# CHAPTER 2

# ON MOORE-GIBSON-THOMPSON THERMOELASTICITY THEORY: DERIVATION OF GALERKIN-TYPE SOLUTION

## 2.1 Introduction[1]

The present chapter of the thesis aims at analyzing the thermoelasticity theory based on the Moore-Gibson-Thompson heat conduction model as proposed by Quintanilla (2019). The MGT heat conduction law is developed by introducing thermal relaxation parameter ($\tau_q$) in the Green-Naghdi-III heat conduction relation (see section 1.3.2.5). Subsequently some researchers have been interested in the MGT thermoelastic model to study the wellposedness and stability of the solution by considering various approximations. Pellicer and Quintanilla (2020) studied the uniqueness of solution and exponential stability of MGT thermoelasticity theory. Bazzara et al. (2021) have discussed the exponential decay of the radially symmetric solutions. In this context, Jangid and Mukhopadhyay (2021) have obtained domain of influence results in the context of this new thermoelasticity theory.

In order to broaden the theoretical analysis of this theory, the present chapter derives the Galerkin-type solution for the MGT thermoelasticity theory. In the studies of elasticity and thermoelasticity theories, the contemporary treatment for several boundary

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value problems normally involves the construction of the Galerkin type representation (Galerkin (1930)) of field equations by means of various elementary functions like harmonic, metaharmonic and biharmonic, etc. These are the foundations to obtain the fundamental solution of the theory. These metaharmonic functions are also known as the solution of Helmholtz's equation. Some dynamical problems on the basis of the classical theory of elasticity can be found in the work reported by Nowacki (1964; 1969), Sandru (1966) and the references therein. In the context of the isothermal theory, the Boussinesq-Papkovitch-Neuber (BPN) (1973), and Green-Lame (GL) (1975) types solutions for materials with voids, was elaborated by Chandrasekharaiah (1987; 1989). Ciarletta (1995) derived fundamental solutions and general solutions for the dynamical theory of binary mixture of an elastic solid. Ciarletta (1999) obtained a Galerkin type representation of the solution for the linear theory of micropolar thermoelasticity (see Eringen (1970; 1999), Boschi and Ieşan (1973), Nowacki (1986)) by considering the GN-II theory. Scalia and Svanadze (2006) presented the Galerkin-type representation of thermoelasticity theory with micro-temperatures. The Galerkin-type solution for the equations for the three-phase-lag thermoelasticity theory was established by Mukhopadhyay et al. (2010). For Kelvin-Voigt material with void, Svanadze (2014) established the representation of solution in case of the linear theory of thermo-viscoelasticity. Recently, Gupta and Mukhopadhyay (2019) derived the Galerkin-type representation for the modified Green-Lindsay (MGL) thermoelasticity theory (Yu et al. (2018)).

The work in the chapter is arranged in the following way. In Section 2.2, the field equations for isotropic elastic material under the Moore-Gibson-Thompson thermoelasticity theory are presented. The Galerkin-type solution of basic governing equations in terms of the elementary functions is derived in Section 2.3. In Section 2.4, a theorem representing the Galerkin-type solution of equations for the steady oscillations is obtained. Finally, in Section 2.5, the general solution of the system of equations in the case of steady oscillations is developed in terms of elementary functions.

## 2.2 Governing Equations

Let $\boldsymbol{x} = (x_1, x_2, x_3)$ represents an arbitrary point in three-dimensional Euclidean space. With the time variable $t$, an isotropic elastic material that occupies the region $W$ is considered. Following Quintanilla (2019), the field equations for the Moore-Gibson-Thompson (MGT) thermoelasticity theory in presence of body force $f$ and heat source $R$ are taken as follows:

$$\mu\left(\Delta \boldsymbol{u}\right) - \beta \,\text{grad}\ \theta + (\lambda+\mu)\left\{\text{grad}\,\text{div}\ \boldsymbol{u}\right\} + \rho \boldsymbol{f} = \rho \ddot{\boldsymbol{u}}, \tag{2.2.1}$$

$$\begin{aligned} &K\Delta\dot{\theta} + K^{*}\Delta\theta - \beta T_0\left\{\text{div}\ \ddot{\boldsymbol{u}} + \tau_q\ \text{div}\ \dddot{\boldsymbol{u}}\right\} \\ &-\rho c_E\left\{\ddot{\theta} + \tau_q\,\dddot{\theta}\right\} = -\left(1+\tau_q\frac{\partial}{\partial t}\right)R. \end{aligned} \tag{2.2.2}$$

Further, the following notations are introduced:

$$\begin{aligned} n_1 &= \left(\frac{\lambda+\mu}{\rho}\right), \quad n_2 = \frac{\mu}{\rho}, \quad n_3 = \frac{\beta}{\rho}, \\ g_1(\Delta, D) &= n_2\Delta - D^2, \quad g_2(\Delta, D) = (K\,D + K^{*})\,\Delta - \rho\,c_E\Box_1, \\ \Box_1(D) &= \left(D^2 + \tau_q\,D^3\right), \quad D^k = \frac{\partial^k}{\partial t^k}\ \text{for}\ k = 1, 2, 3. \end{aligned}$$

Therefore, Eqs. (2.2.1) and (2.2.2) take the following forms:

$$n_1\ \text{grad div}\ \boldsymbol{u} + g_1\boldsymbol{u} - n_3\ \text{grad}\ \theta = -\boldsymbol{f}, \tag{2.2.3}$$

$$g_2\,\theta - \beta\,T_0\,\Box_1\,\text{div}\,\boldsymbol{u} = -\left(1+\tau_q D\right)R. \tag{2.2.4}$$

## 2.3 Galerkin-type Solution of Equations of Motion

Now, the matrix differential operators are introduced as follows:

$$\mathbf{\Gamma}\left(\boldsymbol{D}_{\boldsymbol{x}}, D\right) = \begin{bmatrix} \mathbf{\Gamma}^{(1)} & \mathbf{\Gamma}^{(2)} \\ \mathbf{\Gamma}^{(3)} & \mathbf{\Gamma}^{(4)} \end{bmatrix}_{4\times 4},$$

where

$$\begin{aligned}
&\mathbf{\Gamma}^{(1)}\left(\boldsymbol{D}_{\boldsymbol{x}}, D\right) = \left[\Gamma^{(1)}_{kj}\right]_{3\times 3}, \ \mathbf{\Gamma}^{(2)} = \left[\Gamma^{(2)}_{k1}\right]_{3\mathrm{x}1}, \ \mathbf{\Gamma}^{(3)} = \left[\Gamma^{(3)}_{1j}\right]_{1\times 3}, \ \mathbf{\Gamma}^{(4)} = \left[\Gamma_{44}\right]_{1\times 1}, \\
&\Gamma^{(1)}_{kj}\left(\boldsymbol{D}_{\boldsymbol{x}}, D\right) = g_1\delta_{kj} + n_1\frac{\partial^2}{\partial x_k \partial x_j}, \\
&\Gamma^{(2)}_{k1}\left(\boldsymbol{D}_{\boldsymbol{x}}, D\right) = -n_3\frac{\partial}{\partial x_k}, \\
&\Gamma^{(3)}_{1j}\left(\boldsymbol{D}_{\boldsymbol{x}}, D\right) = \left(-\beta\, T_0\Box_1\right)\frac{\partial}{\partial x_j}, \\
&\Gamma_{44}(\boldsymbol{D}_{\boldsymbol{x}}, D) = g_2, \quad \boldsymbol{D}_{\boldsymbol{x}} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right),
\end{aligned} \tag{2.3.1}$$

where $\delta_{kj}$ represents the Kronecker delta for $k, j = 1, 2, 3$.

After implementing above operators, Eqs. (2.2.3) and (2.2.4) are written as

$$\mathbf{\Gamma}\left(\boldsymbol{D}_{\boldsymbol{x}}, D\right)\boldsymbol{U}\left(\boldsymbol{x}, t\right) = \boldsymbol{F}\left(\boldsymbol{x}, t\right), \tag{2.3.2}$$

where $\boldsymbol{U} = (\boldsymbol{u}, \theta)$, $\boldsymbol{F} = \{-\boldsymbol{f}, -(1+\tau_q\, D)\, R\}$, where $(\boldsymbol{x}, t) \in W \times (0, \infty)$.

Now, in view of the above, the system of equations can be presented as follows:

$$g_1\,\boldsymbol{u} + n_1 \mathrm{grad}\,\mathrm{div}\boldsymbol{u} - \beta\, T_0\,\Box_1\,\mathrm{grad}\,\theta = \boldsymbol{F}', \tag{2.3.3}$$

$$g_2\,\theta - n_3 \mathrm{div}\,\boldsymbol{u} = F_0, \tag{2.3.4}$$

where $\boldsymbol{F'} = (F'_1, F'_2, F'_3)$ and $F_0$ are the vector component and scalar function, respectively defined on the domain $W \times (0, \infty)$.

Using matrix operator as defined above, Eqs. (2.3.3) and (2.3.4) can be reformulated as

$$\boldsymbol{\Gamma}^{\boldsymbol{T}}\left(\boldsymbol{D}_{\boldsymbol{x}}, D\right) \boldsymbol{U}\left(\boldsymbol{x}, t\right)=\boldsymbol{M}\left(\boldsymbol{x}, t\right), \tag{2.3.5}$$

where $\boldsymbol{\Gamma}^{\boldsymbol{T}}$ represents the transpose of the matrix $\boldsymbol{\Gamma}$ and $\boldsymbol{M}$=$(\boldsymbol{F'}, F_0)$.

Now, implementing the operator "div" to Eq. (2.3.3), yields

$$\Omega_1 \text{ div } \boldsymbol{u}-\beta T_0 \,\square_1\, \Delta\theta=\text{div } \boldsymbol{F'}, \tag{2.3.6}$$

where $\Omega_1=\left(\frac{\lambda+2\mu}{\rho}\right)\Delta-D^2$.

Therefore, above Eqs. (2.3.4) and (2.3.6) take the matrix form as

$$\boldsymbol{\Omega}\left(\Delta, D\right) \boldsymbol{V}=\widetilde{\boldsymbol{F}}, \tag{2.3.7}$$

where $\boldsymbol{V}=(\text{div } \boldsymbol{u}, \theta)$, $\widetilde{\boldsymbol{F}}=(f_1, f_2)=\left(\text{div } \boldsymbol{F'}, F_0\right)$,

with

$$\boldsymbol{\Omega}\left(\Delta, D\right)=\left[\Omega_{kj}\left(\Delta, D\right)\right]_{2\times 2}=\begin{bmatrix} \Omega_1 & -\beta T_0 \,\square_1\, \Delta \\ -n_3 & g_2 \end{bmatrix}_{2\times 2}.$$

Now, system (2.3.7) yields

$$\ell_1\left(\Delta, D\right) \boldsymbol{V}=\boldsymbol{\Phi}, \tag{2.3.8}$$

where

$$\boldsymbol{\Phi}=\left(\Phi_1, \Phi_2\right), \quad \Phi_j=\sum_{k=1}^{2} \Omega^*_{kj} f_k, \quad \ell_1\left(\Delta, D\right)=\det\, \boldsymbol{\Omega}\left(\Delta, D\right), \tag{2.3.9}$$

for $j=1,2$ and $\Omega^*_{kj}$ are the co-factors of the elements of the matrix $\boldsymbol{\Omega}$.

Using the operator $\ell_1\left(\Delta, D\right)$ with Eq. (2.3.3) and then utilizing Eq. (2.3.8) gives the relation as follows:

$$\ell_1\left(\Delta, D\right) g_1 \boldsymbol{u}=\boldsymbol{\Phi}', \tag{2.3.10}$$

where

$$\boldsymbol{\Phi}' = \ell_1 \boldsymbol{F}' - \text{grad}(n_1 \Phi_1 - \beta T_0 \,\square_1\, \Phi_2). \tag{2.3.11}$$

Next, in view of Eqs. (2.3.8) and (2.3.10), the following relation is found:

$$\boldsymbol{S}(\Delta, D)\, \boldsymbol{U}(\boldsymbol{x}, t) = \tilde{\boldsymbol{\Phi}}, \tag{2.3.12}$$

where $\widetilde{\boldsymbol{\Phi}} = \left(\boldsymbol{\Phi}', \Phi_2\right)$ and

$$\begin{aligned} &\boldsymbol{S}(\Delta, D) = [\ell_{kj}(\Delta, D)]_{4\times 4}, \\ &\qquad \ell_{yy} = \ell_1(\Delta, D)\, g_1, \ \text{ for } y = 1, 2, 3, \\ &\qquad \ell_{44} = \ell_1(\Delta, D), \ \ell_{kj} = 0, \ \ k, \ j = 1, 2, 3, 4 \ k \neq j. \end{aligned} \tag{2.3.13}$$

Further, introducing the operators as

$$\begin{aligned} &\eta_{k1}(\Delta, D) = -\left(n_1 \Omega^*_{k1} - \beta T_0 \,\square_1\, \Omega^*_{k2}\right), \\ &\eta_{k2}(\Delta, D) = \Omega^*_{k2}, \ k = 1, 2. \end{aligned} \tag{2.3.14}$$

It follows from Eqs. (2.3.9) and (2.3.11) that

$$\boldsymbol{\Phi}' = (\ell_1 \boldsymbol{I} + \eta_{11} \text{ grad div }) \boldsymbol{F}' + \eta_{21} \text{ grad } F_0, \tag{2.3.15}$$

$$\Phi_2 = \eta_{12} \text{ div } \boldsymbol{F}' + \eta_{22} F_0, \tag{2.3.16}$$

where $\boldsymbol{I}$ denotes identity matrix.

Clearly, in view of Eqs. (2.3.15) and (2.3.16), it is obtained that

$$\widetilde{\boldsymbol{\Phi}}(\boldsymbol{x}, t) = \boldsymbol{Z}^{\boldsymbol{T}}(\boldsymbol{D}_{\boldsymbol{x}}, D)\, \boldsymbol{M}(\boldsymbol{x}, t), \tag{2.3.17}$$

where

$$\boldsymbol{Z}\left(\boldsymbol{D_x}, D\right) = \begin{bmatrix} \boldsymbol{Z}^{(1)} & \boldsymbol{Z}^{(2)} \\ \boldsymbol{Z}^{(3)} & \boldsymbol{Z}^{(4)} \end{bmatrix}_{4\times 4},$$

$$\boldsymbol{Z}^{(1)}\left(\boldsymbol{D_x}, D\right) = \left[Z^{(1)}_{kj}\right]_{3\times 3}, \quad \boldsymbol{Z}^{(2)} = \left[Z^{(2)}_{k1}\right]_{3\times 1}, \quad \boldsymbol{Z}^{(3)} = \left[Z^{(3)}_{1j}\right]_{1\times 3}, \quad \boldsymbol{Z}^{(4)} = \left[Z_{44}\right]_{1\times 1},$$

$$Z^{(1)}_{kj}\left(\boldsymbol{D_x}, D\right) = \ell_1\left(\Delta, D\right)\delta_{kj} + \eta_{11}\left(\Delta, D\right)\frac{\partial^2}{\partial x_k \partial x_j},$$

$$Z^{(2)}_{k1}\left(\boldsymbol{D_x}, D\right) = \eta_{12}\left(\Delta, D\right)\frac{\partial}{\partial x_k},$$

$$Z^{(3)}_{1j}\left(\boldsymbol{D_x}, D\right) = \eta_{21}\left(\Delta, D\right)\frac{\partial}{\partial x_j},$$

$$Z_{44} = \eta_{22}\left(\Delta, D\right), \text{ for } k, j = 1, 2, 3. \tag{2.3.18}$$

Now, by virtue of Eqs. (2.3.5), (2.3.12) and (2.3.17), the following equation is obtained:

$$\boldsymbol{S}\,\boldsymbol{U} = \boldsymbol{Z}^T\boldsymbol{\Gamma}^T\boldsymbol{U}.$$

Hence,

$$\boldsymbol{\Gamma}\left(\boldsymbol{D_x}, D\right)\boldsymbol{Z}\left(\boldsymbol{D_x}, D\right) = \boldsymbol{S}\left(\Delta, D\right), \text{ as } \boldsymbol{Z^T\Gamma^T}{=}\boldsymbol{S}. \tag{2.3.19}$$

Thus, the following lemma is proved.

**Lemma-2.3.1:** The matrix differential operators $\boldsymbol{\Gamma}$, $\boldsymbol{Z}$, and $\boldsymbol{S}$ satisfy Eq. (2.3.19), where $\boldsymbol{\Gamma}$, $\boldsymbol{Z}$, and $\boldsymbol{S}$ are defined by Eqs. (2.3.1), (2.3.18) and (2.3.13), respectively.

Now, let $M'_y(\boldsymbol{x}, t)$, $(y = 1, 2, 3)$ and $\widehat{h}(\boldsymbol{x}, t)$ be functions defined on the region $W \times (0, \infty)$ with $\boldsymbol{M'} = M'_y$ for $y = 1, 2, 3$ and $\widetilde{\boldsymbol{M}} = \left(\boldsymbol{M'}, \widehat{h}\right)$.

Then, the following theorem is established that gives the Galerkin-type solution of Eqs. (2.2.3) and (2.2.4).

**Theorem-2.3.1:** Let

$$\boldsymbol{u} = \boldsymbol{Z}^{(1)}\boldsymbol{M'} + \boldsymbol{Z}^{(2)}\widehat{h}, \tag{2.3.20}$$

$$\theta = \boldsymbol{Z}^{(3)}\boldsymbol{M'} + \boldsymbol{Z}^{(4)}\widehat{h}, \tag{2.3.21}$$

where $M'_y$ and $\widehat{h}$ are fields of class $C^7$ and $C^5$, respectively, also satisfy the following equations:

$$\ell_1\left(\Delta, D\right) g_1\boldsymbol{M'} = -\boldsymbol{f}, \tag{2.3.22}$$

$$\ell_1\left(\Delta, D\right)\widehat{h} = -\left(1 + \tau_q\, D\right) R \tag{2.3.23}$$

on the region $W \times (0, \infty)$. Then, $\boldsymbol{U} = (\boldsymbol{u}, \Theta)$ yields as the solution of Eqs. (2.2.3) and (2.2.4).

**Proof:** From Eqs. (2.3.20) and (2.3.21), it is achieved that

$$\boldsymbol{U}(\boldsymbol{x}, t) = \boldsymbol{Z}(\boldsymbol{D_x}, D)\widetilde{\boldsymbol{M}}(\boldsymbol{x}, t). \tag{2.3.24}$$

Also, from Eqs. (2.3.22) and (2.3.23), the following is found:

$$\boldsymbol{S}\left(\Delta, D\right)\widetilde{\boldsymbol{M}}(\boldsymbol{x}, t) = \boldsymbol{F}\left(\Delta, D\right). \tag{2.3.25}$$

Next, in view of Eqs. (2.3.19), (2.3.24) and (2.3.25), $\boldsymbol{\Gamma U} = \boldsymbol{\Gamma Z}\widetilde{\boldsymbol{M}} = \boldsymbol{S}\widetilde{\boldsymbol{M}} = \boldsymbol{F}$ is obtained.

This completes the proof of theorem 2.3.1.

## 2.4 Galerkin-type Solution of System of Equations for Steady Oscillations

It is assumed that

$$\boldsymbol{u}(\boldsymbol{x},t) = \mathrm{Re}[\widetilde{\boldsymbol{u}}(\boldsymbol{x})\, e^{-\mathrm{i}\omega t}], \qquad \theta(\boldsymbol{x},t) = \mathrm{Re}[\widetilde{\theta}(\boldsymbol{x})\, e^{-\mathrm{i}\omega t}],$$

$$\boldsymbol{f}(\boldsymbol{x},t) = \mathrm{Re}[\widetilde{\boldsymbol{f}}(\boldsymbol{x})\, e^{-\mathrm{i}\omega t}], \qquad R(\boldsymbol{x},t) = \mathrm{Re}[\widetilde{R}(\boldsymbol{x})\, e^{-\mathrm{i}\omega t}].$$

Therefore, from Eqs. (2.2.1) and (2.2.2), the system of equations of the steady oscillations for the MGT thermoelasticity theory are obtained as follows:

$$\mu\,(\Delta\widetilde{\boldsymbol{u}}) + (\lambda+\mu)\,\{\mathrm{grad\,div}\,\widetilde{\boldsymbol{u}}\} - \beta\,\mathrm{grad}\;\widetilde{\theta} + \rho\widetilde{\boldsymbol{f}} = -\omega^2\rho\widetilde{\boldsymbol{u}}, \tag{2.4.1}$$

$$\left\{K\Delta(-i\omega) + K^*\Delta + \rho c_E\left(\omega^2 + i\tau_q\omega^3\right)\right\}\widetilde{\theta} + \beta T_0\left\{\omega^2 \text{ div } \widetilde{\boldsymbol{u}} + i\tau_q\omega^3 \text{ div } \widetilde{\boldsymbol{u}}\right\}$$
$$= -\,(1 - i\tau_q\omega)\,\widetilde{R}, \tag{2.4.2}$$

where $(\boldsymbol{x},t)\,\in W\times(0,\infty)$, $\mathrm{i} = \sqrt{-1},$ and $\omega(>0)$ represents the frequency of oscillation.

Now, the following notations are introduced:

$$\boldsymbol{\Upsilon}(\Delta) = [\Upsilon_{kj}(\Delta)]_{2\times 2} = \begin{bmatrix} \omega^2\rho + (\lambda+2\mu)\,\Delta & \beta T_0\,(\omega^2 + i\tau_q\omega^3)\,\Delta \\ -\beta & (K^* - i\omega K)\,\Delta + \rho\, c_E\,(\omega^2 + i\tau_q\omega^3) \end{bmatrix}_{2\times 2},$$

$\widetilde{\ell}_1\,(\Delta) = \det \Upsilon(\Delta)$,

$e_{k1}(\Delta) = -\left[(\lambda+\mu)\,\Upsilon^*_{k1} + \beta T_0\left(\omega^2 + i\tau_q\omega^3\right)\Upsilon^*_{k2}\right],$

$e_{k2}(\Delta) = \Upsilon^*_{k2}\quad$ for $k = 1, 2$.

If the equation $\widetilde{\ell}_1\,(-\lambda) = 0$ has two roots namely $\lambda_1^2$ and $\lambda_2^2$, then $\widetilde{\ell}_1\,(\Delta)$ can be written as

$$\widetilde{\boldsymbol{\ell}}_1\,(\Delta) = \left(\Delta + \lambda_1^2\right)\left(\Delta + \lambda_2^2\right).$$

Further, the matrix differential operators $\boldsymbol{R}$ and $\widetilde{\boldsymbol{\ell}}$ are introduced which are defined by

$$\boldsymbol{R}\left(\boldsymbol{D_x}, D\right) = \begin{bmatrix} \boldsymbol{R}^{(1)} & \boldsymbol{R}^{(2)} \\ \boldsymbol{R}^{(3)} & \boldsymbol{R}^{(4)} \end{bmatrix}_{4\times 4},$$

where $\boldsymbol{R}^{(1)}\left(\boldsymbol{D_x}, D\right) = \left[R^{(1)}_{kj}\right]_{3\times 3}, \ \boldsymbol{R}^{(2)} = \left[R^{(2)}_{k1}\right]_{3\times 1}, \ \boldsymbol{R}^{(3)} = \left[R^{(3)}_{1k}\right]_{1\times 3}, \ \boldsymbol{R}^{(4)} = \left[R_{44}\right]_{1\times 1},$

$$R^{(1)}_{kj}(\boldsymbol{D_x}) = \widetilde{\ell}_1\left(\Delta\right)\delta_{kj} + e_{11}\left(\Delta\right)\frac{\partial^2}{\partial x_k \partial x_j},$$

$$R^{(2)}_{k1}(\boldsymbol{D_x}) = e_{12}\left(\Delta\right)\frac{\partial}{\partial x_k},$$

$$R^{(3)}_{1k}(\boldsymbol{D_x}) = e_{21}\left(\Delta\right)\frac{\partial}{\partial x_k},$$

$$R_{44} = e_{22}\left(\Delta\right), \quad \text{for } k, j = 1, 2, 3 \tag{2.4.3}$$

and

$$\widetilde{\boldsymbol{\ell}}\left(\Delta, D\right) = \left[\widetilde{\ell}_{kj}(\Delta)\right]_{4\times 4}, \quad \widetilde{\ell}_{yy} = \widetilde{\ell}_1(\Delta)\left[\omega^2\rho + \mu\Delta\right], \ \ y = 1, 2, 3,$$

$$\widetilde{\ell}_{44} = \widetilde{\ell}_1(\Delta), \quad \widetilde{\ell}_{kj} = 0, \ \ k, j = 1, 2, 3, 4, \ \ k \neq j. \tag{2.4.4}$$

Let $\widetilde{Q_y}$, for $y = 1, 2, 3$ and $s$ are functions on $W$. Here $\widetilde{\boldsymbol{Q}} = \left(\widetilde{Q_1}, \widetilde{Q_2}, \widetilde{Q_3}\right)$ and $\boldsymbol{Q} = \left(\widetilde{\boldsymbol{Q}}, s\right)$. Hence, by taking into account the Theorem-2.3.1, the Galerkin-type solution to the system of equations for steady oscillations is derived by the following theorem.

**Theorem-2.4.1:** Let

$$\widetilde{\boldsymbol{u}} = \boldsymbol{R}^{(1)}\widetilde{\boldsymbol{Q}} + \boldsymbol{R}^{(2)}s, \tag{2.4.5}$$

$$\widetilde{\theta} = \boldsymbol{R}^{(3)}\widetilde{\boldsymbol{Q}} + \boldsymbol{R}^{(4)}s, \tag{2.4.6}$$

where $\widetilde{Q_y}$ and $s$ are fields of class $C^6$ and $C^4$, respectively, and also satisfy the following equations:

$$\widetilde{\ell}_1(\Delta)\left[\omega^2\rho+\mu\Delta\right]\widetilde{\boldsymbol{Q}}=-\widetilde{\boldsymbol{f}}, \tag{2.4.7}$$

$$\widetilde{\ell}_1(\Delta)\, s=-\left(1-i\tau_q\,\omega\right)\widetilde{R}, \tag{2.4.8}$$

on $W$. Then, $\left(\widetilde{\boldsymbol{u}},\widetilde{\theta}\right)$ is the solution of Eqs. (2.4.1) and (2.4.2).

## 2.5 General Solution of System of Equations for Steady Oscillations

If external body force $\widetilde{\boldsymbol{f}}$ and external heat source $\widetilde{R}$ are assumed to be absent, then Eqs. (2.4.1) and (2.4.2) can be written as

$$\left(\omega^2\rho+\mu\Delta\right)\widetilde{\boldsymbol{u}}+(\lambda+\mu)\left\{\operatorname{grad}\operatorname{div}\widetilde{\boldsymbol{u}}\right\}-\beta\operatorname{grad}\widetilde{\theta}=0, \tag{2.5.1}$$

$$\left\{(K^*-i\omega K)\,\Delta+\rho c_E\left(\omega^2+i\tau_q\omega^3\right)\right\}\widetilde{\theta}+\beta T_0\left\{\omega^2+i\,\tau_q\,\omega^3\right\}\operatorname{div}\widetilde{\boldsymbol{u}}=0. \tag{2.5.2}$$

Firstly, the coming lemma for the above system of equations is proposed.

**Lemma-2.5.1:** If $\left(\widetilde{\boldsymbol{u}},\widetilde{\theta}\right)$ is yielded as a solution of Eqs. (2.5.1) and (2.5.2), then

$$\widetilde{\ell}_1(\Delta)\operatorname{div}\widetilde{\boldsymbol{u}}=0, \tag{2.5.3}$$

$$\widetilde{\boldsymbol{\ell}}_1(\Delta)\,\widetilde{\theta}=0, \tag{2.5.4}$$

$$\left(\omega^2\rho'+\mu\Delta\right)\operatorname{curl}\widetilde{\boldsymbol{u}}=0. \tag{2.5.5}$$

**Proof:** Firstly, by applying the operator div to Eq. (2.5.1), it is acquired that

$$\left\{\omega^2\rho+(\lambda+2\mu)\,\Delta\right\}\operatorname{div}\widetilde{\boldsymbol{u}}-\beta\,\Delta\widetilde{\theta}=0, \tag{2.5.6}$$

After eliminating $\widetilde{\theta}$ from Eqs. (2.5.2) and (2.5.6), the following is obtained:

$$\widetilde{\ell_1}\text{div }\widetilde{\boldsymbol{u}} = 0.$$

Again, elimination of div $\tilde{\boldsymbol{u}}$ from Eqs. (2.5.2) and (2.5.6), gives

$$\widetilde{\boldsymbol{\ell}_1}\widetilde{\theta} = 0.$$

Additionally, by applying the "curl" operator to Eq. (2.5.1), it is found that

$$\left(\omega^2\rho + \mu\Delta\right)\text{curl }\widetilde{\boldsymbol{u}} = 0.$$

This proves the Lemma-2.5.1.

**Theorem-2.5.1:** If $\left(\widetilde{\boldsymbol{u}}, \widetilde{\theta}\right)$ is evaluated as a solution of Eqs. (2.5.1) and (2.5.2), then

$$\widetilde{\boldsymbol{u}}(\boldsymbol{x}) = \beta\,\text{grad}\sum_{k=1}^{2}\phi_k(\boldsymbol{x}) + \boldsymbol{\Psi}(\boldsymbol{x}), \tag{2.5.7}$$

$$\widetilde{\theta}(\boldsymbol{x}) = \sum_{k=1}^{2} a_k\phi_k(\boldsymbol{x}), \tag{2.5.8}$$

where $\phi_k\,(k = 1, 2)$ and $\boldsymbol{\Psi} = (\boldsymbol{\Psi}_1, \boldsymbol{\Psi}_2, \boldsymbol{\Psi}_3)$ satisfy the following equations:

$$\left(\Delta + \lambda_k^2\right)\phi_k(\boldsymbol{x}) = 0, \tag{2.5.9}$$

$$\left(\Delta + \frac{\omega^2\rho}{\mu}\right)\boldsymbol{\Psi}(\boldsymbol{x}) = \boldsymbol{0},\ \boldsymbol{x} \in W \tag{2.5.10}$$

$$\text{div}\,\boldsymbol{\Psi}(\boldsymbol{x}) = 0, \tag{2.5.11}$$

and

$$a_k = -\left(\lambda + 2\mu\right)\lambda_k^2 + \omega^2\rho,\ \text{ for } k = 1, 2. \tag{2.5.12}$$

**Proof:** Suppose $\left(\widetilde{\boldsymbol{u}}, \widetilde{\theta}\right)$ be a solution of Eqs. (2.5.1) and (2.5.2). Then, by taking into consideration $\Delta\widetilde{\boldsymbol{u}} = \text{grad}\,\text{div}\widetilde{\boldsymbol{u}} - \text{curl curl }\widetilde{\boldsymbol{u}}$ and Eq. (2.5.1), it is obtained that

$$\widetilde{\boldsymbol{u}} = \frac{1}{\omega^2 \rho} \left[ \text{grad} \left\{ -(\lambda + 2\mu) \operatorname{div} \widetilde{\boldsymbol{u}} + \beta \widetilde{\theta} \right\} + \mu \text{ curl curl } \widetilde{\boldsymbol{u}} \right]. \tag{2.5.13}$$

Now, introducing the notation $\boldsymbol{\Psi}(\boldsymbol{x})$ as

$$\boldsymbol{\Psi}(\boldsymbol{x}) = \frac{\mu}{\omega^2 \rho} \text{curl curl } \widetilde{\boldsymbol{u}} \tag{2.5.14}$$

with Eq. (2.5.5) and in view of div curl $\widetilde{\boldsymbol{u}} = 0$ for $\boldsymbol{x} \in W$, it is seen that Eqs. (2.5.10) and (2.5.11) can be easily obtained.

Now, let

$$\phi_y = b_y \left[ \prod_{\substack{k=1 \\ k \neq y}}^{2} \left( \Delta + \lambda_k^2 \right) \right] \widetilde{\theta}, \tag{2.5.15}$$

where

$$b_y = \left[ a_y \prod_{\substack{k=1 \\ k \neq y}}^{2} \left( \lambda_k^2 - \lambda_y^2 \right) \right]^{-1}, \; y = 1,\, 2. \tag{2.5.16}$$

Hence, Eqs. (2.5.4) and (2.5.15) yield Eqs. (2.5.8) and (2.5.9).

Further, using Eqs. (2.5.1), (2.5.8), (2.5.9) and (2.5.12), it is obtained that

$$\operatorname{div} \widetilde{\boldsymbol{u}} = -\beta \sum_{k=1}^{2} \lambda_k^2 \phi_k. \tag{2.5.17}$$

Hence, Eq. (2.5.13) yields

$$\widetilde{\boldsymbol{u}} = \frac{1}{\omega^2 \rho} \left[ \text{grad } \left\{ (\lambda + 2\mu)\, \beta \sum_{k=1}^{2} \lambda_k^2 \phi_k + \beta \widetilde{\theta} \right\} + \mu \text{ curl curl } \widetilde{\boldsymbol{u}} \right]. \tag{2.5.18}$$

Finally, with the help of Eqs. (2.5.12) and (2.5.14), Eq. (2.5.18) yields

$$\widetilde{\boldsymbol{u}}(\boldsymbol{x}) = \beta \operatorname{grad} \sum_{i=1}^{2} \phi_k(\boldsymbol{x}) + \boldsymbol{\Psi}(\boldsymbol{x}).$$

It completes the proof of Theorem 2.5.1.

**Theorem-2.5.2:** If $\left(\widetilde{\boldsymbol{u}}, \widetilde{\theta}\right)$ is provided as in Eqs. (2.5.7) and (2.5.8), where $\phi_k$ and $\Psi$ satisfy Eqs. (2.5.9)-(2.5.11), then $\left(\widetilde{\boldsymbol{u}}, \widetilde{\theta}\right)$ is considered as the solution of Eqs. (2.5.1) and (2.5.2) on $W$.

**Proof:** With the help of Eqs. (2.5.9) and (2.5.10), Eq. (2.5.7) can be represented as

$$\Delta \widetilde{\boldsymbol{u}} = -\beta \operatorname{grad} \sum_{k=1}^{2} \lambda_k^2 \phi_k - \frac{\omega^2 \rho}{\mu} \boldsymbol{\Psi}, \tag{2.5.19}$$

$$\operatorname{grad} \operatorname{div} \widetilde{\boldsymbol{u}} = -\beta \operatorname{grad} \sum_{k=1}^{2} \lambda_k^2 \phi_k. \tag{2.5.20}$$

Firstly, changing $\widetilde{\boldsymbol{u}}$ and $\widetilde{\theta}$ as defined in Eqs. (2.5.7) and (2.5.8) on the left-hand side of Eq. (2.5.1). Then, by using Eqs. (2.5.9), (2.5.12) and (2.5.20), it is found that

$$\begin{aligned}
&\left(\omega^2 \rho + \mu \Delta\right) \widetilde{\boldsymbol{u}} + (\lambda + \mu) \operatorname{grad} \operatorname{div} \widetilde{\boldsymbol{u}} - \beta \operatorname{grad} \widetilde{\theta} \\
&= -\beta \operatorname{grad} \sum_{k=1}^{2} \left\{ (\lambda + 2\mu) \lambda_k^2 + a_k \right\} \phi_k - \omega^2 \rho \boldsymbol{\Psi} \\
&\quad + \omega^2 \rho \left( \beta \operatorname{grad} \sum_{k=1}^{2} \phi_k + \boldsymbol{\Psi} \right).
\end{aligned}$$

After rearranging above expression, the following is obtained:

$$\left(\omega^2 \rho + \mu \Delta\right) \widetilde{\boldsymbol{u}} + (\lambda + \mu) \operatorname{grad} \operatorname{div} \widetilde{\boldsymbol{u}} - \beta \operatorname{grad} \widetilde{\theta} = 0,$$

which is the field Eq. (2.5.1).

Similarly, substituting $\widetilde{\boldsymbol{u}}$ and $\widetilde{\theta}$ on the left-hand side of Eq. (2.5.2) as given in Eqs.

(2.5.7) and (2.5.8) and utilizing Eqs. (2.5.9), (2.5.12) and (2.5.17), finally the following is acquired:

$$
\begin{aligned}
&\left\{(K^* - i\omega K)\,\Delta + \rho c_E\left(\omega^2 + i\tau_q\omega^3\right)\right\}\widetilde{\theta} + \beta T_0\left(\omega^2 + i\tau_q\omega^3\right)\,\text{div}\,\widetilde{\boldsymbol{u}} \\
&= \left\{(K^* - i\omega K)\,\Delta + \rho c_E\left(\omega^2 + i\tau_q\omega^3\right)\right\}\left(\sum_{k=1}^{2} a_k\phi_k\right) + \beta^2 T_0\left(\omega^2 + i\tau_q\omega^3\right)\left(-\sum_{k=1}^{2}\lambda_k^2\phi_k\right) \\
&= \sum_{k=1}^{2}\left[a_k\left\{(K^* - i\omega K)\left(-\lambda_k^2\right) + \rho c_E\left(\omega^2 + i\tau_q\omega^3\right)\right\} + \beta^2 T_0\left(\omega^2 + i\tau_q\omega^3\right)\left(-\lambda_k^2\right)\right]\phi_k \\
&= 0\,(\text{by using}\,\widetilde{\ell_1}\left(-\lambda_k^2\right) = 0\ \text{for}\ k = 1, 2).
\end{aligned}
$$

Hence, the field Eq. (2.5.2) is satisfied.

Therefore, the general solution of the system of Eqs. (2.5.1) and (2.5.2) is obtained in terms of the metaharmonic functions $\phi_k$ and $\boldsymbol{\Psi}$.

## 2.6 Conclusion

The Moore-Gibson-Thompson thermoelastic model proposed by Quintanilla (2019) is studied theoretically in the current chapter. The Galerkin-type solution of basic governing equations is acquired in terms of the elementary functions. A theorem representing the Galerkin-type solution of equations for the steady oscillations in the framework of discussed linear thermoelastic model is provided. Finally, the general solution of the system of equations in the case of steady oscillations is also derived in terms of metaharmonic functions.