CHAPTER 1

INTRODUCTION AND LITERATURE REVIEW

1.1 Mathematical Modeling and Applications

Mathematical modeling is the characterization and simplification of real-world problems and the behavior of objects in terms of mathematics. In other words, the process of developing mathematical formulae, equations, and algorithms to represent events or systems of the real world is known as mathematical modeling. There are several different fields where mathematical modeling is utilized, including:

- In physics, mathematical models are frequently employed to explain physical mechanisms and predict results.

- In complex structures like bridges, electrical circuits, and buildings, engineers use mathematical models to estimate performance, optimize designs, and ensure security.

- Environmental systems, such as ecosystem dynamics, air pollution, water pollution, and climate change are studied and managed using mathematical models.

The formulation of the mathematical model for any device and phenomenon is based on their behavior in “real world” and “conceptual world”. The external world where we observe behaviors and various phenomena is called the real world. These behaviors and phenomena are either natural or produced by artifacts. However, the world of our mind which tries to understand what is going on in the real or external world is the conceptual world. We see the conceptual world in three different ways: observation, modeling, and prediction. Fig. 1.1.1 can illustrate this concept of mathematical modeling:

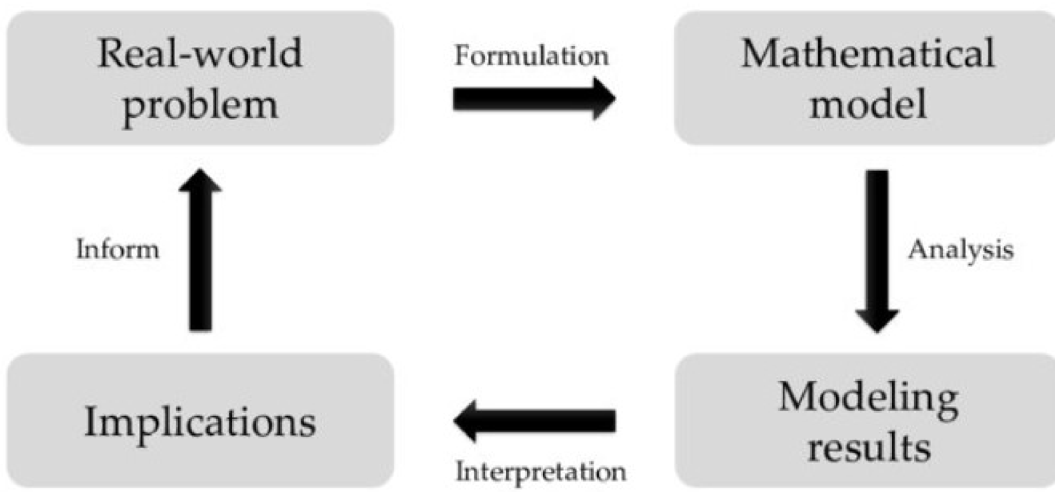

Figure 1.1.1: Graphical illustration of Mathematical modeling

Modeling, therefore, requires mathematical ideas and methods to investigate, assess and anticipate the behavior of complex systems. The accuracy of the model depends on both the state of knowledge about a system and how well the modeling is done. The present thesis is aimed at studying some important aspects of various thermoelastic systems using some recent thermoelastic models and understanding their effects in vibration analysis of various micro and nano mechanical structures.

1.2 Fourier's Law of Heat Conduction and its Drawback

In order to study Fourier's Law of heat conduction, we need to discuss the meaning of heat, heat conduction and heat flux.

- **Heat** is a type of energy that may be transmitted from the boundaries of a system when there is a temperature difference. There are only three ways to transmit heat: conduction, convection, and radiation.

- **Heat conduction** refers to the process of heat transfer when one area of a material is hotter than the other, which is the result of direct contact between the materials.

- **Heat flux** is the rate of heat transfer per unit area. The following formula determines the heat flux ($\overrightarrow{q}$) mathematically:

$$\overrightarrow{q} = \lim_{A \to 0} \frac{Q}{A},$$

where Q is the total amount of heat transfer and A refers to the area of surface.

1.2.1 Fourier's law of heat conduction:

In an isotropic and homogeneous medium, Fourier's law of heat conduction states that the heat flux $\overrightarrow{q}(\overrightarrow{\wp}, t)$ is the instantaneous result of negative temperature gradient $\overrightarrow{\nabla} T(\overrightarrow{\wp}, t)$:

$$\overrightarrow{q}(\overrightarrow{\wp}, t) = -K\overrightarrow{\nabla} T(\overrightarrow{\wp}, t),$$

where K is a thermal conductivity, $\vec{\nabla}$ is gradient operator and $\overrightarrow{\wp}$ denotes position vector in body.

1.2.2 Drawbacks of fourier's law:

Even though Fourier's Law of heat conduction is a fundamental concept that exhibits heat transfer by conduction, it has several restrictions and disadvantages. Consider the following drawbacks:

- In some circumstances, especially when there are significant temperature changes or when dealing with non-linear materials, this law may fail to operate.
- At extremely short times (i.e., $10^{-12}s$ to $10^{-15}s$), this Fourier's law is physically unrealistic for heat conduction because it predicts an infinite speed of heat propagation
- Various materials, such as anisotropic materials, have varying thermal conductivity along various axes, so Fourier's law is not an adequate explanation for the heat transfer behavior of those materials.

1.3 Thermoelasticity: Definition and Applications

The subject thermoelasticity illustrates how the size and shape of a deformable solid body can vary when it is placed in an elevated temperature field. This considers that even in the absence of any external mechanical forces, deformation and, consequently, stress may develop in the body. On the other hand, the time-varying deformation of a body is always associated with a change in the heat content and, consequently, with a change in the body temperature. Thermoelasticity theory is therefore an expansion of elasticity theory that considers thermal effects, including thermal stress, strain, and also deformation. It generalizes both the elasticity theory and the heat conduction theory. The internal energy of the body under this theory, therefore, becomes a function of the deformation and the temperature. It means that the concept of thermoelasticity admits that the heat conduction equation contains a deformation element alongside the temperature term that appears in the equation of motion.

Due to its broad applications across different sectors, thermoelasticity theory has attracted a great deal of interest from engineers and researchers in a variety of scientific and technological disciplines. The design and analysis of structural components that are subject to heat and mechanical loads make use of thermoelasticity theories. Thermoelasticity theory is employed in the aerospace industry to assess and create spacecraft and aircraft parts that are impacted by extreme temperature changes. Various material characterization techniques, such as measuring thermal expansion, use thermoelasticity theory. Thermoelasticity also serves as the foundation for several other scientific fields, including aerothermoelasticity, viscothermoelasticity, magnetothermoelasticity, porothermoelasticity, and thermo-piezoelectric theory.

1.3.1 Classical coupled theory of thermoelasticity and its limitations

Thermoelasticity theory has made considerable progress through extensive research work carried out in this area during the last several decades. However, investigations in the field of thermoelasticity have been originated from extensive research work carried out under the so-called theory of thermal stresses. This theory is based on the simplifying assumption that the influence of the strain on the temperature field may be neglected and this theory is therefore termed as uncoupled theory of thermoelasticity. This theory is not well adapted to two unnatural occurrences because of its physical behavior. This theory is based on Fourier's law of heat conduction and the parabolic heat equation leads to an infinite speed of thermal waves, which contradicts the real-world phenomenon. Furthermore, the absence of an elastic term in the heat equation under this theory suggests that elastic changes do not result in heat effects, and vice versa. The second issue of this uncoupled theory has been addressed by the thermoelasticity theory given by Biot (1956) for the first time. Based on irreversible thermodynamic processes, Biot's theory states that any difference in temperature results in a change in strain measurements of elastic bodies and vice versa; it is therefore called fully classical coupled thermoelasticity theory (CTE). It is worth recalling that Biot's theory was based on the concept of coupling between strain and temperature fields first postulated by Duhamel (1837) and Neumann (1843) independently. Biot's theory is considered to be an elegant model for studying the coupling effects of elastic and thermal fields. However, it has been reported in the literature during the last several years that Fourier's law involved in this theory leads to a diffusion-type heat conduction model that indicates a fatal paradox regarding the infinite speed of thermal waves. Hence, according to this theory, it is possible to successfully explain thermo-mechanical interactions in thermoelastic problems with low heat fluxes and large time responses. This classical coupled thermoelasticity theory, however, fails to properly depict the thermoelastic response of solids when exposed to short laser pulses, rapidly propagating crack tips, and description of thermoelastic material at low temperatures, etc. Additionally, the devices at the micro- and nanoscale have shown different behavior that is incompatible with the classical thermoelasticity theory.

1.3.2 Generalized thermoelasticity

The paradox related to the infinite speed behavior of thermal propagation has aroused immense focus from the mathematical and mechanical perspective, which suggests that the classical theory needs a well-grounded modification. It is worth to be mentioned that as early as in 1867, Maxwell (1867) postulated for the first time that "the thermal disturbance is a wave like phenomenon rather than diffusion phenomenon" and indicated the modification of Fourier law (see Chandrasekharaiah (1998)). Later on several eminent researchers including Tisza (1947), Landau (1941), and Peshkov (1944) discussed the wave like behavior of heat propagation, which is now referred to as the "second sound effect". Experimental evidence of 'second sound effect' in materials have also been reported by various other researchers like Maurer and Herlin (1949), Pellam and Scott (1949), Atkins and Osborne (1950), Ackerman et al. (1966), Ackerman and Overton (1969), Bertman and Sandiford (1970), McNelly et al. (1970), Jackson et al. (1970), Jackson and Walker (1971), Rogers (1971), etc. In view of this, several researchers have contributed and modified Fourier's law as well as coupled thermoelasticity in order to address their shortcomings. The first group of generalized theories incorporates the concept of phase-lags/thermal relaxation parameters for constitutive variables in the Fourier law of heat conduction. The second group of thermoelasticity theories uses an alternate improvement method to get reconditioned constitutive equations by keeping Fourier's law unchanged. Another thermoelastic theory is developed by an alternative formulation of the coupled theory by introducing new constitutive field variables in the derivation of governing equations. The objective of such generalized theories has been to overcome the classical theory's apparent paradox related to infinite speed behavior. Because of their experimental evidence of the finite speed of heat wave propagation, generalized thermoelasticity theories are considered to be more efficient in solving practical thermodynamical problems than conventional thermoelasticity theory, particularly those involving short time intervals and high heat fluxes. The following subsections will provide a brief introduction to the generalized thermoelasticity theories that are relevant to the present study.

1.3.2.1 Lord-Shulman thermoelasticity (LS) theory based on Cattaneo-Vernotte law:

By including a time relaxation parameter τ_q with the heat flux rate $\vec{q}$ term in Fourier's law, Cattaneo (1958) and Vernotte (1958; 1961) have proposed the first modified version to Fourier's law of heat conduction. This modified law, also known as CV law, leads to a hyperbolic type heat transport equation allowing a finite speed of the thermal signal. Subsequently, Lord and Shulman (1967) introduced the first generalized thermoelasticity theory (LS theory) with the non-Fourier effect-based Cattaneo-Vernotte (CV) heat conduction law. This generalized theory establishes the finite speed of thermal signals and resolves the paradox of infinite heat propagation of Biot's theory. The heat conduction law (CV's law) under LS thermoelasticity theory for a homogeneous and isotropic medium can be written as

$$\overrightarrow{q}(\overrightarrow{\wp}, t) + \tau_q \frac{\partial \overrightarrow{q}(\overrightarrow{\wp}, t)}{\partial t} = -K\overrightarrow{\nabla} T(\overrightarrow{\wp}, t).$$

Here, τ_q is a time lag parameter, which is required to establish a steady-state heat conduction in the medium.

1.3.2.2 Temperature-rate dependent thermoelasticity theory

The second generalization to the coupled theory of thermoelasticity is known as the theory of thermoelasticity with two relaxation times that also admits finite speed of heat propagation. On the basis of a new entropy production inequality developed by Muller (1971), a generalization of constitutive relations of thermoelasticity theory has been proposed by Green and Laws (1972). Subsequently, Green and Lindsay (1972) obtained an explicit version of these constitutive equations and developed a new modification of coupled thermoelasticity theory. This theory is also referred to as temperature-rate-dependent thermoelasticity (TRDTE) theory as it includes temperature-rate terms among the constitutive variables and two constants that act as thermal relaxation times. This theory modifies all the equations of the coupled theory, not only the heat conduction equation. The classical Fourier's law of heat conduction is not violated in this theory if the medium under consideration has a center of symmetry.

1.3.2.3 Green-Naghdi (GN) thermoelasticity theory

By introducing thermal displacement variable, as the time integral of absolute temperature function, in Fourier's law of heat conduction, Green and Naghdi (1991; 1992; 1993) proposed three versions of alternative thermoelasticity theory (GN theory) for homogeneous and isotropic material. The GN theory is divided into three distinct unconventional types of generalized theories, called as GN-I, GN-II, and GN-III. As the linearized form of the GN-I produces the infinite speed of thermal wave, it is the same as the well-known CTE. The GN-II model admits a finite speed of heat propagation and the GN-III model is a merger of the two earlier models (GN-I and GN-II) and admits the dissipation of energy in general. The heat conduction law for type GN-III theory is of the form:

$$\overrightarrow{q}(\overrightarrow{\wp}, t) = -\left(K\overrightarrow{\nabla}T(\overrightarrow{\wp}, t) + K^{*}\overrightarrow{\nabla}\upsilon(\overrightarrow{\wp}, t)\right).$$

Here, the essential material parameter K^{*} is the conductivity rate of the material and υ represents thermal displacement, where the relation $\dot{\upsilon} = T$ is available. The above heat conduction law can be reduced into the heat conduction relation of other two theories (GN-1 and GN-II) as follows:

- When $K^{*} = 0$, then $\overrightarrow{q}(\overrightarrow{\wp}, t) = -K\overrightarrow{\nabla}T(\overrightarrow{\wp}, t)$ (**GN-I theory**)
- When $K = 0$, then $\overrightarrow{q}(\overrightarrow{\wp}, t) = -K^{*}\overrightarrow{\nabla}\upsilon(\overrightarrow{\wp}, t)$ (**GN-II theory**)

1.3.2.4 Dual-Phase-Lag (DPL) and Three-Phase-Lag (TPL) thermoelasticity theories

During 1995-1998, another modification in Fourier's law and also in thermoelasticity theory is proposed. This modified model is now called as dual phase-lag model that can explain all fundamental properties in diffusion, thermal waves, and phonon-electron scattering in heat transport phenomenon associated with small-time responses. Tzou

(1995) developed the theory of heat conduction by introducing the idea of phase-lag. To account for the microscopic effects in the heat transport phenomenon and the nature of interaction of phonon-electron, he introduced two phase-lag parameters, one for the heat flux vector and the other for the temperature gradient vector in the Fourier's law. This modified heat conduction relation is termed as dual-phase-lag heat conduction model. Later on, Chandrasekharaiah (1998) extended this DPL heat conduction theory and established an alternative thermoelasticity theory called the dual-phase-lag thermoelasticity theory. The heat conduction law for DPL thermoelasticity theory is expressed as follows:

$$\overrightarrow{q}(\overrightarrow{\wp},\, t+\tau_q) = -K\overrightarrow{\nabla}T(\overrightarrow{\wp},\, t+\tau_T),$$

where, the phase-lag (τ_q) of the heat flux vector captures the thermal wave behavior, a small-scale reaction in time. Moreover, the phase-lag τ_T of temperature gradient captures the effect of phonon-electron interactions.

Subsequently, based on the type-III model developed by Green and Naghdi, Roychoudhuri (2007) introduced three phase-lag parameters in the heat flux vector ($\overrightarrow{q}$), temperature gradient $\left(\overrightarrow{\nabla}T\right)$ and thermal displacement gradient $\left(\overrightarrow{\nabla}\upsilon\right)$ to elaborate the effect of phono-electron interactions and phonon-scattering at a microscopic level. The heat conduction constitutive relation for TPL (three phase-lag) thermoelasticity theory involving three different phase-lag parameters is given as follows:

$$\overrightarrow{q}(\vec{\wp}, t+\tau_q) = -\left[K\overrightarrow{\nabla}T(\vec{\wp}, t+\tau_T) + K^{*}\overrightarrow{\nabla}\upsilon(\vec{\wp}, t+\tau_\upsilon)\right].$$

Here, τ_υ is a phase-lag of thermal displacement gradient vector.

1.3.2.5 MGT thermoelasticity theory

Recently, Quintanilla (2019) has developed an innovative Moore-Gibson-Thompson (MGT) thermoelasticity theory that gained serious attention from researchers, which is a combination (or generalization) of the Lord-Shulman and Green-Naghdi thermoelasticity theory of type III. LS and GN-III models can be recovered when we omit the dependence with respect to suitable variables in this model. The following is the heat conduction law due to the MGT thermoelasticity theory:

$$\overrightarrow{q}(\overrightarrow{\wp}, t) + \tau_q \frac{\partial \overrightarrow{q}(\overrightarrow{\wp}, t)}{\partial t} = -\left(K\overrightarrow{\nabla}T(\overrightarrow{\wp}, t) + K^*\overrightarrow{\nabla}\upsilon(\overrightarrow{\wp}, t)\right).$$

1.4 Development of Micro/Nano-Electromechanical Systems

Nowadays, Microelectromechanical systems (MEMS) and nanoelectromechanical systems (NEMS), which operate at the micro- and nanoscale, combine mechanical components with sensors, actuators, and electronics. Due to their excellent characteristics, Micro and Nanoelectromechanical systems (MEMS/NEMS) draw considerable attention for a great number of applications across various rapidly expanding mechanical and civil engineering nanotechnology sectors. Nanotechnology has emerged in the last decade, allowing the accurate prediction of the natural frequency of the nanobeam for specialized applications such as micro-electromechanical resonators, nanoscale actuation, resonator sensors and detection, etc. MEMS-based devices are used in the manipulation and control of fluid at small ranges in the order of microliters or picoliters. In optical systems, these devices are often used to develop digital projectors and optical filters with micro-mirror arrays to adjust the light path. A significant impact of MEMS and NEMS on the healthcare industry has been reported to enhance the functionality of biomedical diagnostic equipment, as well as improve their compatibility. In the analysis of blood-cell separation, the MEMS specialized devices gained tremendous popularity and evolved in the past two decades. The actual performance of dynamic systems can be captured by the modeling of such MEMS/NEMS devices. Due to the interaction of mechanical and electrical characteristics at the nanoscale, resonators are the most common structures used in MEMS/NEMS systems. The micro and nano resonators maximize operating speed and sensitivity while utilizing minimal energy during vibration.

1.4.1 Micro and nanomechanical resonators

Micro and nanomechanical resonators are devices that exhibit the resonance phenomenon at a specific frequency. The rectangular beam and thin plates are the most common structures for designing the sensor and resonators. The creation of nanoscale resonators has gained significant attention from researchers, recently. When resonators are developed, one of the essential features to ensure high sensitivity and resolution is the quality factor (QF), which interprets dissipated energy due to both extrinsic and intrinsic energy loss mechanisms in the system. Therefore, QF control is an essential task for microelectromechanical systems (MEMS). In the real world, maximum energy dissipation and a low-quality factor are fatal flaws that hinder the practical problem. During the vibration process, a high QF for micro and nano resonators indicates low energy dissipation, implying higher detection resolution and indicates excellence performance. In an another way, a resonator's efficiency and selectivity are measured by contrasting the energy stored there with the energy lost during each cycle. By utilizing appropriate intrinsic and extrinsic support technologies, quality factor enhancement is mainly focused on reducing the various loss factors during oscillation. The QF of resonators can be defined in the following way:

$$\text{Quality factor (QF)} = 2\pi \left(\frac{\text{Stored energy}}{\text{Loss of energy(dissipated energy)}} \right).$$

1.4.2 Thermoelastic damping (TED)

The comprise of several dissipation mechanisms like thermoelastic damping, support loss, air damping, etc. could administer the performance of the resonator. A high QF (a dimensionless parameter) is required as a priority to reduce energy loss during vibration, thus being the goal of MEMS design. The following flow chart characterizes the several energy loss factors in the small-scale resonator.

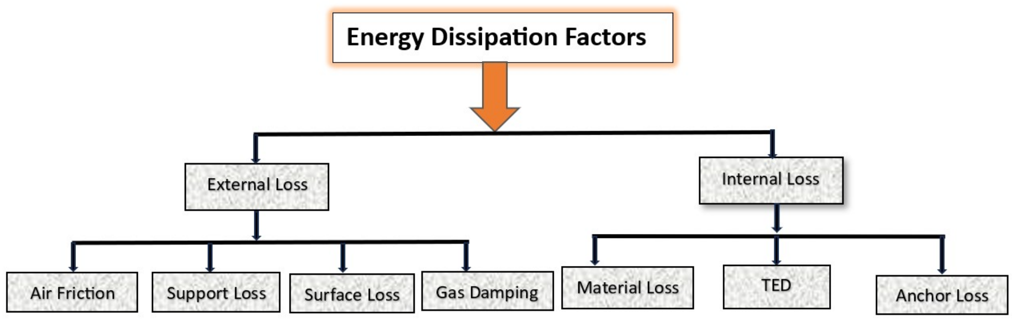

Figure 1.4.1: Energy loss factors in resonators

The irreversible heat flow within an elastic body is a significant source of TED, among many other energy dissipation mechanisms. The bending structure goes under tension and compression at temperature increments or decrements, which produce a thermal gradient inside the material of the deforming system. According to the practical findings of the research, thermoelastic damping eventually came to be acknowledged as one of the unavoidable sources of energy loss in microstructures. Moreover, there is no way to recover the energy loss during the process of the deforming structure returning to its initial zero-bending stress. Structural design is the only way to optimize TED, but it cannot be completely eliminated. Therefore, in order to design high-performance small-size resonators, it is required to examine the thermo-mechanical coupling phenomenon and construct a theoretical TED model. In many micro devices, maintaining the various energy loss factors results in improved stability, less power consumption, and increased resonator sensitivity.

1.4.3 Some micro-structure based beam and plate theories

1.4.3.1 Classical Euler-Bernoulli beam theory

A fundamental model is used to examine the performance of beam when loads are imposed, and it is referred to as the classical Euler-Bernoulli beam theory or engineer beam theory. In the 18th century, Leonhard Euler and Daniel Bernoulli developed it, and it remains an important discovery in structural engineering and mechanics. The Euler-Bernoulli beam theory analyses the small deflection caused by lateral stress by obeying Hooke's law for homogeneous and isotropic material. The classical beam theory implies that irrespective of deformation, the cross-sections of the beam stay plane and normal to the longitudinal axis of the beam. This implies that shear deformation and rotational effect phenomenons do not occur in this beam model. The comprehensive study of small deflection, bending stress, and the effect of applied forces can be analyzed utilizing the Euler-Bernoulli beam theory.

The fourth-order linear differential equation of motion which governs the beam's behavior can be written as

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} - f(x,t) = 0.$$

Here, E, I, ρ and A denote Young's modulus, moment of inertia, density, and beam cross section, respectively. w is a small deflection in the beam and $f(x,t)$ represents the lateral load.

1.4.3.2 Timoshenko beam theory

In the twentieth century, the Timoshenko beam theory was one of the remarkable development of structural dynamics consisting of shear and rotatory inertia effects as contributed by Stephen Timoshenko. Timoshenko beam theory is an extension of the Euler-Bernoulli beam theory that gives an improved representation of thick beam behavior under high shear forces. Timoshenko beam theory, which makes more accurate predictions than the conventional beam theory, is frequently utilized in engineering constructions with thick cross-sections or when the height-to-length ratio of the beam is small. Shear deformations and rotational effects play a key role in the difference between the Timoshenko beam and Euler-Bernoulli beam theories. In addition, this beam theory allows the shape of the cross-section can not remain plane and normal after deformation because of shear forces. The Timoshenko beam theory leads to the following partial differential equation to predict the motion of the beam:

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} - \rho I\left(1 + \frac{E}{KG}\right)\frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{\rho^2 I}{KG}\frac{\partial^4 w}{\partial t^4} - f(x,t) = 0.$$

Here, G is a shear modulus.

1.4.3.3 Kirchhoff plate theory

Kirchhoff plates are one of the most basic and crucial structural components in solid mechanics because of their practical importance. Kirchhoff plate theory, also known as Kirchhoff-Love theory, is named after the German physicist Gustav Kirchhoff and is used to describe stress and deformation behavior in thin elastic plates under various loading conditions. When aspect ratios (height-to-length) are low, the Kirchhoff plate theory neglects shear stress and shear deformation. The following assumptions form the foundation of this theory:

- The thickness of plate is thin as compared to its length and width, and it does not alter when deformed.

- A plate formed of homogeneous and isotropic material experiences only small deformation, guaranteeing its linear elastic properties.

- Deformation does not affect the middle surface of the plate, which remains plane and normal to the surface.

Kirchhoff plate theory provides the following equation of motion for moments, shears, and uniform or non-uniform loads:

$$Z\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial z^4}\right)+2\rho h\frac{\partial^2 w}{\partial t^2}+f(x,y,t)=0.$$

Here, bending stiffness and load at a plate surface are denoted by Z and $f(x,y,t)$, respectively. Also, h refers to the plate thickness.

1.4.4 Background of non-classical continuum theories

There has been significant interest placed on small-scaled structures due to their exceptional characteristics, however, it may be challenging to comprehend their mechanical properties. For a full understanding of mechanical behavior and capture of size effects, it is therefore vital to incorporate these advanced small-scaled structures into classical and non-classical continuum elasticity theories, including couple stress theory (Toupin (1962), Mindlin and Tiersten (1962), Koiter (1964)), nonlocal elasticity theory (Eringen (1972; 1983)), strain gradient theory (Lam et al. (2003)), modified couple stress theory (Yang et al. (2002)), and nonlocal strain gradient theory (Lim et al. (2015)). These higher-order continuum theories incorporate the additional material length scale parameters, which capture the size effect at the micro and nanoscale.

1.4.4.1 Nonlocal elasticity theory

The nonlocal continuum theory is an attractive approach for modeling of nanostructures' mechanical properties. Firstly, the literature of Kroner (1967) and Kunin (1968) can be traced as the origins of nonlocal elastic continuum mechanics for long-range interactions. Thereafter, Eringen (1972; 1983) proposed the improved formulation of nonlocal elasticity theory to overcome the drawback related to sharp geometrical singularities in classical (local) elasticity theory. Also, it was well known that the Eringen's nonlocal elasticity theory is applicable for both linear isotropic and homogeneous materials.

Eringen's nonlocal elasticity theory is a mathematical framework widely employed for investigating the behaviors of micro/nano structural elements that exhibit nonlocal effects. The nonlocal theory of elasticity differs from the classical theory of elasticity in that the stress here is influenced by strain at nearby points as well because of atomic forces and micro/nanoscale effects. The integral type constitutive equations of Eringen's theory were introduced first, afterwards, a differential form with a kernel function was introduced to capture nonlocal effects as follows:

$$\left(1 - \kappa\nabla^2\right)\sigma_{ij}^{nonlocal} = \sigma_{ij}^{local},\ \kappa = (e_0 a)^2 .$$

Here, σ_{ij}^{local}, $\sigma_{ij}^{nonlocal}$ and ∇^2 are local stress, nonlocal stress and Laplacian operator, respectively. κ is a non-local parameter, where a represents internal characteristic length and e_0 is a material constant.

1.4.4.2 Modified couple stress theory (MCST)

The classical continuum theory assumes that the matter in a body is distributed continuously. The purpose of this uniform distribution is to investigate the behavior of materials at a macroscale. However, the mechanical behavior of micro/nano structures in engineering fields became questionable to many researchers due to the lack of an internal material length-scale parameter in classical deformation theory for MEMS. To solve the difficulties in the classical continuum theory (CCT) regarding material length-scale parameters, it was an advantage to reformulate and extend the CCT theory with the material length-scale parameters. Therefore, the size dependent continuum theories were introduced with the conjunction of couple stresses. During the early 1960s, researchers such as Koiter (1964), Mindlin and Tiersten (1962) developed the couple stress theory with the merit of accounting for two additional material constants. Later on, alternative elasticity theory, such as modified couple stress theory (MCST) has been proposed to adequately account for the accurate size effect. MCST is characterized by only one additional material length scale parameter, instead of two classical material constants. Based on the couple stress theory and containing only one length size parameter, Yang et al. (2002) introduced the MCST for isotropic materials where the curvature tensor and couple stress tensor are both symmetric. The defined strain energy function U_s in MCST satisfies

$$U_s = \iiint_V \left(\sigma_{ij} e_{ij} + \Upsilon_{ij} \chi_{ij}\right) dV,$$

where

$$\Upsilon_{ij} = 2\mu l^2 \chi_{ij}.$$

Here, $\boldsymbol{\Upsilon}$ represents the couple stress tensor and χ_{ij} represents the component of the rotation gradient tensor $\boldsymbol{\chi}$. l refers to the material length size parameter.

1.4.4.3 Nonlocal strain gradient theory (NSGT)

The discrepancy between experimental and classical continuum mechanics results arising from a wide variety of nanoscale investigations demonstrates that the classical theory might not be able to accurately predict the response of nanostructures since it does not take into account the size-scale parameters. An improved size-dependent strain gradient theory based on three length scale parameters for the dilatation gradient vector, the curvature tensor and the deviatoric stretch gradient tensor were proposed by Lam et al. (2003). However, more length scale parameters involved in this non-classical theory have created significant confusion among researchers over recent years and efforts are made to reduce the number of independent material constants. Several studies have confirmed that the size effect in nonlocal theory leads to stiffness softening, whereas the size-dependent strain gradient theory illustrates the stiffening-hardening influence in microstructural deformation. By combining nonlocal elasticity theory and strain gradient theory, Lim et al. (2015) developed a comprehensive higher-order nonlocal strain gradient theory (NSGT) based on the observation that materials at small scales exhibit either softening or stiffening behaviors in a more accurate way. Two non-classical scale parameters (the nonlocal parameter and the strain gradient coefficient) are used in this pure nonlocal strain and strain gradient driven theory. Total stress is a key postulate of this theory, which is determined by the nonlocal effects of strain and strain gradient fields. According to this proposed theory, total stress tensor $\sigma_{ij}^{T}(x)$ is fusion of both non local elastic stress tensor $\sigma_{ij}^{(0)}(x)$ and higher order strain gradient stress tensor $\sigma_{ij}^{(1)}(x)$, which is defined as follows:

$$\sigma_{ij}^{T}(x) = \sigma_{ij}^{(0)}(x) - \nabla\sigma_{ij}^{(1)}(x),$$

where ∇ is a gradient operator.

1.5 Literature Review

In view of the wide applications to numerous problems in engineering and technology, thermoelasticity theories have aroused much interest in the last few decades and several researchers have applied coupled thermoelasticity theories to investigate various problems concerning thermoelastic interactions in different kinds of media. Extensive review works carried out on various thermoelasticity theories are available in the review articles and books written by Chandrasekharaiah ((1986); (1998)), Hetnarski and Eslami (2009) and Ignaczak and Ostoja-Starzewski (2009) and also in the Ph.D. theses of Roushan Kumar (2010), Harendra Kumar (2022). The present section of the thesis will give the state of art of the literature relevant to the present thesis and gain an understanding of some recent generalized thermoelasticity theories and their uses in vibration analysis of various small scale structures.

Studies of elasticity and thermoelasticity theory normally use Galerkin representation (Galerkin (1930)) of field equations to solve boundary value problems. These representations are constructed using several elementary functions, including harmonic, metaharmonic, and biharmonic functions. These are the foundations to obtain the fundamental solution of the theory. Therefore, the Fundamental solution and Galerkin-type representation of the solution of thermoelasticity theory are of much interest of researchers. First, Iacovache (1949) proposed the solution of the equations of classical elastokinetics in the form of a Galerkin-type solution (1930). Nowacki (1964; 1969) and Sandru (1966) formulated Galerkin's and Papkovitch's type representations of the solution in the thermodynamics and micropolar elasticity theories. Gurtin (1973) and Nowacki (1975) provided the representation of solutions to some dynamical problems in the case of classical elasticity. Chandrasekharaiah (1987; 1989) formulated a complete solution of coupled systems for homogeneous and isotropic elastic material with voids. Galerkin-type representations were established by Ciarletta (1991; 1995; 1999) in various contexts, including thermoplastic materials with voids, micropolar theories without energy loss, and binary mixture dynamics, respectively. Svanadze (1993) obtained the Galerkin-type solution of steady-state oscillations of an elastic mixture. Svanadze and de Boer (2005) further proposed the Galerkin-type representation for a liquid-saturated porous medium consisting of an incompressible solid skeleton. Scalia and Svanadze

(2006) established the Galerkin-type solution of equation of motion in the context of elementary functions. Mukhopadhyay et al. (2010) derived the Galerkin-type solution for the linear theory of TPL thermoelasticity. In the context of theory of generalized thermoelastic diffusion, Kothari and Mukhopadhyay (2012) obtained the Galerkin-type representation of field equations of motion and established various theoretical results. By utilizing the Kelvin-Voigt materials with voids, Svanadze (2014; 2017) established the Galerkin-type solution for linear thermoelasticity and micropolar viscoelasticity, respectively. Recently, Gupta and Mukhopadhyay (2019) derived the Galerkin-type solution of field equations in the context of modified Green-Lindsay (MGL) thermoelasticity theory (Yu et al. (2018))

The fundamental solution was first studied by Hetnarski (1964a; 1964b) in the context of the classical theory of coupled thermoelasticity. Dragocs (1979a; 1979b; 1984) proposed the fundamental solution in thermoelasticity and micropolar elasticity. Svanadze (1988) obtained the fundamental matrix in order to obtain the fundamental solutions of linearized equations in the context of theory of elastic mixtures. Sherief (1986) established the short-time approximated fundamental solutions with a continuous heat source under generalized thermoelasticity theory. Under the spherically continuous heat source, Sherief (1992) further derived the fundamental solution for thermoelasticity theory containing two relaxation times. Wang and Dhaliwal (1993) obtained the fundamental solution of a system of generalized thermoelastic equations under arbitrarily distributed body forces and heat sources. By using Hankel and Laplace transform methods, Ezzat (1995) derived the fundamental solution for stress and temperature distribution under cylindrical regions. Svanadze (1996) established the fundamental solution of the oscillation equation utilizing the theory of a mixture of two elastic solids. Ieşn (1998) used the Galerkin-type solution to determine the fundamental solution in the frame of a linear theory of thermoelasticity. By means of elementary functions, Svanadze (2004) constructed the fundamental solution of the equation of equilibrium under the micro-temperature effect. Svanadze and De Cicco (2005) derived the fundamental solution of a set of differential equations in the case of thermomicrostretch elastic solids. Svanadze et al. (2006) constructed the fundamental solution of steady oscillations utilizing the theory of micropolar thermoelasticity in the case of theory without energy dissipation. By utilizing elementary functions in the linear theory of micropolar thermoelasticity, Ciarletta et al. (2007) obtained the fundamental solution to a system of steady oscillation for materials with voids. For the case of homogeneous and isotropic bodies, Kothari et al. (2010) constructed the fundamental solution for TPL thermoelasticity theory. Subsequently, by Kumar and Kansal (2012), elementary functions were used to derive fundamental solutions to differential equations in micropolar thermoelastic diffusion with voids under steady oscillations. Svanadze and De Cicco (2013) derived the fundamental solution of dynamical equations in the context of linear theory of elasticity with double porosity. Scarpetta (1990) established the fundamental solutions for micropolar elasticity and the system of steady vibration using the thermoelasticity theory of double porosity. Sherief and Hussein (2017) used the instantaneous point heat source to obtain the fundamental solution of thermoelasticity with two relaxation times. Kumari and Mukhopadhyay (2017) derived the fundamental solution under the heat source and body force for the heat conduction model with a delay term. Biswas and Sarkar (2018) and Biswas (2020) constructed the fundamental solution for DPL thermoelasticity in a homogeneous porous medium for the system of steady oscillations. Under the Bio-thermoelastic medium, Kumar et al. (2020) computed the fundamental solutions of steady vibration with DPL thermoelasticity theory. By means of a perfect isotropic conductor, El-Bary and Atef (2021) derived the fundamental solution of magneto-thermo-viscoelasticity with two relaxation time parameters. Under the effect of micro temperature and micro concentrations, Tarun (2022) constructed the fundamental solution for isotropic micromorphic thermoelastic diffusion materials.

Various thermoelasticity theories have been employed to study the vibration problems of beams and plates. In the work by Zener (1937; 1938), internal friction caused by thermoelastic damping was examined on the basis of quality factor (QF) for the first time. For the rectangular cross-section of a microbeam, Zener's analytical model has been verified at the micro-scale level. Shieh (1975) studied the thermoelastic vibration and damping behavior within the framework of a circular Timoshenko beam. Alblas (1981) investigated the TED in vibrating three-dimensional elastic beams. By extending Zener's work on simple beams, Lifshitz and Roukes (2000) introduced TED as an important fundamental dissipation source of energy. Srikar and Senturia (2002) estimated the upper bound of QF by designing the polysilicon flexural beam resonator. Hao et al. (2003) presented the closed-form analytical expression for support loss with flexural vibration. Vengallatore (2005) evaluated the TED in a micromechanical three-layered Euler-Bernoulli beam with silicon coating. Lepage and Golinval (2005) considered finite element modeling in order to assess TED in MEMS.

Efforts have also been made by researchers to accurately study the vibration problems of small-scale structures considering the size effects. In order to account for the size effect, Park and Gao (2006) developed an Euler-Bernoulli beam model based on the modified couple stress theory (MCST). Park and Gao (2008) derived the closed-form expression of honeycomb micromechanics structure by using MCST. Later, Ma et al. (2008) developed the non-classical Timoshenko beam model which incorporated both bending and axial deformations. Wang and Feng (2009) studied the surface effect on buckling and vibration behavior by using the refined Timoshenko beam theory. Asghari et al. (2010) formulated the nonlinear Timoshenko beam model in the context of MCST to delineate size effects. A numerical and analytical solution is presented by Şimşek (2010) for the vibration of a microbeam under the action of moving particles based on MCST. Akgöz and Civalek (2011) employed the higher-order continuum theories in the mechanical modeling of micro-scaled beams. Asghari et al. (2011) analyzed the size-dependent behavior of functionally graded Timoshenko beam on the basis of MCST. Ke and Wang (2011) investigated the dynamic stability of functionally graded beam via MCST. Roque et al. (2011) studied the bending and buckling of Timoshenko beams using nonlocal formulations of Eringen's elasticity theory. Sharma (2011) derived the analytical formula for frequency shift in an anisotropic beam at micro/nano scale. Rezazadeh et al. (2012) used the plane stress and strain condition to obtain an analytical expression for the QF of TED. Guo et al. (2012) derived the explicit expression for TED of beam resonator based on DPL thermoelasticity theory. Şimşek et al. (2013) developed the Timshenko beam model to predict the bending analysis in the context of MCST. Şimşek and Reddy (2013) examined the bending and free vibration of functionally graded beam based on MCST. Akgöz and Civalek (2013) studied the vibration response of Euler-Bernoulli beam in conjunction with MCST. Barretta et al. (2015) studied the bending formulation of functionally graded Euler-Bernoulli beam in the framework of nonlocal thermodynamic theory. Mohammadabadi et al. (2015) investigated the size-dependent thermal buckling effects of micro composite laminated beams in the context of MCST. Fathalilou and Rezazadeh (2016) investigated the effect of the length scale parameter on the TED in microbeam using MCST. Shafiei et al. (2017) used the rotary tapered Euler-Bernoulli microbeam to study the transverse vibration based on MCST. Li et al. (2018) proposed a standard experimental method to calculate the internal material length scale parameter based on MCST. Borjalilou et al. (2019) studied a small-scale TED analysis in micro-beams utilizing the MCST and DPL heat conduction model. Hamidi and Hosseini (2020) derived the exact solution of a gold microbeam resonator via TED considering MCST and GN III theory. Rahi (2021) illustrated the analytical approach to capture vibration analysis of multi-layer microbeam based on MCST. Kumar and Mukhopadhyay (2023) studied the TED in a nanobeam resonator using the non-classical continuum and the generalized thermoelasticity theories.

Gao (2015) developed the new Timoshenko beam model using surface elasticity theory incorporating microstructure. Dehrouyeh-Semnani and Bahrami (2016) examined the accuracy of the size effect in the complex Timoshenko beam model based on MCST. Karttunen et al. (2016) derived the general closed-form solution of the Timoshenko beam model in terms of plane stress distributions by using MCST. Ghayesh et al. (2019) studied an asymmetric vibration analysis of a functionally graded deformable Timoshenko beam. In this respect, we also refer the work by Esen (2020), Kumar (2020), Kumar and Mukhopadhyay (2021), Weng et al. (2021), Abdelrahman et al. (2021), Liu and Peng (2022), Yuan et al. (2022), Zhang et al. (2022), Loya et al. (2022) and Ye et al. (2023).

Ma et al. (2011) developed a size-dependent non-classical Mindlin plate model using MCST. Jomehzadeh et al. (2011) used the variational approach to study a vibration analysis of micro-plate via MCST. Later, Ke et al. (2012a) captured the free vibration of the Mindlin micro plate by MCST taking into account. Moreover, the bending and buckling characterization of functionally graded annular microplates are studied by Ke et al. (2012b). Thai and Choi (2013) presented the size-dependent vibration of Kirchhoff and Mindlin by using MCST. Shaat et al. (2014) studied the bending analysis of the Kirchhoff plate resonator in the presence of a surface effect with MCST. Zhong et al. (2014) estimated the dynamic properties and TED in microplate resonators via MCST. The study of static bending and free vibration for piezoelectric microplate involving MCST was developed by Li and Pan (2015). Taati (2016) derived an analytical solution of the functionally graded microplate in the context of buckling and post-buckling. Guo et al. (2016) obtained three-dimensional analytical solutions of layer-based composite plates considering MCST. Alinaghizadeh et al. (2017) investigated the bending response of the annular sector of functionally graded microplate based on MCST. Devi and Kumar (2018) studied the damping and frequency shift of the Kirchhoff-Love plate resonator via MCST. Borjalilou and Asghari (2018) investigated the TED of microplate for DPL thermoelasticity under MCST. Kumar and Mukhopadhyay (2020) considered the Silicon microplate resonator to study size-dependent TED in the frame of TPL and modified couple stress theories. Ma et al. (2020) used the finite element formulation to study the free vibration of composite laminated Reddy plate based on the MCST. A laminated microplate shear deformation theory based on the MCST was proposed by He et al. (2021). Kaur and Singh (2021) studied the TED, time-harmonic displacement, and temperature in isotropic circular Kirchhoff plate considering GN III theory. Wang et al. (2022) derived the analytical solution of viscoelastic nanoplate to capture bending analysis based on the MCST. Fu et al. (2023) explained the bending and buckling of the partially covered laminated micro components. As a generalization of the classical nonlocal elasticity theory, the nonlocal strain gradient theory (NSGT) introduces a strain tensor with a nonlocality effect into the stored energy function. Li and Hu (2015) derived the buckling analysis of linear beam analytically in the context of NSGT. The investigation on TED under NSGT as reported by Lu et al. (2017; 2019), Ebrahimi and Barati (2017a; 2017b), Li et al. (2017), Ge et al. (2021) and Yue et al. (2021) are also worth to be mentioned here. One can further refer to the analysis of buckling responses of nano beams with nonlocal effects by NSGT as reported by Jena et al. (2021; 2022) who implemented numerical methods for their study. Picard (2016) and Picard and Watson (2018) studied the some problem on boundary damping analysis for inhomogeneous Timoshenko. beam. Abouelregal et al. (2023) introduced the new thermal vibrational model using DPL thermoelastic model under nonlocal elasticity theory. Marin et al. (2020) studied the some mixed problems in SGT with the Lagrange identity. Chakraverty and Pradhan (2014) investigated the free vibration for rectangular plate within the frame work of thermal environment. In this respect, we also refer the work by Gupta et al. (2023a; 2023b) and Steigmann (2007; 2008).

1.6 Objective of the Thesis

The objective of the present thesis is aimed at broadening the theoretical analysis of the thermoelasticity theory based on the Moore-Gibson-Thompson heat conduction model proposed by Quintanilla (2019). The thesis primarily begins by analyzing the Galerkin-type representation of the field equations in terms of elementary function (metaharmonic). It is well known that the Galerkin-type representation plays a dominant role in probing various challenges of contemporary mathematical analysis of different boundary value problems. These Galerkin-type representations are the foundations to obtain the fundamental solution in the studies of elasticity and thermoelasticity theories. Using these Galerkin-type solutions, the thesis deals with the formulation of the fundamental solution for Moore-Gibson-Thompson thermoelasticity theory for the distributions of displacement components and temperature for two cases: concentrated body force and concentrated heat source. Also, the present thesis attempts to estimate the thermoelastic damping (TED) in micro- and nano-mechanical resonators by utilizing non-conventional theory to take into account of the size effects in the context of the recently proposed Moore-Gibson-Thompson (MGT) generalized thermoelasticity theory. This thesis examines the prediction of bending and vibration characteristics of the Timoshenko beam and Euler-Bernoulli beam using Moore-Gibson-Thompson thermoelasticity theory in conjunction with MSCT and NSGT. The impact of material length-scale parameters on TED for the micro-beam resonator under the modified couple stress theory (MCST) and MGT theory is investigated. Finally, the influences of various parameters including size effects on vibration and TED in micro/nano resonators in the present context are investigated by carrying out the numerical computation of the present analytical results.