# CHAPTER 3

# FUNDAMENTAL SOLUTIONS OF MOORE-GIBSON-THOMPSON (MGT) THERMOELASTICITY THEORY

## 3.1 Introduction[1]

The fundamental solution of a differential equation provides the building blocks for solving problems involving impulse responses. Also, fundamental solution plays a significant role in obtaining solutions to several problems on elastodynamic theory. It has been discovered that there are several methods for constructing fundamental solutions in the classical theories of elasticity and thermoelasticity. The Potential (boundary integral equation) method is a powerful tool for constructing the fundamental solution of boundary value problems. In this process, Galerkin-type representation (Galerkin (1930)) plays a vital role. In the previous chapter, Galerkin-type representation of solution under MGT thermoelasticity theory has been derived.

The present Chapter continues the theoretical investigation on the MGT thermoelastic model by establishing the fundamental solution of field equations for this theory. Proceeding with addressing the field equations in the context of MGT model, the fundamental solutions in a short time approximate manner are derived for the coupled thermoelastic equations using Laplace transform. In the case of steady vibrations, the

[1]The content of this chapter is is published in ***Zeitschrift für angewandte Mathematik und Physik***, 74(105), 2023.

fundamental solutions of the field equations are also acquired. The following is a list of related works that are available in the literature. In the classical theory of coupled thermoelasticity, Hetnarski (1964a; 1964b) was the first to study the short-time approximated fundamental solutions. The books by Hormander provide comprehensive information on fundamental solutions of differential equations (see Hormander (1963; 1983)). The work of Knops (1981) describes the methods for constructing fundamental solutions to systems of differential equations involving elasticity and thermoelasticity. A fundamental solution for the generalized thermoelastic LS model has been developed by Wang and Dhaliwal (1993) for arbitrary distributions of body forces and heat sources. Using the Galerkin representation, Ciarletta (1995) developed fundamental solutions for a binary mixture of gases and elastic solids. Later on, Kothari et al. (2010) derived the fundamental solutions for generalized thermoelasticity with three-phase lags based on homogeneous and isotropic bodies. For Kelvin-Voigt materials with voids, Svanadze (2014) presented a fundamental solution to the system of equations of steady vibrations in the context of elementary functions. Using generalized thermoelasticity theory with a single delay term, Kumari and Mukhopadhyay (2017) formulated fundamental solutions in an unbounded medium. An analysis of the fundamental solution of the steady oscillation equations in a porous thermoelastic medium and the dual-phase lag model has been carried out by Biswas and Sarkar (2018).

## 3.2 Governing Equations

An Euclidean three-dimensional space $W$ is considered where a homogeneous isotropic thermoelastic body occupies the surface boundary $\partial W$ at a time $t_0$. Let $\boldsymbol{x} = (x_1, x_2, x_3)$ be a reference point in the Euclidean space $W$. A fixed system of rectangular Cartesian axes $Ox_k$ for $k =(1, 2, 3)$ is considered to refer to the motion of the body. The following are the field equations (as given in Chapter-2) under the Moore-Gibson-Thompson

(MGT) thermoelasticity theory for homogeneous and isotropic material (Quintanilla (2019)):

$$\mu\,(\Delta \boldsymbol{u}) - \beta\,\text{grad}\ \theta + (\lambda+\mu)\,\{\text{grad}\,\text{div}\ \boldsymbol{u}\} + \rho \boldsymbol{f} = \rho \ddot{\boldsymbol{u}}, \tag{3.2.1}$$

$$K\Delta\dot{\theta} + K^{*}\Delta\theta - \beta T_0\,\{\text{div}\ \ddot{\boldsymbol{u}} + \tau_q\ \text{div}\ \dddot{\boldsymbol{u}}\}$$
$$-\rho c_E\left\{\ddot{\theta} + \tau_q\,\dddot{\theta}\right\} = -\left(1+\tau_q\frac{\partial}{\partial t}\right)R. \tag{3.2.2}$$

To get simplified forms of Eqs. (3.2.1) and (3.2.2), the following notations and operators are introduced:

$$n_1 = \left(\frac{\lambda+\mu}{\rho}\right)^{\frac{1}{2}},\quad n_2 = \left(\frac{\mu}{\rho}\right)^{\frac{1}{2}},\quad n_3 = \frac{\beta}{\rho},$$
$$\phi = \left(D^2 + \tau_q D^3\right),\ \ D^i = \frac{\partial^i}{\partial t^i},\ \text{for}\ i=1,2,3,\ \ g_1 = n_2^2\triangle - D^2,\ g_2 = (KD + K^{*})\triangle - \rho c_E\phi,$$
$$\mathcal{H} = \begin{bmatrix} L_1 & -\beta T_0\phi\triangle \\ -n_3 & g_2 \end{bmatrix}_{2\times 2},\ \ L_d = \text{determinant of}\ \mathcal{H},\quad L_1 = n_1^2\triangle\ - D^2,$$
$$C_{i1} = -\left\{\left(n_1^2 - n_2^2\right)L_{i1}^{*} - \beta T_0\phi L_{i2}^{*}\right\},\quad C_{i2} = L_{i2}^{*},$$

where $L_{ij}^{*}\ \ (i,j=1,2)$ denote the co-factors of matrix $\mathcal{H}$.

Using the aforementioned notations, Eqs. (3.2.1) and (3.2.2) are converted into the following forms:

$$\left(n_1^2 - n_2^2\right)\text{grad div}\ \boldsymbol{u} + g_1\boldsymbol{u} - n_3\text{grad}\ \theta = -\boldsymbol{f}, \tag{3.2.3}$$

$$g_2\theta - \beta T_0\phi\,\text{div}\boldsymbol{u} = -\left(1+\tau_q D\right)R. \tag{3.2.4}$$

Then, the following theorem is established:

**Theorem-3.2.1:** Let

$$\boldsymbol{u} = L_d\boldsymbol{F} + C_{11}\mathbf{grad\,div}\boldsymbol{F} + C_{12}\,\mathbf{grad}r, \tag{3.2.5}$$

$$\theta = C_{21}\mathbf{div}\boldsymbol{F} + C_{22}r, \tag{3.2.6}$$

where $F_i$ and $r$ are the fields of class of $\mathbb{C}^7$ and $\mathbb{C}^5$, respectively. Also, $(F_i, r)$ satisfy the following equations in the domain of definition $W \times (0, t_0)$ :

$$L_d g_1 \boldsymbol{F} = -\boldsymbol{f}, \tag{3.2.7}$$

$$L_d r = -\left(1 + \tau_q D\right) R, \tag{3.2.8}$$

then $(\boldsymbol{u}, \theta)$ satisfy Eqs. (3.2.3) and (3.2.4).

**Proof:** This is the Galerkin-type solution of the governing equations of the MGT theory. The complete proof of this theorem is given in Chapter-2.

## 3.3 Fundamental Solution

This section determines the fundamental solutions of the field Eqs. (3.2.3) and (3.2.4) based on Theorem-3.2.1 with the following initial conditions:

$$\boldsymbol{u}\left(x,0\right) = \dot{\boldsymbol{u}}\left(x,0\right) = 0,$$
$$\theta\left(x,0\right) = \dot{\theta}\left(x,0\right) = 0, \quad x \in W, \tag{3.3.1}$$

with

$$\lim_{|x|\to\infty} u_j \to 0, \quad \lim_{|x|\to\infty} u_{j,k} \to 0,$$
$$\lim_{|x|\to\infty} \theta \to 0, \quad \lim_{|x|\to\infty} \theta_{,j} \to 0. \tag{3.3.2}$$

The initial conditions for the functions $F_i$ and $r$ can be assumed as

$$\frac{\partial^j F_i}{\partial t^j}(x,0)=0, \quad \frac{\partial^k r}{\partial t^k}(x,0)=0, \; x\in W \; \text{ for } (j=1,2,..,6;\; k=1,2,3)\,. \tag{3.3.3}$$

Now, it is straight-forward that condition (3.3.3) implies the initial condition (3.3.1) in view of Eqs. (3.2.5) and (3.2.6).

By using above notations and operators, the generalization of Theorem-3.2.1 can be written in the following way:

$$\begin{aligned}\boldsymbol{u} &= \left[\left(n_1^2\,\triangle - D^2\right)\left\{(KD+K^*)\,\triangle - \rho c_E\phi\right\} - \beta T_0 n_3\phi\triangle\right]\boldsymbol{F} + n_3\text{grad } r\\ &\quad - \left[\left(n_1^2-n_2^2\right)\left((KD+K^*)\,\triangle - \rho c_E\phi\right) - \beta T_0 n_3\phi\right]\text{grad div}\boldsymbol{F}, \qquad (3.3.4)\\ \theta &= \left[\beta T_0\phi\left(n_2^2\triangle - D^2\right)\right]\text{div }\mathbf{F} + \left[n_1^2\triangle - D^2\right]r, \qquad (3.3.5)\end{aligned}$$

together with the corresponding fields $F_i$ and $r$ satisfying the following equations:

$$\left[\left\{\left(\triangle - \frac{D^2}{n_1^2}\right)((KD+K^*)\triangle - \rho c_E\phi) - \frac{\beta T_0 n_3\phi\triangle}{n_1^2}\right\}\left(\triangle - \frac{D^2}{n_2^2}\right)\right]F_i = \frac{-f_i}{n_1^2 n_2^2}, \tag{3.3.6}$$

$$\left[\left(\triangle - \frac{D^2}{n_1^2}\right)((KD+K^*)\triangle - \rho c_E\phi) - \frac{\beta T_0 n_3\phi\triangle}{n_1^2}\right]r = \frac{-(1+\tau_q D)\,R}{n_1^2}. \tag{3.3.7}$$

In order to solve the above system of differential equations, the Laplace transform is used which is defined as

$$\overline{F}(x,p) = \mathcal{L}\left[f(x,t)\right] = \int_0^\infty e^{-(pt)} f(x,t)dt,$$

where $p$ is the Laplace transform parameter.

After using Laplace transform, Eqs. (3.3.4) – (3.3.7) yield the following equations:

$$\overline{u} = \left[\left(n_1^2 \triangle - p^2\right)\left\{\left(Kp + K^*\right)\triangle - \rho c_E \overline{\phi}\right\} - \beta T_0 n_3 \overline{\phi} \triangle\right] \overline{F} + n_3 \text{grad } \overline{r}$$
$$- \left[\left(n_1^2 - n_2^2\right)\left(\left(Kp + K^*\right)\triangle - \rho c_E \overline{\phi}\right) - \beta T_0 n_3 \overline{\phi}\right] \text{grad div} \overline{F}, \tag{3.3.8}$$
$$\overline{\theta} = \left[\beta T_0 \overline{\phi} n_2^2 \left(\triangle - \frac{p^2}{n_2^2}\right)\right] \text{div} \overline{F} + \left[n_1^2 \left(\triangle - \frac{p^2}{n_1^2}\right)\right] \overline{r}, \tag{3.3.9}$$

with $\overline{F}_i$ and $\overline{r}$ satisfy the following equations:

$$\left(\triangle - \frac{p^2}{n_2^2}\right)\left[\left(\triangle - \frac{p^2}{n_1^2}\right)\left(\triangle - K_1 G_1(p)\overline{\phi}\right) - K_1 K_2 G_1(p) \overline{\phi} \triangle\right] \overline{F}_i = -\frac{\overline{f}_i G_1(p)}{n_1^2 n_2^2}, \tag{3.3.10}$$
$$\left[\left(\triangle - \frac{p^2}{n_1^2}\right)\left(\triangle - K_1 G_1(p)\overline{\phi}\right) - K_1 K_2 G_1(p) \overline{\phi} \triangle\right] \overline{r} = -\frac{G_1(p) G_2(p)}{n_1^2} \overline{R}, \tag{3.3.11}$$

where

$$K_1 = \rho c_E, \quad K_2 = \frac{\beta T_0 n_3}{\rho c_E n_1^2}, \quad G_1(p) = \frac{1}{(Kp + K^*)}, \quad G_2(p) = \left(1 + \tau_q p\right).$$

Let $\widetilde{a}_1^2$ and $\widetilde{a}_2^2$ be the roots of the following equation:

$$M^2 - \left(\frac{p^2}{n_1^2} + K_1 G_1(p)\overline{\phi} + K_1 K_2 G_1(p) \overline{\phi}\right) M + \frac{K_1}{n_1^2} G_1(p) \overline{\phi} p^2 = 0.$$

Then, Eqs. (3.3.10) and (3.3.11) can be written into the following forms:

$$\left(\triangle - \frac{p^2}{n_2^2}\right)\left(\triangle - \widetilde{a}_1^2\right)\left(\triangle - \widetilde{a}_2^2\right) \overline{F}_i = -\frac{\overline{f}_i G_1(p)}{n_1^2 n_2^2}, \tag{3.3.12}$$
$$\left(\triangle - \widetilde{a}_1^2\right)\left(\triangle - \widetilde{a}_2^2\right) \overline{r} = -\frac{G_1(p) G_2(p)}{n_1^2} \overline{R}, \tag{3.3.13}$$

where

$$\widetilde{a}_1^2 = \sqrt{\frac{m_1 + m_2}{2}}, \quad \widetilde{a}_2^2 = \sqrt{\frac{m_1 - m_2}{2}},$$

$$m_1 = \frac{p^2}{n_1^2} + K_1 G_1(p)\overline{\phi}\left(1 + K_2\right), \quad m_2 = \sqrt{\frac{\left(p^2 + G_1(p)K_1\left(1 + K_2\right) n_1^2\overline{\phi}\right)^2 - 4G_1(p)K_1 n_1^2\overline{\phi}p^2}{n_1^4}}.$$

Now, the fundamental solution is obtained for two separate cases namely: Concentrated body force and Concentrated heat source in the upcoming section.

## Case 1. Concentrated body force

For convenience, time-dependent body force $f_i$ is assumed as $f_i^j$, which acts in $x_i\,(i = 1, 2, 3)$ directions at a fixed point $y$ and the external heat source term $R = 0$ is considered. Therefore, it is found that

$$f_i^j = \delta_{ij}\delta\left(x - y\right) d_1(t), \quad \textbf{and} \quad R = 0,$$

$$F_i = \delta_{ij}\Psi, \quad \textbf{and} \quad r = 0, \tag{3.3.14}$$

where $d_1(t)$ is a function of $t$ and $\delta(.)$ is the Dirac delta function.

Then, Eq. (3.3.12) takes the following form:

$$\left(\triangle - \frac{p^2}{n_2^2}\right)\left(\triangle - \widetilde{a}_1^2\right)\left(\triangle - \widetilde{a}_2^2\right)\overline{\Psi} = Z_1,$$

where

$$Z_1 = -\frac{G_1(p)\delta\left(x - y\right)\overline{d_1}(p)}{n_1^2 n_2^2}.$$

If $\overline{\Psi}_i$ satisfy the equations

$$\left(\triangle - \frac{p^2}{n_2^2}\right)\overline{\Psi}_1 = Z_1, \quad \left(\triangle - \widetilde{a}_1^2\right)\overline{\Psi}_2 = Z_1, \quad \left(\triangle - \widetilde{a}_2^2\right)\overline{\Psi}_3 = Z_1. \tag{3.3.15}$$

then, by virtue of Eq. (3.3.15), $\overline{\Psi}$ can be written as

$$\overline{\Psi} = \sum_{n=1}^{3} \beta_n(p)\overline{\Psi}_n, \tag{3.3.16}$$

where

$$\begin{aligned}
\beta_1(p) &= \frac{1}{\left(\frac{p^2}{n_2^2} - \widetilde{a}_1^2\right)\left(\frac{p^2}{n_2^2} - \widetilde{a}_2^2\right)}, \\
\beta_2(p) &= \frac{1}{\left(\frac{p^2}{n_2^2} - \widetilde{a}_1^2\right)\left(\widetilde{a}_2^2 - \widetilde{a}_1^2\right)}, \\
\beta_3(p) &= \frac{1}{\left(\frac{p^2}{n_2^2} - \widetilde{a}_2^2\right)\left(\widetilde{a}_1^2 - \widetilde{a}_2^2\right)}.
\end{aligned}$$

Using the condition at infinity (3.3.3), the solutions of Eq. (3.3.15) can be expressed as

$$\overline{\Psi}_1 = \frac{G_1(p)\overline{d_1}(p)}{4\pi n_1^2 n_2^2 d} e^{\left(-\frac{p}{n_2}d\right)}, \tag{3.3.17}$$

$$\overline{\Psi}_2 = \frac{G_1(p)\overline{d_1}(p)}{4\pi n_1^2 n_2^2 d} e^{(-\widetilde{a}_1 d)}, \tag{3.3.18}$$

$$\overline{\Psi}_3 = \frac{G_1(p)\overline{d_1}(p)}{4\pi n_1^2 n_2^2 d} e^{(-\widetilde{a}_2 d)}, \tag{3.3.19}$$

where $d = |x - y|$.

When $x \neq y$, we have

$$\left[\left(n_1^2 - n_2^2\right)\left((Kp + K^*)\triangle - \rho c_E \overline{\phi}\right) - \beta T_0 n_3 \overline{\phi}\right] = \gamma_1(p)\overline{\Psi_1} + \gamma_2(p)\overline{\Psi_2} + \gamma_3(p)\overline{\Psi_3}, \tag{3.3.20}$$

$$\left(\triangle - \frac{p^2}{n_2^2}\right)\overline{\Psi} = \frac{1}{\left(\widetilde{a}_1^2 - \widetilde{a}_2^2\right)}\left(\overline{\Psi}_2 - \overline{\Psi}_3\right), \quad \left(\triangle - \widetilde{a}_1^2\right)\left(\triangle - \widetilde{a}_2^2\right)\overline{\Psi} = \overline{\Psi}_1, \tag{3.3.21}$$

where

$$\gamma_1(p) = \frac{\beta_1(p)}{G_1(p)} \left[ \left(n_1^2 - n_2^2\right) \left( \frac{p^2}{n_2^2} - G_1(p) K_1 \overline{\phi} \right) - \beta T_0 n_3 G_1(p) \overline{\phi} \right],$$
$$\gamma_2(p) = \frac{\beta_2(p)}{G_1(p)} \left[ \left(n_1^2 - n_2^2\right) \left( \widetilde{a}_1^2 - G_1(p) K_1 \overline{\phi} \right) - \beta T_0 n_3 G_1(p) \overline{\phi} \right],$$
$$\gamma_3(p) = \frac{\beta_3(p)}{G_1(p)} \left[ \left(n_1^2 - n_2^2\right) \left( \widetilde{a}_2^2 - G_1(p) K_1 \overline{\phi} \right) - \beta T_0 n_3 G_1(p) \overline{\phi} \right].$$

In view of Eqs. (3.3.8), (3.3.9), (3.3.20), and (3.3.21), the fundamental solutions for the displacement components and temperature in the Laplace transform domain are obtained as

$$\overline{u}_i^{(j)} = \frac{n_1^2}{G_1(s)} \overline{\Psi}_1 \delta_{ij} - \left( \gamma_1(p) \overline{\Psi_1} + \gamma_2(p) \overline{\Psi_2} + \gamma_3(p) \overline{\Psi_3} \right)_{,ij}, \tag{3.3.22}$$
$$\overline{\theta}^{(j)} = \frac{\beta T_0 n_2^2 \overline{\phi}}{\left(\widetilde{a}_1^2 - \widetilde{a}_2^2\right)} \left( \overline{\Psi}_2 - \overline{\Psi}_3 \right)_{,j}. \tag{3.3.23}$$

## Case 2. Concentrated heat source

For this case $f_i = 0$ and $R = \delta(x-y) d_2(t)$ are considered, where $d_2(t)$ is any time dependent function. Here, $F_i = 0$ and $\overline{r}$ satisfies the following equation:

$$\left( \triangle - \widetilde{a}_1^2 \right) \left( \triangle - \widetilde{a}_2^2 \right) \overline{r} = -\frac{G_1(p) G_2(p)}{n_1^2} \delta(x-y) \overline{d_2}(p). \tag{3.3.24}$$

Now, let $\Lambda_1$ and $\Lambda_2$ satisfy the equations

$$\left( \triangle - \widetilde{a}_1^2 \right) \overline{\Lambda_1} = Z_2, \quad \left( \triangle - \widetilde{a}_2^2 \right) \overline{\Lambda_2} = Z_2, \tag{3.3.25}$$

where

$$Z_2 = -\frac{G_1(p) G_2(p)}{n_1^2} \delta(x-y) \overline{d_2}(p).$$

Then, $\overline{r}$ is obtained as

$$\bar{r} = \frac{1}{(\widetilde{a}_1^2 - \widetilde{a}_2^2)} \left( \overline{\Lambda_1} - \overline{\Lambda_2} \right), \tag{3.3.26}$$

where $\overline{\Lambda_i}$, $(i = 1, 2)$ can be obtained as

$$\overline{\Lambda_i} = \frac{G_1(p) G_2(p) \overline{d_2}(p)}{4\pi n_1^2 d} e^{(-\widetilde{a}_i d)}, \quad i = 1, 2. \tag{3.3.27}$$

From Eqs. (3.3.8), (3.3.9) and (3.3.27), the fundamental solutions for the displacement component $\overline{u}_i^{(4)}$ and temperature component $\overline{\theta}^{(4)}$ with the presence of concentrated heat source is obtained in the Laplace transform domain as follows:

$$\overline{u}_i^{(4)} = \frac{n_3}{(\widetilde{a}_1^2 - \widetilde{a}_2^2)} \left( \overline{\Lambda_1} - \overline{\Lambda_2} \right)_{,i}, \tag{3.3.28}$$

$$\overline{\theta}^{(4)} = \frac{1}{(\widetilde{a}_1^2 - \widetilde{a}_2^2)} \left[ \left( n_1^2 \widetilde{a}_1^2 - p^2 \right) \overline{\Lambda_1} - \left( n_2^2 \widetilde{a}_2^2 - p^2 \right) \overline{\Lambda_2} \right]. \tag{3.3.29}$$

## 3.4 Short-time Approximated Fundamental Solutions

Since inversion of Laplace transforms involved in the solutions of different field variables as derived in previous sections is highly complicated, derivation of closed form analytical fundamental solution is a formidable task. Hence, the closed form fundamental solutions in the physical domain is obtained for small time. Therefore, assuming the Laplace transform parameter $p$ to be very large and neglecting the higher powers of $\frac{1}{p}$, the roots $\widetilde{a}_1$and $\widetilde{a}_1$ are acquired in the following forms:

$$\widetilde{a}_1 \approx \vartheta_{11} p + \vartheta_{12} + \frac{\vartheta_{13}}{p},$$

$$\widetilde{a}_2 \approx \vartheta_{21} p + \vartheta_{22} + \frac{\vartheta_{23}}{p},$$

where

$$\vartheta_{11} = \sqrt{\frac{A_1}{2}}, \quad \vartheta_{12} = \frac{1}{2\sqrt{2}}\frac{B_1}{\sqrt{A_1}}, \quad \vartheta_{13} = \frac{1}{2\sqrt{2}}\frac{C_1}{\sqrt{A_1}} - \frac{1}{8\sqrt{2}}\frac{B_1^2}{A_1^{3/2}},$$

$$\vartheta_{21} = \sqrt{\frac{A_2}{2}}, \quad \vartheta_{22} = \frac{1}{2\sqrt{2}}\frac{B_2}{\sqrt{A_2}}, \quad \vartheta_{23} = \frac{1}{2\sqrt{2}}\frac{C_2}{\sqrt{A_2}} - \frac{1}{8\sqrt{2}}\frac{B_2^2}{A_2^{3/2}},$$

$$A_1 = b_1 + \sqrt{b_1^2 - b_5}, \quad B_1 = b_2 + \frac{b_1 b_2}{\sqrt{b_1^2 - b_5}} - \frac{b_6}{2\sqrt{b_1^2 - b_5}},$$

$$C_1 = \frac{8b_1^2 b_3\sqrt{b_1^2 - b_5} - 4b_2^2 b_5 - 8b_3 b_5\sqrt{b_1^2 - b_5} + 4b_1 b_2 b_6 - b_6^2 + 4\left(b_1^2 - b_5\right)\left(2b_1 b_3 - b_7\right)}{8\left(b_1^2 - b_5\right)^{3/2}},$$

$$A_2 = b_1 - \sqrt{b_1^2 - b_5}, \quad B_2 = b_2 - \frac{b_1 b_2}{\sqrt{b_1^2 - b_5}} + \frac{b_6}{2\sqrt{b_1^2 - b_5}},$$

$$C_2 = \frac{8b_1^2 b_3\sqrt{b_1^2 - b_5} + 4b_2^2 b_5 - 8b_3 b_5\sqrt{b_1^2 - b_5} - 4b_1 b_2 b_6 + b_6^2 - 4\left(b_1^2 - b_5\right)\left(2b_1 b_3 - b_7\right)}{8\left(b_1^2 - b_5\right)^{3/2}},$$

$$b_1 = \left(\frac{1}{n_1^2} + \frac{\rho c_E}{K}\tau_q + \frac{\beta T_0 n_3}{K n_1^2}\tau_q\right), \quad b_2 = \left(1 - \tau_q\frac{K^*}{K}\right)\left(\frac{\rho c_E}{K} + \frac{\beta T_0 n_3}{K n_1^2}\right),$$

$$b_3 = \left\{\tau_q\left(\frac{K^*}{K}\right)^2 - \frac{K^*}{K}\right\}\left(\frac{\rho c_E}{K} + \frac{\beta T_0 n_3}{K n_1^2}\right), \quad b_4 = \left\{\left(\frac{K^*}{K}\right)^2 - \tau_q\left(\frac{K^*}{K}\right)^3\right\}\left(\frac{\rho c_E}{K} + \frac{\beta T_0 n_3}{K n_1^2}\right),$$

$$b_5 = \frac{4\rho c_E \tau_q}{K n_1^2}, \quad b_6 = \frac{4\rho c_E}{K n_1^2}\left(1 - \tau_q\frac{K^*}{K}\right), \quad b_7 = -\frac{4\rho c_E K^*}{K^2 n_1^2}\left(1 - \tau_q\frac{K^*}{K}\right).$$

By substituting the approximated roots into the Eqs. (3.3.22), (3.3.23), (3.3.28) and (3.3.29), the following fundamental solutions for displacement components and temperature field in two cases are obtained:

**Case 1:**

$$\begin{aligned}\overline{u}_i^{(j)} =& \frac{d_1(p)}{4\pi n_2^2 d} e^{-\frac{p}{n_2}d}\delta_{ij} - \frac{\partial^2}{\partial x_i \partial x_j}\left[\frac{d_1(p)}{4\pi d K n_1^2}\left\{\left(\frac{Y_{11}}{p^2} + \frac{Y_{12}}{p^3}\right)e^{-\frac{p}{n_2}d}\right.\right. \\ &\left.\left. + \left(\frac{Y_{13}}{p^2} + \frac{Y_{14}}{p^3}\right)e^{-\left(\vartheta_{11}p + \vartheta_{12} + \frac{\vartheta_{13}}{p}\right)d} + \left(\frac{Y_{15}}{p^2} + \frac{Y_{16}}{p^3}\right)e^{-\left(\vartheta_{21}p + \vartheta_{22} + \frac{\vartheta_{23}}{p}\right)d}\right\}\right], \quad (3.4.1)\end{aligned}$$

$$\overline{\theta}^{(j)} = \frac{d_1(p)}{4\pi d K n_1^2}\frac{\partial}{\partial j}\left[\left(Y_{17} + \frac{Y_{18}}{p} + \frac{Y_{19}}{p^2}\right)\left\{e^{-\left(\vartheta_{11}p + \vartheta_{12} + \frac{\vartheta_{13}}{p}\right)d} - e^{-\left(\vartheta_{21}p + \vartheta_{22} + \frac{\vartheta_{23}}{p}\right)d}\right\}\right]. \quad (3.4.2)$$

**Case 2:**

$$\overline{u}_i^{(4)} = \frac{d_2(p)}{4\pi n_1^2 d}\frac{\partial}{\partial x_i}\left[\left(\frac{Z_{11}}{p^2}+\frac{Z_{12}}{p^3}\right)\left\{e^{-\left(\vartheta_{11}p+\vartheta_{12}+\frac{\vartheta_{13}}{p}\right)d}-e^{-\left(\vartheta_{21}p+\vartheta_{22}+\frac{\vartheta_{23}}{p}\right)d}\right\}\right], \tag{3.4.3}$$

$$\overline{\theta}^{(4)} = \frac{d_2(p)}{4\pi n_1^2 d}\left[\left(Z_{13}+\frac{Z_{14}}{p}+\frac{Z_{15}}{p^2}\right)e^{-\left(\vartheta_{11}p+\vartheta_{12}+\frac{\vartheta_{13}}{p}\right)d}-\left(Z_{16}+\frac{Z_{17}}{p}+\frac{Z_{18}}{p^2}\right)e^{-\left(\vartheta_{21}p+\vartheta_{22}+\frac{\vartheta_{23}}{p}\right)d}\right]. \tag{3.4.4}$$

## 3.5 Laplace Inversion

The inversion of Laplace transforms are carried out by using the following formulas:

$$\mathscr{L}^{-1}[\delta(t)] = 1, \quad \mathscr{L}^{-1}[e^{-\gamma p d}] = \delta\left(t-\gamma d\right),$$

$$\mathscr{L}^{-1}\left[\frac{e^{-\gamma p d}}{p^i}\right] = X_i\left(t-\gamma d\right) = \begin{cases} 0 & t \le \gamma d \\ \frac{(t-\gamma d)^{i-1}}{(i-1)!} & t > \gamma d \end{cases},$$

$$\mathscr{L}^{-1}\left[\frac{e^{-\frac{\gamma}{p}}}{p^{v+1}}\right] = \left(\frac{t}{\gamma}\right)^{v/2} J_v\left(2\sqrt{\gamma t}\right), \quad \mathrm{Re}(v)>-1, \gamma>0,$$

$$\mathscr{L}^{-1}\left[e^{-\frac{\gamma}{p}}\right] = \delta(t) - \sqrt{\frac{\gamma}{t}}J_1\left(2\sqrt{\gamma t}\right), \quad \gamma>0,$$

where $J_v()$ is the Bessel function.

Now, the time-dependent functions $d_i(t)$, $i=1,2$ are assumed as $d_1(t)=d_2(t)=\delta(t)$. After taking Laplace inverse of the Eqs. (3.4.1)-(3.4.4), the fundamental solutions in the physical domain are obtained for both cases as follows:

**Case 1:**

$$
\begin{aligned}
u_i^{(j)} =& \frac{1}{4\pi n_2^2 d}\left[\delta\left(t-\frac{d}{n_2}\right)\right]\delta_{ij} - \frac{\partial^2}{\partial x_i \partial x_s}\left[\frac{1}{4\pi d}\left\{Y_{11}X_2\left(t-\frac{d}{n_2}\right)+Y_{12}X_3\left(t-\frac{d}{n_2}\right)\right.\right. \\
&+ Y_{13}\left\{e^{-(\vartheta_{12}d)}\sqrt{\frac{t-\vartheta_{11}d}{\vartheta_{13}d}}J_1\left(2\sqrt{\vartheta_{13}\left(t-\vartheta_{11}d\right)d}\right)\right\} \\
&+ Y_{14}\left\{e^{-(\vartheta_{12}d)}\left(\frac{t-\vartheta_{11}d}{\vartheta_{13}d}\right)J_2\left(2\sqrt{\vartheta_{13}\left(t-\vartheta_{11}d\right)d}\right)\right\} \\
&+ Y_{15}\left\{e^{-(\vartheta_{22}d)}\sqrt{\frac{t-\vartheta_{21}d}{\vartheta_{23}d}}J_1\left(2\sqrt{\vartheta_{23}\left(t-\vartheta_{21}d\right)d}\right)\right\} \\
&\left.\left.+ \ Y_{16}\left\{e^{-(\vartheta_{22}d)}\left(\frac{t-\vartheta_{21}d}{\vartheta_{23}d}\right)J_2\left(2\sqrt{\vartheta_{23}\left(t-\vartheta_{21}d\right)d}\right)\right\}\right\}\right],
\end{aligned}
\tag{3.5.1}
$$

$$
\begin{aligned}
\theta^{(j)} =& \frac{1}{4\pi n_1^2 d}\frac{\partial}{\partial j}\left[Y_{17}\left\{e^{-(\vartheta_{12}d)}\left(\delta\left(t-\vartheta_{11}d\right)-\sqrt{\frac{\vartheta_{13}d}{t-\vartheta_{11}d}}J_1\left(2\sqrt{\vartheta_{13}\left(t-\vartheta_{11}d\right)d}\right)\right)\right.\right. \\
&\left.- \ e^{-(\vartheta_{22}d)}\left(\delta\left(t-\vartheta_{21}d\right)-\sqrt{\frac{\vartheta_{23}d}{t-\vartheta_{21}d}}J_1\left(2\sqrt{\vartheta_{23}\left(t-\vartheta_{21}d\right)d}\right)\right)\right\} \\
&+ Y_{18}\left\{e^{-(\vartheta_{12}d)}J_0\left(2\sqrt{\vartheta_{13}\left(t-\vartheta_{11}d\right)d}\right)-e^{-(\vartheta_{22}d)}J_0\left(2\sqrt{\vartheta_{23}\left(t-\vartheta_{21}d\right)d}\right)\right\} \\
&+ Y_{19}\left\{e^{-(\vartheta_{12}d)}\sqrt{\frac{t-\vartheta_{11}d}{\vartheta_{13}d}}J_1\left(2\sqrt{\vartheta_{13}\left(t-\vartheta_{11}d\right)d}\right)\right. \\
&\left.\left.- \ e^{-(\vartheta_{22}d)}\sqrt{\frac{t-\vartheta_{21}d}{\vartheta_{23}d}}J_1\left(2\sqrt{\vartheta_{23}\left(t-\vartheta_{21}d\right)d}\right)\right\}\right].
\end{aligned}
\tag{3.5.2}
$$

**Case 2:**

$$
\begin{aligned}
u_i^{(4)} =& \frac{1}{4\pi n_1^2 d}\frac{\partial}{\partial x_i}\left[Z_{11}\left\{e^{-(\vartheta_{12}d)}\sqrt{\frac{t-\vartheta_{11}d}{\vartheta_{13}d}}J_1\left(2\sqrt{\vartheta_{13}\left(t-\vartheta_{11}d\right)d}\right)\right.\right. \\
&\left.- \ e^{-(\vartheta_{22}d)}\sqrt{\frac{t-\vartheta_{21}d}{\vartheta_{23}d}}J_1\left(2\sqrt{\vartheta_{23}\left(t-\vartheta_{21}d\right)d}\right)\right\} \\
&+ Z_{12}\left\{e^{-(\vartheta_{12}d)}\left(\frac{t-\vartheta_{11}d}{\vartheta_{13}d}\right)J_2\left(2\sqrt{\vartheta_{13}\left(t-\vartheta_{11}d\right)d}\right)\right. \\
&\left.\left.- \ e^{-(\vartheta_{22}d)}\left(\frac{t-\vartheta_{21}d}{\vartheta_{23}d}\right)J_2\left(2\sqrt{\vartheta_{23}\left(t-\vartheta_{21}d\right)d}\right)\right\}\right],
\end{aligned}
\tag{3.5.3}
$$

$$
\begin{aligned}
\theta^{(4)} =& \frac{1}{4\pi n_1^2 d}\Bigg[Z_{13}e^{-(\vartheta_{12}d)}\left\{\delta\left(t-\vartheta_{11}d\right)-\sqrt{\frac{\vartheta_{13}d}{t-\vartheta_{11}d}}J_1\left(2\sqrt{\vartheta_{13}\left(t-\vartheta_{11}d\right)d}\right)\right\} \\
&+Z_{14}\left\{e^{-(\vartheta_{12}d)}J_0\left(2\sqrt{\vartheta_{13}\left(t-\vartheta_{11}d\right)d}\right)\right\} \\
&+Z_{15}\left\{e^{-(\vartheta_{12}d)}\sqrt{\frac{t-\vartheta_{11}d}{\vartheta_{13}d}}J_1\left(2\sqrt{\vartheta_{13}\left(t-\vartheta_{11}d\right)d}\right)\right\} \\
&-Z_{16}e^{-(\vartheta_{22}d)}\left\{\delta\left(t-\vartheta_{21}d\right)-\sqrt{\frac{\vartheta_{23}d}{t-\vartheta_{21}d}}J_1\left(2\sqrt{\vartheta_{23}\left(t-\vartheta_{21}d\right)d}\right)\right\} \\
&-Z_{17}\left\{e^{-(\vartheta_{22}d)}J_0\left(2\sqrt{\vartheta_{23}\left(t-\vartheta_{21}d\right)d}\right)\right\}-Z_{18}\left\{e^{-(\vartheta_{22}d)}\sqrt{\frac{t-\vartheta_{21}d}{\vartheta_{23}d}}J_1\left(2\sqrt{\vartheta_{23}\left(t-\vartheta_{21}d\right)d}\right)\right\}\Bigg].
\end{aligned}
\tag{3.5.4}
$$

## 3.6 Representation of Fundamental Solutions for Steady Oscillation

In this section, Eqs. (3.2.5) and (3.2.6) are used to determine the fundamental solutions of the field equations in the case of steady vibration. The complex-valued functions can be assumed in the periodic form as

$$
\boldsymbol{f} = \mathrm{Re}\big[\boldsymbol{f}^* e^{(-i\omega t)}\big], \quad R = \mathrm{Re}\big[R^* e^{(-i\omega t)}\big], \tag{3.6.1}
$$

$$
\boldsymbol{F} = \mathrm{Re}\big[\boldsymbol{F}^* e^{(-i\omega t)}\big], \quad r = \mathrm{Re}\big[r^* e^{(-i\omega t)}\big], \tag{3.6.2}
$$

along with displacement vector $\boldsymbol{u}$ and the temperature $\theta$ given as

$$
\boldsymbol{u} = \mathrm{Re}\big[\boldsymbol{u}^* e^{(-i\omega t)}\big], \quad \theta = \mathrm{Re}\big[\theta^* e^{(-i\omega t)}\big], \tag{3.6.3}
$$

where $i = \sqrt{-1}$ and $\omega$ represents the oscillation frequency.

In view of Eq. (3.6.3), the field Eqs. (3.2.5) and (3.2.6) reduce to the following forms:

$$\boldsymbol{u}^* = \left[\left(n_1^2\triangle + \omega^2\right)\left\{\left(K(-i\omega) + K^*\right)\triangle - \rho c_E\left(-\omega^2 + i\tau_q\omega^3\right)\right\} - \beta T_0 n_3 \triangle\left(-\omega^2 + i\tau_q\omega^3\right)\right]\boldsymbol{F}^*$$
$$- \left[\left(n_1^2 - n_2^2\right)\left\{\left(K(-i\omega) + K^*\right)\triangle - \rho c_E\left(-\omega^2 + i\tau_q\omega^3\right)\right\}\right.$$
$$\left. - \beta T_0 n_3\left(-\omega^2 + i\tau_q\omega^3\right)\right] \text{grad div}\boldsymbol{F}^* + n_3 \text{grad } r^*, \tag{3.6.4}$$
$$\theta^* = \left[\beta T_0 n_2^2\left(-\omega^2 + i\tau_q\omega^3\right)\left(\triangle + \frac{\omega^2}{n_2^2}\right)\right]\text{div}\boldsymbol{F}^* + \left[n_1^2\left(\triangle + \frac{\omega^2}{n_1^2}\right)\right]r^*. \tag{3.6.5}$$

In addition, the complex valued functions $(\boldsymbol{F}^*, r^*)$ satisfy the following system of equations:

$$\left(\triangle + \sigma_1^2\right)\left[\left(\triangle + \sigma_2^2\right)\left(\triangle + \sigma_3^2\right) - \frac{\beta T_0 n_3\left(-\omega^2 + i\tau_q\omega^3\right)}{n_1^2\left(K(-i\omega) + K^*\right)}\triangle\right]\boldsymbol{F}^* = -\frac{\boldsymbol{f}^*}{n_1^2 n_2^2\left(K(-i\omega) + K^*\right)}, \tag{3.6.6}$$
$$\left[\left(\triangle + \sigma_2^2\right)\left(\triangle + \sigma_3^2\right) - \frac{\beta T_0 n_3\left(-\omega^2 + i\tau_q\omega^3\right)}{n_1^2\left(K(-i\omega) + K^*\right)}\triangle\right]r^* = -\frac{R^*\left(1 - i\tau_q\omega\right)}{n_1^2\left(K(-i\omega) + K^*\right)}, \tag{3.6.7}$$

where

$$\sigma_1 = \frac{\omega}{n_2}, \quad \sigma_2 = \frac{\omega}{n_1}, \quad \sigma_3 = \sqrt{\frac{\rho c_E\left(\omega^2 - i\tau_q\omega^3\right)}{\left(K(-i\omega) + K^*\right)}}.$$

Here, the operator $N(\triangle)$ is defined by

$$N(\triangle) = \triangle^2 + \left\{\left(\sigma_2^2 + \sigma_3^2\right) - \frac{\beta T_0 n_3\left(-\omega^2 + i\tau_q\omega^3\right)}{n_1^2\left(K(-i\omega) + K^*\right)}\right\}\triangle + \sigma_2^2\sigma_3^2. \tag{3.6.8}$$

If $\xi_1^2$ and $\xi_2^2$ are the roots of $N(-\lambda) = 0$, then above operator can be rewritten as

$$N(\triangle) = \left(\triangle + \xi_1^2\right)\left(\triangle + \xi_2^2\right). \tag{3.6.9}$$

Therefore, Eqs. (3.6.6) and (3.6.7) can be expressed in the following forms:

$$\left(\triangle+\sigma_1^2\right)\left(\triangle+\xi_1^2\right)\left(\triangle+\xi_2^2\right)\boldsymbol{F}^*=-\frac{\boldsymbol{f}^*Q_1^*}{n_1^2n_2^2}, \tag{3.6.10}$$

$$\left(\triangle+\xi_1^2\right)\left(\triangle+\xi_2^2\right)r^*=-\frac{R^*Q_1^*Q_2^*}{n_1^2}, \tag{3.6.11}$$

where $Q_1^*=(K(-i\omega)+K^*)^{-1}$ and $Q_2^*=(1-i\tau_q\omega)$.

Now consider following two cases:

## Case 1:

Here $f_i^*=\delta(x-y)\delta_{ij}$, $R^*=0$, $F_i^*=\Phi\delta_{ij}$, and $r^*=0$ are considered. Therefore, the Eq. (3.6.10) takes the following form:

$$\left(\triangle+\lambda_1^2\right)\left(\triangle+\lambda_2^2\right)\left(\triangle+\lambda_3^2\right)\Phi=-\mathfrak{R}, \tag{3.6.12}$$

where $\mathfrak{R}=\frac{\delta(x-y)Q_1^*}{n_1^2n_2^2}$, $\lambda_1=\sigma_1^2$, $\lambda_2=\xi_1^2$, and $\lambda_3=\xi_2^2$.

If the functions $\Phi_i$ $(i=1,2,3)$ satisfy the following equations:

$$\left(\triangle+\lambda_1^2\right)\Phi_1=-\mathfrak{R},\quad\left(\triangle+\lambda_2^2\right)\Phi_2=-\mathfrak{R},\quad\left(\triangle+\lambda_3^2\right)\Phi_3=-\mathfrak{R}. \tag{3.6.13}$$

then $\Phi$ can be expressed in the form as

$$\Phi=\sum_{n=1}^{3}D_n\Phi_n,$$

where

$$D_n^{-1}=\prod_{j=1,(j\neq n)}^{3}\left(\lambda_j^2-\lambda_n^2\right),\quad(n=1,2,3).$$

Hence, the following is found:

$$\Phi = \frac{Q_1^*}{4\pi d n_1^2 n_2^2} \sum_{n=1}^{3} D_n e^{(i\lambda_n d)}. \tag{3.6.14}$$

In view of Eqs. (3.6.14), (3.6.4) and (3.6.5), one can find

$$\begin{aligned} u_j^{*(n)} =& \delta_{jn} n_1^2 \left(K(-i\omega) + K^*\right) \left[ \left(\triangle + \xi_1^2\right) \left(\triangle + \xi_2^2\right) \right] \Phi \\ & - \left[ \left(n_1^2 - n_2^2\right) \left\{ \left(K(-i\omega) + K^*\right) \triangle - \rho c_E \left(-\omega^2 + i\tau_q \omega^3\right) \right\} \right. \\ & \left. - \beta T_0 n_3 \left(-\omega^2 + i\tau_q \omega^3\right) \right] \Phi_{,jn}, \qquad (3.6.15) \\ \theta^{*(n)} =& \beta T_0 n_2^2 \left(-\omega^2 + i\tau_q \omega^3\right) \left(\triangle + \sigma_1^2\right) \Phi_{,n}. \qquad (3.6.16) \end{aligned}$$

## Case 2:

Now, let $f_i^* = 0$, $R^* = \delta(x - y)$ and $F_i^* = 0$, $r^* = \Omega$ are assumed, where $\Omega$ satisfies the following equation:

$$\left(\triangle + \lambda_2^2\right) \left(\triangle + \lambda_3^2\right) \Omega = -\frac{Q_1^* Q_2^*}{n_1^2} \delta(x - y), \tag{3.6.17}$$

then, $\Omega$ can be obtained as follows:

$$\Omega = \frac{Q_1^* Q_2^*}{4\pi d n_1^2 \left(\lambda_3^2 - \lambda_2^2\right)} \left[ e^{(i\lambda_2 d)} - e^{(i\lambda_3 d)} \right]. \tag{3.6.18}$$

In view of Eqs. (3.6.4), (3.6.5) and (3.6.18), the solutions for displacement $u_j^{*(4)}$ and temperature distribution $\theta^{*(4)}$ are obtained in the following forms:

$$u_j^{*(4)} = n_3 \Omega_{,j}\,, \tag{3.6.19}$$

$$\theta^{*(4)} = n_1^2 \left(\triangle + \sigma_2^2\right) \Omega. \tag{3.6.20}$$

This completes the solution of steady oscillations.

## 3.7 Conclusion

The present Chapter examines a non-classical thermoelastic model in accordance with the MGT heat conduction law. This includes fundamental solutions in the context of MGT theory for the homogeneous and isotropic body. Two cases are considered; namely concentrated body force and concentrated heat source for investigating the fundamental solutions for thermoelasticity in the context of MGT model. The problem is formulated to obtain solutions for the distributions of displacement components and temperature in the Laplace transform domain. Then fundamental solutions are established in the physical domain for the short-time approximated displacement components and temperature fields by applying the Laplace inversion technique. Lastly, the system of equations as a fundamental solutions in the case of steady oscillations are obtained in the present context.