(w) = \frac{1}{a}(\mathcal{F}_\alpha \phi)\Big(\frac{w}{a^\alpha}\Big), \quad w \in \mathbb{R}. \tag{1.4.6}$$

6. If $\phi \in \Phi(\mathbb{R})$, $\beta \in \mathbb{N}$ and $\alpha > 0$, then we have

$$\mathcal{F}_\alpha(D^\beta \phi(x))(w) = ((\operatorname{sign} w)|w|^{\frac{1}{\alpha}})^\beta (\mathcal{F}_\alpha \phi)(w), \quad w \in \mathbb{R}. \tag{1.4.7}$$

7. Let $\phi, \psi \in L^2(\mathbb{R})$, then the Parseval formula of the fractional Fourier transform is given by

$$\langle \phi, \psi \rangle = \frac{1}{2\pi\alpha}\Big\langle |.|^{\frac{1}{\alpha}-1}(\mathcal{F}_\alpha \phi), (\mathcal{F}_\alpha \psi)\Big\rangle. \tag{1.4.8}$$

8. If $\phi \in L^1(\mathbb{R})$ and $\mathcal{F}(\phi) \in L^1(\mathbb{R})$, then $\mathcal{F}_\alpha^{-1}(\mathcal{F}_\alpha \phi) = \phi \quad$ a.e. on $\mathbb{R}$.

9. If $\phi \in L^1(\mathbb{R})$, then $(\mathcal{F}_\alpha \widetilde{\phi})(w) = \widetilde{\mathcal{F}_\alpha \phi}(w),\ w \in \mathbb{R}$, where $\widetilde{\phi}(x) = \phi(-x)$, for all $x \in \mathbb{R}$.

1.5 Pseudo-differential operators involving the fractional Fourier transform

In this section, from [14], definitions and properties of pseudo-differential operators are discussed by exploiting the theory of the fractional Fourier transform.

Definition 1.5.1. Let $m \in \mathbb{R}$ and $\alpha \in (0,1]$. We define $S^{\frac{m}{\alpha}}$ is the set of all functions $\sigma_\alpha \in C^\infty(\mathbb{R} \times \mathbb{R})$ such that for all $\beta, \gamma \in \mathbb{N}_0$, there exists a constant $A_{\beta,\gamma} > 0$ depending on β and γ only, such that

$$|D_x^\beta D_w^\gamma \sigma_\alpha(x,w)| \leq A_{\beta,\gamma}(1+|w|)^{\frac{m}{\alpha}-|\gamma|}. \tag{1.5.1}$$

We call any function $\sigma_\alpha \in \cup_{m\in\mathbb{R}} S^{\frac{m}{\alpha}}$ a symbol.

Definition 1.5.2. Let $\sigma_\alpha \in C^\infty(\mathbb{R} \times \mathbb{R})$ be a symbol. Then the pseudo-differential operator $T_{\sigma,\alpha}$ associated with symbol $\sigma_\alpha(x,w)$ is defined by

$$(T_{\sigma,\alpha}\phi)(x) := \frac{1}{2\pi\alpha} \int_{\mathbb{R}} e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}} x} |w|^{\frac{1}{\alpha}-1} \sigma_\alpha(x,w)(\mathcal{F}_\alpha \phi)(w)dw, \tag{1.5.2}$$

for all $\phi \in S(\mathbb{R})$.

Definition 1.5.3. (**Formal adjoint**)

For any pair of functions ϕ and ψ in $S(\mathbb{R})$, we define (ϕ, ψ) by

$$(\phi, \psi) := \int_{\mathbb{R}} \phi(x)\overline{\psi(x)}dx. \tag{1.5.3}$$

Let σ_α be a symbol in $S^{\frac{m}{\alpha}}$ and $T_{\sigma,\alpha}$ its associated pseudo differential operator. Suppose there exists a linear operator $T^*_{\sigma,\alpha} : S(\mathbb{R}) \to S(\mathbb{R})$ such that

$$(T_{\sigma,\alpha}\phi, \psi) = (\phi, T^*_{\sigma,\alpha}\psi), \quad \phi, \psi \in S(\mathbb{R}). \tag{1.5.4}$$

Then we call $T^*_{\sigma,\alpha}$ a formal adjoint of the operator $T_{\sigma,\alpha}$.

Definition 1.5.4. Let $T_{\sigma,\alpha}$ be a pseudo differential operator on $S(\mathbb{R})$ and $u \in S'(\mathbb{R})$, then $T_{\sigma,\alpha}$ is defined on $S'(\mathbb{R})$ by

$$(T_{\sigma,\alpha}u)(\phi) = u(\overline{T^*_{\sigma,\alpha}\overline{\phi}}), \quad \phi \in S(\mathbb{R}), \tag{1.5.5}$$

where $T^*_{\sigma,\alpha}$ is the formal adjoint of $T_{\sigma,\alpha}$.

Properties of pseudo-differential operator

1. The operator $T_{\sigma,\alpha}$ is a bounded linear map from $S(\mathbb{R})$ into itself.

2. The operator $T_{\sigma,\alpha}$ is a bounded linear map from $S^{'}(\mathbb{R})$ into itself.

1.6 The continuous fractional wavelet transform involving the fractional Fourier transform

From [16], definitions and properties of the continuous fractional wavelet transform which are helpful in the subsequent chapters of this thesis are given below :

Definition 1.6.1. The fractional wavelet $\psi_{\alpha,a,b}(t)$ of order α $(0 < \alpha \leq 1)$, where $\psi \in L^2(\mathbb{R})$, is defined as follows:

$$\psi_{\alpha,a,b}(t) = \frac{1}{|a|^{\frac{1}{2\alpha}}}\psi\left(\frac{t-b}{|a|^{\frac{1}{\alpha}}}\right), \quad a, b \in \mathbb{R}, \; a \neq 0, \tag{1.6.1}$$

and satisfies the following admissibility condition:

$$C_{\psi_\alpha} = \int_{-\infty}^{+\infty} \frac{|\hat{\psi}_\alpha(w)|^2}{|w|} dw < \infty. \tag{1.6.2}$$

Definition 1.6.2. If $\psi \in L^2(\mathbb{R})$, then the integral transformation given by

$$\big(W_{\psi_\alpha}\phi\big)(b,a) = \langle \phi, \psi_{\alpha,a,b} \rangle = \int_{-\infty}^{+\infty} \phi(t) \frac{1}{|a|^{\frac{1}{2\alpha}}} \overline{\psi\left(\frac{t-b}{|a|^{\frac{1}{\alpha}}}\right)} dt \tag{1.6.3}$$

is called continuous fractional wavelet transformation of a given signal $\phi \in L^2(\mathbb{R})$.

Properties of the continuous fractional wavelet transform

For $0 < \alpha \leq 1$, following properties hold:

1. If $\psi \in L^2(\mathbb{R})$, we have

$$F_\alpha(\psi_{\alpha,a,b}(t))(w) = |a|^{\frac{1}{2\alpha}} e^{-i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}} b} \hat{\psi}_\alpha(aw). \tag{1.6.4}$$

2. Let $\psi \in L^2(\mathbb{R})$, then for any signal $\phi \in L^2(\mathbb{R})$, we have following integral representation of $\big(W_{\psi_\alpha}\phi\big)(b,a)$

$$\big(W_{\psi_\alpha}\phi\big)(b,a) = \frac{|a|^{\frac{1}{2\alpha}}}{2\pi\alpha} \int_{-\infty}^{+\infty} e^{i(\operatorname{sign} w)|w|^{\frac{1}{\alpha}} b} |w|^{\frac{1}{\alpha}-1} \hat{\phi}_\alpha(w) \overline{\hat{\psi}_\alpha(aw)} dw. \tag{1.6.5}$$

3. Let $\psi \in L^2(\mathbb{R})$, then for any $\phi_1, \phi_2 \in L^2(\mathbb{R})$, the Parseval's formula for the continuous fractional wavelet transform is given by

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \big(W_{\psi_\alpha}\phi_1\big)(b,a) \overline{\big(W_{\psi_\alpha}\phi_2\big)(b,a)} \frac{dbda}{|a|^{\frac{1}{\alpha}+1}} = C_{\psi_\alpha} \langle \phi_1, \phi_2 \rangle. \tag{1.6.6}$$

4. Let $\psi \in L^2(\mathbb{R})$, then a signal $\phi \in L^2(\mathbb{R})$ can be reconstructed by the following inversion formula for the continuous fractional wavelet transform

$$\phi(t) = \frac{1}{C_{\psi_\alpha}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \big(W_{\psi_\alpha}\phi\big)(b,a) \psi_{\alpha,a,b}(t) \frac{dbda}{|a|^{\frac{1}{\alpha}+1}}. \tag{1.6.7}$$

5. Let $\psi \in L^2_\sigma(I)$. Then the discrete fractional wavelet transform of a signal $\phi \in L^2_\sigma(I)$ is given by

$$\big(W_{\psi_\alpha}\phi\big)(m,n) = \int_0^\infty \phi(t) \, \overline{\psi_{\alpha,m,n}(t)} \, \mathrm{d}\sigma(t), \tag{1.6.8}$$

where

$$\psi_{\alpha,m,n}(t) = |a_0|^{-\frac{m}{2\alpha}}\psi\Big(a_0^{-\frac{m}{\alpha}}\ t - nb_0\Big),\ m,n \in \mathbb{N}. \tag{1.6.9}$$

1.7 Hankel transform

From Haimo [17] and Pathak et al. [32], definitions, properties and formulae of the Hankel transform are given in this section.

Let μ be a positive real number. Take

$$\sigma(x) = \frac{x^{2\mu+1}}{2^{\mu+\frac{1}{2}}\Gamma(\mu+\frac{3}{2})} \tag{1.7.1}$$

and

$$j(x) = C_\mu x^{\frac{1}{2}-\mu}J_{\mu-\frac{1}{2}}(x), \quad C_\mu = 2^{\mu-\frac{1}{2}}\Gamma\left(\mu+\frac{1}{2}\right), \tag{1.7.2}$$

where $J_{\mu-\frac{1}{2}}$ denotes the Bessel function of order $\mu - \frac{1}{2}$.
The space $L^p_\sigma(I),\ I = (0,\infty)$ and $1 \le p \le \infty$, is the space of those real measurable functions ϕ on $(0,\infty)$ for which

$$||\phi||_{p,\sigma} = \left[\int_0^\infty |\phi(x)|^p d\sigma(x)\right]^{\frac{1}{p}} < \infty, \quad 1 \le p < \infty, \tag{1.7.3}$$

$$||\phi||_{\infty,\sigma} = \underset{0<x<\infty}{ess\ sup}\ |\phi(x)| < \infty. \tag{1.7.4}$$

Definition 1.7.1. For each $\phi \in L^1_\sigma(I)$, the Hankel transform of the function ϕ is defined by

$$(h_\mu\phi)(w) \equiv \hat{\phi}(w) := \int_0^\infty j\,(wx)\ \phi(x)\ \mathrm{d}\sigma(x),\ 0 < x < \infty. \tag{1.7.5}$$

If $\phi \in L^1_\sigma(I)$ and $h_\mu\phi \in L^1_\sigma(I)$, then the inverse Hankel transform is defined by

$$\phi(x) := \int_0^\infty j\,(wx)\,(h_\mu\phi)\,(w)\mathrm{d}\sigma(w) \qquad (0 < x < \infty). \tag{1.7.6}$$

If $\phi \in L^1_\sigma(I)$, then $\hat{\phi}$ is continuous and bounded on $[0, \infty)$ and

$$||h_\mu\phi||_{\infty,\sigma} \leq ||\phi||_{1,\sigma}. \tag{1.7.7}$$

If $\phi,\ \psi \in L^1_\sigma(I)$, then the following Parseval formula holds:

$$\int_0^\infty \hat{\phi}(w)\hat{\psi}(w)d\sigma(x) = \int_0^\infty \phi(x)\psi(x)d\sigma(x). \tag{1.7.8}$$

To define the Hankel convolution # we need to introduce Hankel translation. Define

$$D(x, y, z) := \int_0^\infty j(wx)j(wy)j(wz)d\sigma(w). \tag{1.7.9}$$

Using (1.7.5), (1.7.6) and (1.7.9), we get

$$\int_0^\infty j(wx)D(x, y, z)d\sigma(x) = j(wy)j(wz),\ \ 0 < x, y < \infty,\ \ 0 \leq w < \infty. \tag{1.7.10}$$

Setting $w = 0$, we obtain

$$\int_0^\infty D(x, y, z)d\sigma(z) = 1. \tag{1.7.11}$$

The Hankel translation τ_y of $\phi \in L^p_\sigma(I)$, $1 \leq p \leq \infty$ is defined by

$$(\tau_y\phi)(x) = \phi(x, y) := \int_0^\infty \phi(z)D(x, y, z)d\sigma(z), \quad 0 < x, y < \infty. \tag{1.7.12}$$

For each fixed x, we have

$$||\phi(x,.)||_{p,\sigma} \leq ||\phi||_{p,\sigma}. \tag{1.7.13}$$

Let $p, q, r \in [1, \infty)$ and $1/r = (1/p)+(1/q)-1$. The Hankel convolution of $\phi \in L^p_\sigma(I)$ and $\psi \in L^q_\sigma(I)$ is defined by

$$(\phi\#\psi)(x) := \int_0^\infty \phi(x,y)\psi(y)d\sigma(y), \tag{1.7.14}$$

and

$$||\phi\#\psi||_{r,\sigma} \leq ||\phi||_{p,\sigma}||\psi||_{q,\sigma}. \tag{1.7.15}$$

If $\phi, \psi \in L^1_\sigma(I)$, then

$$h_\mu(\phi\#\psi)(w) = (h_\mu\phi)(w)(h_\mu\psi)(w), \quad 0 \leq w < \infty. \tag{1.7.16}$$

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