Chapter 5

Two semi-infinite moving cracks situated at two different interfaces of four semi-infinite orthotropic strips in a composite medium

5.1 Introduction

In fracture mechanics, many crack propagation problems can be converted to the W-H equation in a complex transformed plane. The W-H Technique approach implemented in this chapter, was initially described by B. Noble in his book [113], and since then, it has been extensively studied by other scholars and engineers. The hardest part of this method is factorizing the kernel. Nilsson [114,115] has provided a method for determining the asymptotic expression of SIF by using the W-H methodology without the kernel's visible factorization, which was revolutionary and used in several research articles. Meanwhile, the applications of the W-H technique are further studied by Abrahams [116]. The use of W-H technique for a semi-infinte crack for square lattice has been studied by Sharma [118]. SIF for a moving semi-infinite crack has been studied by Basak and Mandal [119]. Furthermore, the scattering effect on two semi-infinite cracks has been studied by Maurya and Sharma [117]. Ustinov et al. [120] studied the effect of arbitrary loading on a central occurred semi-infinite crack using the W-H Technique.

In this chapter, the problem of two moving interfacial semi-infinite cracks at different interfaces under different normal loadings have been converted into a standard W-H equation by applying defined Fourier transformation. The asymptotic analytical expressions of SIFs and CODs have been determined using the W-H technique. The graphical representation of SIFs and CODs justifies the physical nature of the problem for various values of depths of strips and crack velocities. The model of two interfacial semi-infinite cracks moving from opposite directions at different interfaces has been solved for the very first time in this chapter. Thus, the problem is the first of its kind where the W-H technique has been used to find SIF and COD for a four-layered orthotropic strip system weakened by two moving semi-infinite cracks.

5.2 Mathematical problem formulation

Let us consider two semi infinite cracks ($h < X < \infty, Y = h$ and $-\infty < X < -h, Y = -h$) situated at the interfaces of two different semi infinite strips of different orthotropic materials. Consider the first crack is propogating with a constant velocity c, along the positive X-axis and the second crack is propogating in the direction along negative X-axis with same constant velocity c. At any time t, the position of the first crack is $ct < X < \infty$, $Y = h$, the position for second crack is $-\infty < X < ct$, $Y = -h$ as depicted through Fig.5.1.

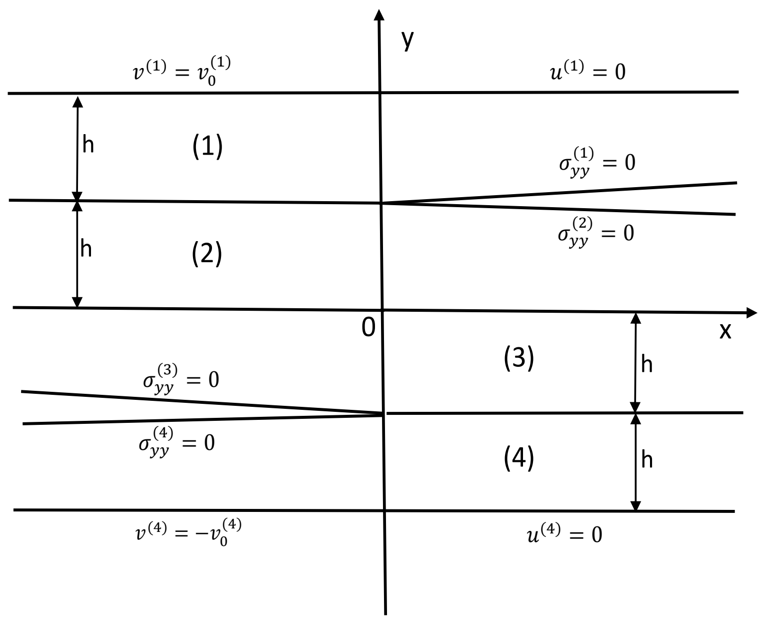

Figure 5.1: Geometry of the original problem

The equation of motion for the displacement components $U^{(j)}(X, Y, t)$ and $V^{(j)}(X, Y, t)$, along X and Y directions, respectively is as follows

$$C_{11}^{(j)}\frac{\partial^2 U^{(j)}}{\partial X^2}+\mu_{12}^{(j)}\frac{\partial^2 U^{(j)}}{\partial Y^2}+(C_{12}^{(j)}+\mu_{12}^{(j)})\frac{\partial^2 V^{(j)}}{\partial X\partial Y}=\rho^{(j)}\frac{\partial^2 U^{(j)}}{\partial t^2},$$

$$\mu_{12}^{(j)}\frac{\partial^2 V^{(j)}}{\partial X^2}+C_{22}^{(j)}\frac{\partial^2 V^{(j)}}{\partial Y^2}+(C_{12}^{(j)}+\mu_{12}^{(j)})\frac{\partial^2 U^{(j)}}{\partial X\partial Y}=\rho^{(j)}\frac{\partial^2 V^{(j)}}{\partial t^2}, \tag{5.2.1}$$

where $C_{11}^{(j)}, C_{12}^{(j)}, C_{22}^{(j)}$ and $\mu_{12}^{(j)}$ are material constants and $\rho^{(j)}$ are material densities and $j = 1, 2, 3$ and 4, correspond to the medium-1, 2, 3 and 4, respectively, are related by the relation $C_{12}^{(j)} = \nu_{21}^{(j)} C_{11}^{(j)}$, where $\nu_{21}^{(j)}$ are Poisson's ratios.

The following transformation has been used to make the system free from t, such as $X = x - ct, Y = y$ and $t = t$, we get

$$(C_{11}^{(j)}-\rho^{(j)}c^2)\frac{\partial^2 u^{(j)}}{\partial x^2}+\mu_{12}^{(j)}\frac{\partial^2 u^{(j)}}{\partial y^2}+(C_{12}^{(j)}+\mu_{12}^{(j)})\frac{\partial^2 v^{(j)}}{\partial x\partial y}=0,$$

$$(\mu_{12}^{(j)}-\rho^{(j)}c^2)\frac{\partial^2 v^{(j)}}{\partial x^2}+C_{22}^{(j)}\frac{\partial^2 v^{(j)}}{\partial y^2}+(C_{12}^{(j)}+\mu_{12}^{(j)})\frac{\partial^2 u^{(j)}}{\partial x\partial y}=0, \tag{5.2.2}$$

where $u^{(j)}(x, y) = U^{(j)}(X, Y, t)$ and $v^{(j)}(x, y) = V^{(j)}(X, Y, t)$ are displacement components. The expressions for stresses related to displacement components are as follows:

$$\sigma_{yy}^{(j)} = C_{12}^{(j)}\frac{\partial u^{(j)}}{\partial x}+C_{22}^{(j)}\frac{\partial v^{(j)}}{\partial y}, \tag{5.2.3}$$

$$\sigma_{xy}^{(j)} = \mu_{12}^{(j)}\left(\frac{\partial u^{(j)}}{\partial y}+\frac{\partial v^{(j)}}{\partial x}\right), \qquad j = 1, 2, 3, 4. \tag{5.2.4}$$

The boundary and continuity conditions for the problem are follows as

$$
\begin{aligned}
&\sigma_{yy}^{(1)}(x,h+)=\sigma_{yy}^{(2)}(x,h-)=0, && x>0,\\
&\sigma_{xy}^{(1)}(x,h+)=\sigma_{xy}^{(2)}(x,h-)=0, && x>0,\\
&\sigma_{yy}^{(2)}(x,0)=\sigma_{yy}^{(3)}(x,0), && -\infty<x<\infty,\\
&\sigma_{xy}^{(2)}(x,0)=\sigma_{xy}^{(3)}(x,0), && -\infty<x<\infty,\\
&\sigma_{yy}^{(3)}(x,-h+)=\sigma_{yy}^{(4)}(x,-h-)=0, && x<0,\\
&\sigma_{xy}^{(3)}(x,-h+)=\sigma_{xy}^{(4)}(x,-h-)=0, && x<0,\\
&u^{(1)}(x,h)=u^{(2)}(x,h), && -\infty<x<0,\\
&v^{(1)}(x,h)=v^{(2)}(x,h), && -\infty<x<0,\\
&u^{(2)}(x,0)=u^{(3)}(x,0), && -\infty<x<\infty,\\
&v^{(2)}(x,0)=v^{(3)}(x,0), && -\infty<x<\infty,\\
&u^{(3)}(x,-h)=u^{(4)}(x,-h), && 0<x<\infty,\\
&v^{(3)}(x,-h)=v^{(4)}(x,-h), && 0<x<\infty,\\
&u^{(1)}(x,2h)=0, && -\infty<x<\infty,\\
&v^{(1)}(x,2h)=v_0^{(1)}, && -\infty<x<\infty,\\
&u^{(4)}(x,-2h)=0, && -\infty<x<\infty,\\
&v^{(4)}(x,-2h)=-v_0^{(4)}, && -\infty<x<\infty, \qquad (5.2.5)
\end{aligned}
$$

where $v_0^{(1)}$ and $v_0^{(4)}$ are the displacement constants. The boundary and continuity conditions need to be slightly modified to use the W-H Technique by superimposing normal constant loadings $\sigma_0^{(1)}$ and $\sigma_0^{(3)}$ on first and second crack surfaces, respectively. Hence the modified boundary and continuity conditions are as follows:

$$\sigma_{yy}^{(1)}(x,h+)=\sigma_{yy}^{(2)}(x,h-)=\sigma_0^{(1)}, \quad x>0, \tag{5.2.6}$$

$$\sigma_{xy}^{(1)}(x,h+)=\sigma_{xy}^{(2)}(x,h-)=0, \quad x>0, \tag{5.2.7}$$

$$\sigma_{yy}^{(2)}(x,0)=\sigma_{yy}^{(3)}(x,0), \quad -\infty<x<\infty, \tag{5.2.8}$$

$$\sigma_{xy}^{(2)}(x,0)=\sigma_{xy}^{(3)}(x,0), \quad -\infty<x<\infty, \tag{5.2.9}$$

$$\sigma_{yy}^{(3)}(x,-h+)=\sigma_{yy}^{(4)}(x,-h-)=-\sigma_0^{(3)}, \quad x<0, \tag{5.2.10}$$

$$\sigma_{xy}^{(3)}(x,-h+)=\sigma_{xy}^{(4)}(x,-h-)=0, \quad x<0, \tag{5.2.11}$$

$$u^{(1)}(x,h)=u^{(2)}(x,h), \quad -\infty<x<0, \tag{5.2.12}$$

$$v^{(1)}(x,h)=v^{(2)}(x,h), \quad -\infty<x<0, \tag{5.2.13}$$

$$u^{(2)}(x,0) = u^{(3)}(x,0), \qquad -\infty < x < \infty, \tag{5.2.14}$$
$$v^{(2)}(x,0) = v^{(3)}(x,0), \qquad -\infty < x < \infty, \tag{5.2.15}$$
$$u^{(3)}(x,-h) = u^{(4)}(x,-h), \qquad 0 < x < \infty, \tag{5.2.16}$$
$$v^{(3)}(x,-h) = v^{(4)}(x,-h), \qquad 0 < x < \infty, \tag{5.2.17}$$
$$u^{(1)}(x,2h) = 0, \qquad -\infty < x < \infty, \tag{5.2.18}$$
$$v^{(1)}(x,2h) = 0, \qquad -\infty < x < \infty, \tag{5.2.19}$$
$$u^{(4)}(x,-2h) = 0, \qquad -\infty < x < \infty, \tag{5.2.20}$$
$$v^{(4)}(x,-2h) = 0, \qquad -\infty < x < \infty. \tag{5.2.21}$$

The values of $v_0^{(1)}$, $v_0^{(4)}, \sigma_0^{(1)}$ and $\sigma_0^{(3)}$ are should be picked in a manner that the normal loadings on the crack surfaces are uniform, as follows

$$(C_{22}^{(1)} + \nu_{21}^{(1)} C_{12}^{(1)}) v_0^{(1)} = (C_{22}^{(2)} + \nu_{21}^{(2)} C_{12}^{(2)}) v_0^{(2)}, \tag{5.2.22}$$

Addionally, the specific values of $\sigma_0^{(1)}$ and $\sigma_0^{(3)}$, as given by Georgiadis and Papadopoulos [107], the two sets of boundary conditions previously defined refer to the same cracked problem are as follows:

$$\sigma_0^{(1)} = -\frac{(C_{22}^{(1)} + \nu_{21}^{(1)} C_{12}^{(1)}) v_0^{(1)}}{h} = -\frac{(C_{22}^{(2)} + \nu_{21}^{(2)} C_{12}^{(2)}) v_0^{(2)}}{h}, \tag{5.2.23}$$
$$\sigma_0^{(3)} = -\frac{(C_{22}^{(3)} + \nu_{21}^{(3)} C_{12}^{(3)}) v_0^{(3)}}{h} = -\frac{(C_{22}^{(4)} + \nu_{21}^{(4)} C_{12}^{(4)}) v_0^{(4)}}{h}. \tag{5.2.24}$$

Now the boundary conditions (5.2.6) and (5.2.10) can be modified as

$$\sigma_{yy}^{(1)}(x,h) = \sigma_0^{(1)} e^{\epsilon_1 x}, \qquad x > 0, \tag{5.2.25}$$
$$\sigma_{yy}^{(3)}(x,-h) = -\sigma_0^{(3)} e^{\epsilon_3 x}, \qquad x < 0, \tag{5.2.26}$$

where $\epsilon_1$and ϵ_3 are very small positive quantites, which are tending to zero.

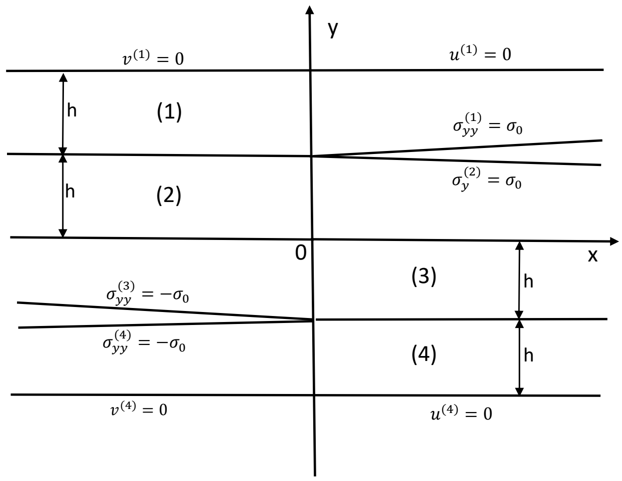

Figure 5.2: Geometry of the transformed problem

The solutions of Eq.(5.2.2) after applying the Fourier transformation by Eq.(1.4.1) are given by

$$\overline{u}^{(j)}(\xi, y) = A^{(j)}(\xi)e^{\gamma_1^{(j)}\xi y} + B^{(j)}(\xi)e^{-\gamma_1^{(j)}\xi y} + C^{(j)}(\xi)e^{\gamma_2^{(j)}\xi y} + D^{(j)}(\xi)e^{-\gamma_2^{(j)}\xi y}, \quad (5.2.27)$$

$$\begin{aligned}\overline{v}^{(j)}(\xi, y) &= i\ \alpha_1^{(j)}(A^{(j)}(\xi)e^{\gamma_1^{(j)}\xi y} - B^{(j)}(\xi)e^{-\gamma_1^{(j)}\xi y}) \\ &+ i\ \alpha_2^{(j)}(C^{(j)}(\xi)e^{\gamma_2^{(j)}\xi y} - D(\xi)e^{-\gamma_2^{(j)}\xi y}), \end{aligned} \quad (5.2.28)$$

where $A^{(j)}(\xi), B^{(j)}(\xi), C^{(j)}(\xi)$ and $D^{(j)}(\xi)$ are the unknown functions and $\alpha_k^{(j)} = \frac{C_{11}^{(j)} - \mu_{12}^{(j)}\gamma_k^{(j)2} - \rho^{(j)}c^2}{(C_{12}^{(j)} + \mu_{12}^{(j)})\gamma_k^{(j)}}$, $k = 1, 2$ and $(\gamma_1^{(j)})^2$, $(\gamma_2^{(j)})^2$ are the positive roots of the following equation

$$\begin{aligned}\mu_{12}^{(j)}C_{22}^{(j)}\gamma^{(j)4} &+ \left\{C_{12}^{(j)2} + 2C_{12}^{(j)}\mu_{12}^{(j)} - C_{11}^{(j)}C_{12}^{(j)} + \left(\mu_{12}^{(j)} + C_{22}^{(j)}\right)\rho^{(j)}c^2\right\}\gamma^{(j)2} \\ &+ (C_{11}^{(j)} - \rho^{(j)}c^2)(\mu_{12}^{(j)} - \rho^{(j)}c^2) = 0. \end{aligned} \quad (5.2.29)$$

The expressions for the stresses after taking Fourier transform are given as

$$\begin{aligned}\overline{\sigma}_{yy}^{(j)}(\xi, y) &= i\xi(C_{22}^{(j)}\ \alpha_1^{(j)}\gamma_1^{(j)} - C_{12}^{(j)})\ (A^{(j)}(\xi)e^{\gamma_1^{(j)}\xi y} + B^{(j)}(\xi)e^{-\gamma_1^{(j)}\xi y}) \\ &+ i\xi(C_{22}^{(j)}\ \alpha_2^{(j)}\gamma_2^{(j)} - C_{12}^{(j)})\ (C^{(j)}(\xi)e^{\gamma_2^{(j)}\xi y} + D^{(j)}(\xi)e^{-\gamma_2^{(j)}\xi y}), \end{aligned} \quad (5.2.30)$$

$$\begin{aligned}\overline{\sigma}_{xy}^{(j)}(\xi, y) &= \xi\mu_{12}^{(j)}(\alpha_1^{(j)} + \gamma_1^{(j)})(A^{(j)}(\xi)e^{\gamma_1^{(j)}\xi y} - B^{(j)}(\xi)e^{-\gamma_1^{(j)}\xi y}) \\ &+ \xi\mu_{12}^{(j)}(\alpha_2^{(j)} + \gamma_2^{(j)})(C^{(j)}(\xi)e^{\gamma_2^{(j)}\xi y} - D^{(j)}(\xi)e^{-\gamma_2^{(j)}\xi y}). \end{aligned} \quad (5.2.31)$$

5.3 Solution of the mathematical problem

Using the boundary and continuity conditions from (5.2.12)-(5.2.21) to reduce the unknowns into $A^{(j)}(\xi)'s$ and $B^{(j)}(\xi)'s$,we get

$$C^{(1)}(\xi) = a_1 A^{(1)}(\xi) + b_1 B^{(1)}(\xi), \tag{5.3.1}$$

$$D^{(1)}(\xi) = a_2 A^{(1)}(\xi) + b_2 B^{(1)}(\xi), \tag{5.3.2}$$

$$C^{(2)}(\xi) = a_7 A^{(1)}(\xi) + b_7 B^{(1)}(\xi) + a_8 A^{(2)}(\xi) + b_8 B^{(2)}(\xi), \tag{5.3.3}$$

$$D^{(2)}(\xi) = a_9 A^{(1)}(\xi) + b_9 B^{(1)}(\xi) + a_{10} A^{(2)}(\xi) + b_{10} B^{(2)}(\xi), \tag{5.3.4}$$

$$\begin{aligned} C^{(3)}(\xi) = {} & a_{15} A^{(1)}(\xi) + b_{15} B^{(1)}(\xi) + a_{16} A^{(2)}(\xi) + b_{16} B^{(2)}(\xi) \\ & + a_{17} A^{(3)}(\xi) + b_{17} B^{(3)}(\xi), \end{aligned} \tag{5.3.5}$$

$$\begin{aligned} D^{(3)}(\xi) = {} & a_{18} A^{(1)}(\xi) + b_{18} B^{(1)}(\xi) + a_{19} A^{(2)}(\xi) + b_{19} B^{(2)}(\xi) \\ & + a_{20} A^{(3)}(\xi) + b_{20} B^{(3)}(\xi), \end{aligned} \tag{5.3.6}$$

$$C^{(4)}(\xi) = a_3 A^{(4)}(\xi) + b_3 B^{(4)}(\xi), \tag{5.3.7}$$

$$D^{(4)}(\xi) = a_4 A^{(4)}(\xi) + b_4 B^{(4)}(\xi). \tag{5.3.8}$$

Now Using the conditions from (5.2.7)-(5.2.11) to reduce the rest of the unknowns in terms of $A^{(1)}(\xi)$ are obtained as

$$A^{(2)}(\xi) = \frac{(a_{29}b_{32} - a_{31}b_{30})A^{(1)}(\xi) + (b_{29}b_{32} - b_{31}b_{30})B^{(1)}(\xi)}{(a_{32}b_{30} - a_{30}b_{32})}, \tag{5.3.9}$$

$$B^{(2)}(\xi) = \frac{(a_{29}a_{32} - a_{31}a_{30})A^{(1)}(\xi) + (b_{29}a_{32} - b_{31}a_{30})B^{(1)}(\xi)}{(b_{32}a_{30} - b_{30}a_{32})}, \tag{5.3.10}$$

$$A^{(3)}(\xi) = \frac{(a_{35}b_{38} - a_{37}b_{36})A^{(1)}(\xi) + (b_{35}b_{38} - b_{37}b_{36})B^{(1)}(\xi)}{(a_{38}b_{36} - a_{36}b_{38})}, \tag{5.3.11}$$

$$B^{(3)}(\xi) = \frac{(a_{35}a_{38} - a_{37}a_{36})A^{(1)}(\xi) + (b_{35}a_{38} - b_{37}a_{36})B^{(1)}(\xi)}{(b_{38}a_{36} - b_{36}a_{38})}, \tag{5.3.12}$$

$$A^{(4)}(\xi) = \frac{(a_{41}b_{44} - a_{43}b_{42})A^{(1)}(\xi) + (b_{41}b_{44} - b_{43}b_{42})B^{(1)}(\xi)}{(a_{44}b_{42} - a_{42}b_{44})}, \tag{5.3.13}$$

$$B^{(4)}(\xi) = \frac{(a_{41}a_{44} - a_{43}a_{42})A^{(1)}(\xi) + (b_{41}a_{44} - b_{43}a_{42})B^{(1)}(\xi)}{(b_{44}a_{42} - b_{42}a_{44})}, \tag{5.3.14}$$

$$B^{(1)}(\xi) = \frac{-(a_{21} + a_{22}a_{33} + b_{22}a_{34} + a_{23}a_{39} + b_{23}a_{40} + a_{24}a_{45} + b_{24}a_{46})A^{(1)}(\xi)}{(b_{21} + a_{22}b_{33} + b_{22}b_{34} + a_{23}b_{39} + b_{23}b_{40} + a_{24}b_{45} + b_{24}b_{46})}, \tag{5.3.15}$$

The values of all the unknown coefficients involved here can be found in Appendix A. Let us define $\sigma_{yy}^{(1)}(x,h)$, $\sigma_{yy}^{(3)}(x,-h)$ and displacements in $y - axis$ as

$$\sigma_{yy}^{(1)}(x,h) = m^{(1)}(x), \qquad x < 0, \tag{5.3.16}$$

$$v^{(1)}(x,h) - v^{(2)}(x,h) = n^{(1)}(x), \qquad x > 0, \tag{5.3.17}$$

$$\sigma_{yy}^{(3)}(x,-h) = m^{(3)}(x), \qquad x > 0, \tag{5.3.18}$$

$$v^{(3)}(x,-h) - v^{(4)}(x,-h) = n^{(3)}(x), \qquad x < 0, \tag{5.3.19}$$

with the Fourier transforms as

$$\overline{m}_{-}^{(1)}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} m^{(1)}(x) e^{i\xi x} dx, \tag{5.3.20}$$

$$\overline{n}_{+}^{(1)}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} n^{(1)}(x) e^{i\xi x} dx, \tag{5.3.21}$$

$$\overline{m}_{+}^{(3)}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} m^{(3)}(x) e^{i\xi x} dx, \tag{5.3.22}$$

$$\overline{n}_{-}^{(3)}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} n^{(3)}(x) e^{i\xi x} dx. \tag{5.3.23}$$

The stresses and displacements are known to be bounded at infinity according to the physical nature of the problem. Hence we have

$$|m^{(1)}(x)| < M_1|x|^{-lm_1}, \qquad as \quad x \to -\infty, \tag{5.3.24}$$

$$|n^{(1)}(x)| < N_1|x|^{-ln_1}, \qquad as \quad x \to -\infty, \tag{5.3.25}$$

$$|m^{(3)}(x)| < M_3 x^{-lm_3}, \qquad as \quad x \to \infty, \tag{5.3.26}$$

$$|n^{(3)}(x)| < N_3 x^{-ln_3}, \qquad as \quad x \to \infty, \tag{5.3.27}$$

for some $lm_1, lm_3, ln_1, ln_3 > 0$ with $M_1, M_3, N_1$and N_3 are finite positive numbers. These conditions ensure that Eqs.(5.3.20)-(5.3.23) exist. It is also seen that $\overset{-(1)}{m_-}(\xi)$, $\overset{-(3)}{m_+}(\xi)$, $\overset{-(1)}{n_+}(\xi)$ and $\overset{-(3)}{n_-}(\xi)$ are analytic for $\tau \leq 0$, $\tau \geq 0$, $\tau \geq 0$ and $\tau \leq 0$.

5.3.1 Wiener-Hopf technique

The expressions of stresses and displacements with the help of Eq.(5.2.25)-(5.2.26) and (5.3.16)-(5.3.19), we get

$$\overline{\sigma}^{(1)}_{yy}(\xi, h) = \overline{m}^{(1)}_-(\xi) + \frac{\sigma_0^{(1)}}{\sqrt{2\pi}(\epsilon_1 + i\xi)}, \tag{5.3.28}$$

$$\overline{v}^{(1)}(\xi, h) - \overline{v}^{(3)}(\xi, h) = \overline{n}^{(1)}_+(\xi), \tag{5.3.29}$$

$$\overline{\sigma}^{(3)}_{yy}(\xi, -h) = \overline{m}^{(3)}_+(\xi) - \frac{\sigma_0^{(3)}}{\sqrt{2\pi}(\epsilon_3 + i\xi)}, \tag{5.3.30}$$

$$\overline{v}^{(3)}(\xi, -h) - \overline{v}^{(4)}(\xi, -h) = \overline{n}^{(3)}_-(\xi). \tag{5.3.31}$$

We get the following Wiener-Hopf Equations for both the cracks after some mathematical computations by using Eqs.(5.3.1)-(5.3.15) and relations from Eqs.(5.3.28)-(5.3.31) as

$$\overline{m}^{(1)}_-(\xi) = K^{(1)}(\xi)\overline{n}^{(1)}_+(\xi) - \frac{\sigma_0^{(1)}}{\sqrt{2\pi}(\epsilon_1 + i\xi)}, \tag{5.3.32}$$

$$\overline{m}^{(3)}_+(\xi) = K^{(3)}(\xi)\overline{n}^{(3)}_-(\xi) + \frac{\sigma_0^{(3)}}{\sqrt{2\pi}(\epsilon_3 + i\xi)}, \tag{5.3.33}$$

where $K^{(1)}(\xi) = \xi k_1^{(1)}(\xi)/k_2^{(1)}(\xi)$ and $K^{(3)}(\xi) = \xi k_1^{(3)}(\xi)/k_2^{(3)}(\xi)$ and

$$
\begin{aligned}
k_1^{(1)}(\xi) &= (C_{22}^{(1)}\,\alpha_1^{(1)}\gamma_1^{(1)} - C_{12}^{(1)})(e^{\gamma_1^{(1)}\xi h} + a_{47}e^{-\gamma_1^{(1)}\xi h}) + (C_{22}^{(1)}\,\alpha_2^{(1)}\gamma_2^{(1)} \\
&\quad - C_{12}^{(1)})((a_1 + b_1 a_{47})e^{\gamma_2^{(1)}\xi h} + (a_2 + b_2 a_{47})e^{-\gamma_2^{(1)}\xi h}),
\end{aligned} \tag{5.3.34}
$$

$$
\begin{aligned}
k_2^{(1)}(\xi) &= \alpha_1^{(1)}(e^{\gamma_1^{(1)}\xi h} + a_{47}e^{-\gamma_1^{(1)}\xi h}) + \alpha_2^{(1)}((a_1 + a_{47}b_1)e^{\gamma_2^{(1)}\xi h} - (a_2 + b_2 a_{47})e^{-\gamma_2^{(1)}\xi h}) \\
&\quad - \alpha_1^{(2)}((a_{33} + b_{33}a_{47})e^{\gamma_1^{(2)}\xi h} - (a_{34} + b_{34}a_{47})e^{-\gamma_1^{(2)}\xi h}) - \alpha_2^{(2)}((a_7 + b_7 a_{47} \\
&\quad + a_8(a_{33} + b_{33}a_{47}) + b_8(a_{34} + b_{34}a_{47}))e^{\gamma_2^{(2)}\xi h} - (a_9 + b_9 a_{47} + a_{10}(a_{33} + b_{33}a_{47}) \\
&\quad + b_{10}(a_{34} + b_{34}a_{47}))e^{\gamma_2^{(2)}\xi h},
\end{aligned} \tag{5.3.35}
$$

$$
\begin{aligned}
k_1^{(3)}(\xi) &= i\omega(C_{22}^{(3)}\,\alpha_1^{(3)}\gamma_1^{(3)} - C_{12}^{(3)})(e^{-\gamma_1^{(3)}\xi h}(a_{39} + b_{39}a_{47}) + e^{\gamma_1^{(3)}\xi h}(a_{40} + b_{40}a_{47})) \\
&\quad + i\omega(C_{22}^{(3)}\,\alpha_2^{(3)}\gamma_2^{(3)} - C_{12}^{(3)})(e^{-\gamma_2^{(3)}\xi h}(a_{15} + b_{15}a_{47} + a_{16}(a_{33} + b_{33}a_{47}) + b_{16}(a_{34} \\
&\quad + b_{34}a_{47}) + a_{17}(a_{39} + b_{39}a_{47}) + b_{17}(a_{40} + b_{40}a_{47})) + e^{\gamma_2^{(3)}\xi h}(a_{18} + b_{18}a_{47} \\
&\quad + a_{19}(a_{33} + b_{33}a_{47}) + b_{19}(a_{34} + b_{34}a_{47}) + a_{20}(a_{39} + b_{39}a_{47}) + b_{20}(a_{40} + b_{40}a_{47}))),
\end{aligned} \tag{5.3.36}
$$

$$
\begin{aligned}
k_2^{(3)}(\xi) &= i\alpha_1^{(3)}((a_{39} + b_{39}a_{47})e^{-\gamma_1^{(3)}\xi h} - (a_{40} + b_{40}a_{47})e^{\gamma_1^{(3)}\xi h}) + i\alpha_2^{(3)}(e^{-\gamma_2^{(3)}\xi h}(a_{15} \\
&\quad + b_{15}a_{47} + a_{16}(a_{33} + b_{33}a_{47}) + b_{16}(a_{34} + b_{34}a_{47}) + a_{17}(a_{39} + b_{39}a_{47}) + b_{17}(a_{40} \\
&\quad + b_{40}a_{47})) - e^{\gamma_2^{(3)}\xi h}(a_{18} + b_{18}a_{47} + a_{19}(a_{33} + b_{33}a_{47}) + b_{19}(a_{34} + b_{34}a_{47}) \\
&\quad + a_{20}(a_{39} + b_{39}a_{47}) + b_{20}(a_{40} + b_{40}a_{47}))) - i\alpha_1^{(4)}((a_{45} + b_{45}a_{47})e^{-\gamma_1^{(4)}\xi h} \\
&\quad - (a_{46} + b_{46}a_{47})e^{\gamma_1^{(4)}\xi h}) - i\alpha_2^{(4)}(a_3(a_{45} + b_{45}a_{47}) + b_3(a_{46} + b_{46}a_{47}))e^{-\gamma_2^{(4)}\xi h} \\
&\quad - (a_4(a_{45} + b_{45}a_{47}) + b_4(a_{46} + b_{46}a_{47}))e^{\gamma_2^{(4)}\xi h}).
\end{aligned} \tag{5.3.37}
$$

The Eqs.(5.3.32)-(5.3.33) represent the standard form of Wiener-Hopf Equation.In order to solve these equations the factorization of both the kernals $K^{(1)}(\xi)$ and $K^{(3)}(\xi)$ are given as

$$
K^{(1)}(\xi) = k_+^{(1)}(\xi)k_-^{(1)}(\xi), \tag{5.3.38}
$$

$$
K^{(3)}(\xi) = k_+^{(3)}(\xi)k_-^{(3)}(\xi), \tag{5.3.39}
$$

where $k_+^{(1)}(\xi)$, $k_+^{(3)}(\xi)$ are non zero analytic in upper half plane for any $\tau \geq 0$ and $k_-^{(1)}(\xi)$, $k_-^{(3)}(\xi)$ are non zero analytic in lower half plane for any $\tau \leq 0$.

The Eqs.(5.3.32)-(5.3.33) can be rewritten by using the factors from Eqs.(5.3.38)-

(5.3.39), as

$$\frac{\overline{m}_-^{(1)}(\xi)}{k_-^{(1)}(\xi)} = k_+^{(1)}(\xi)\overline{n}_+^{(1)}(\xi) - \frac{\sigma_0^{(1)}}{\sqrt{2\pi}(\epsilon_1 + i\xi)k_-^{(1)}(\xi)}, \tag{5.3.40}$$

$$\frac{\overline{m}_+^{(3)}(\xi)}{k_+^{(3)}(\xi)} = k_-^{(3)}(\xi)\overline{n}_-^{(3)}(\xi) + \frac{\sigma_0^{(3)}}{\sqrt{2\pi}(\epsilon_3 + i\xi)k_+^{(3)}(\xi)}. \tag{5.3.41}$$

Now, we decompose the last term in the RHS of the Eqs.(5.3.40) and (5.3.41) as

$$\frac{\sigma_0^{(1)}}{\sqrt{2\pi}(\epsilon_1 + i\xi)k_-^{(1)}(\xi)} = H^{(1)}(\xi) = H_+^{(1)}(\xi) + H_-^{(1)}(\xi), \tag{5.3.42}$$

$$\frac{\sigma_0^{(3)}}{\sqrt{2\pi}(\epsilon_3 + i\xi)k_+^{(3)}(\xi)} = H^{(3)}(\xi) = H_+^{(3)}(\xi) + H_-^{(3)}(\xi), \tag{5.3.43}$$

where

$$H_-^{(1)}(\xi) = \frac{\sigma_0^{(1)}}{\sqrt{2\pi}(\epsilon_1 + i\xi)}\left(\frac{1}{k_-^{(1)}(\xi)} - \frac{1}{k_-^{(1)}(i\epsilon_1)}\right), \tag{5.3.44}$$

$$H_+^{(1)}(\xi) = \frac{\sigma_0^{(1)}}{\sqrt{2\pi}(\epsilon_1 + i\xi)k_-^{(1)}(i\epsilon_1)}, \tag{5.3.45}$$

$$H_+^{(3)}(\xi) = \frac{\sigma_0^{(3)}}{\sqrt{2\pi}(\epsilon_3 + i\xi)}\left(\frac{1}{k_+^{(3)}(\xi)} - \frac{1}{k_+^{(3)}(i\epsilon_3)}\right), \tag{5.3.46}$$

$$H_-^{(3)}(\xi) = \frac{\sigma_0^{(3)}}{\sqrt{2\pi}(\epsilon_3 + i\xi)k_+^{(3)}(i\epsilon_3)}, \tag{5.3.47}$$

where $H_+^{(1)}(\xi)$ and $H_+^{(3)}(\xi)$ are non zero analytic for any $\tau > \tau_0(\tau_0 \geq 0)$, and $H_-^{(1)}(\xi)$ and $H_-^{(3)}(\xi)$ are non zero analytic for any $\tau < \tau_0(\tau_0 \leq 0)$.

The Eqs.(5.3.40) and (5.3.41) with the help of Eqs.(5.3.42)-(5.3.47) becomes

$$\frac{\overline{m}_{-}^{(1)}(\xi)}{k_{-}^{(1)}(\xi)} + H_{-}^{(1)}(\xi) = k_{+}^{(1)}(\xi)\overline{n}_{+}^{(1)}(\xi) - H_{+}^{(1)}(\xi), \tag{5.3.48}$$

$$\frac{\overline{m}_{+}^{(3)}(\xi)}{k_{+}^{(3)}(\xi)} - H_{+}^{(3)}(\xi) = k_{-}^{(3)}(\xi)\overline{n}_{-}^{(3)}(\xi) + H_{-}^{(3)}(\xi). \tag{5.3.49}$$

Here the LHS of Eq.(5.3.48) and RHS of Eq.(5.3.49) i.e., $\overline{m}_{-}^{(1)}(\xi)$, $k_{-}^{(1)}(\xi)$, $H_{-}^{(1)}(\xi)$, $\overline{n}_{-}^{(3)}(\xi)$, $k_{-}^{(3)}(\xi)$ and $H_{-}^{(3)}(\xi)$ are analytic in the lower half plane for $\tau < \tau_0(\tau_0 \leq 0)$. Again, the RHS of Eq.(5.3.48) and LHS of Eq.(5.3.49) i.e., $k_{+}^{(1)}(\xi)$, $\overline{n}_{+}^{(1)}(\xi)$, $H_{+}^{(1)}(\xi)$, $\overline{m}_{+}^{(3)}(\xi)$, $k_{+}^{(3)}(\xi)$ and $H_{+}^{(3)}(\xi)$ are analytic in the lower upper plane for $\tau > \tau_0(\tau_0 \geq 0)$. Hence, the common region of analyticity for all the functions is the line $\tau = 0$. Thus by using the analytic continuation it can be said that both the Eqs.(5.3.48) and (5.3.49) are single valued analytic in the entire complex plane and will be assumed to be equal to entire functions $P^{(1)}(\xi)$ and $P^{(3)}(\xi)$,respectively. The functions $\overline{m}_{-}^{(1)}(\xi)$, $\overline{n}_{+}^{(1)}(\xi)$, $\overline{m}_{+}^{(3)}(\xi)$ and $\overline{n}_{-}^{(3)}(\xi)$ are bounded for large value of ξ and $k_{+}^{(1)}(\xi)$, $k_{-}^{(1)}(\xi)$, $k_{+}^{(3)}(\xi)$ and $k_{-}^{(3)}(\xi)$ will tend to $\xi^{1/2}$. Hence, LHS and RHS of Eqs.(5.3.48) and Eq.(5.3.49) tend to $\xi^{-1/2}$, and RHS and LHS of Eq.(5.3.48) and Eq.(5.3.49) tend to $\xi^{1/2}$. Thus, the zero function is the only analytic function, as determined by the extended Liouville theorem theorem, to satisfy both sides of Eq.(5.3.48) as well as of Eq.(5.3.49). Hence, we get

$$P^{(1)}(\xi) = 0, \tag{5.3.50}$$

$$P^{(3)}(\xi) = 0. \tag{5.3.51}$$

Now by using Eqs.(5.3.48)-(5.3.49) with the help of Eqs.(5.3.50)-(5.3.51) and Eqs.(5.3.42)-(5.3.47), we get the following relations as

$$\overline{m}_{-}^{(1)}(\xi) = \frac{\sigma_0^{(1)}}{\sqrt{2\pi}(\epsilon_1 + i\xi)}\left(\frac{k_{-}^{(1)}(\xi)}{k_{-}^{(1)}(i\epsilon_1)} - 1\right), \tag{5.3.52}$$

$$\overline{n}_{+}^{(1)}(\xi) = \frac{\sigma_0^{(1)}}{\sqrt{2\pi}(\epsilon_1 + i\xi)k_{-}^{(1)}(i\epsilon_1)k_{+}^{(1)}(\xi)}, \tag{5.3.53}$$

$$\overline{m}_{+}^{(3)}(\xi) = \frac{-\sigma_0^{(3)}}{\sqrt{2\pi}(\epsilon_3 + i\xi)}\left(\frac{k_{+}^{(3)}(\xi)}{k_{+}^{(3)}(i\epsilon_3)} - 1\right), \tag{5.3.54}$$

$$\overline{n}_{-}^{(3)}(\xi) = \frac{-\sigma_0^{(3)}}{\sqrt{2\pi}(\epsilon_3 + i\xi)k_{+}^{(3)}(i\epsilon_3)k_{-}^{(3)}(\xi)}. \tag{5.3.55}$$

As per constant loading conditions, ϵ_1 and ϵ_3 will be tending to 0. Hence the above relations become

$$\overline{m}_{-}^{(1)}(\xi) = \frac{\sigma_0^{(1)}}{\sqrt{2\pi}(i\xi)}\left(\frac{k_{-}^{(1)}(\xi)}{k_{-}^{(1)}(0)} - 1\right), \tag{5.3.56}$$

$$\overline{n}_{+}^{(1)}(\xi) = \frac{\sigma_0^{(1)}}{\sqrt{2\pi}(i\xi)k_{-}^{(1)}(0)k_{+}^{(1)}(\xi)}, \tag{5.3.57}$$

$$\overline{m}_{+}^{(3)}(\xi) = \frac{-\sigma_0^{(3)}}{\sqrt{2\pi}(i\xi)}\left(\frac{k_{+}^{(3)}(\xi)}{k_{+}^{(3)}(0)} - 1\right), \tag{5.3.58}$$

$$\overline{n}_{-}^{(3)}(\xi) = \frac{-\sigma_0^{(3)}}{\sqrt{2\pi}(i\xi)k_{+}^{(3)}(0)k_{-}^{(3)}(\xi)}. \tag{5.3.59}$$

To find the values of SIFs without factorizing the kernals $K^{(1)}(\xi)$ and $K^{(3)}(\xi)$, we are using the method given by Nilsson [114, 115]. In this method by only knowing the values of kernals at small and large values of ξ, the expression for SIF can be found. Therefore, the asymptotic values for both the kernals are

$$\lim_{\xi\to 0} K^{(1)}(\xi) = \frac{C_{22}^{(1)}(2C_{12}^{(2)} - C_{22}^{(2)}(\alpha_1^{(2)}\gamma_1^{(2)} + \alpha_2^{(2)}\gamma_2^{(2)}))}{h(2C_{12}^{(2)} + C_{22}^{(1)}(\alpha_1^{(2)}\gamma_1^{(2)} - \alpha_2^{(2)}\gamma_2^{(2)}) - C_{22}^{(2)}(\alpha_1^{(2)}\gamma_1^{(2)} + \alpha_2^{(2)}\gamma_2^{(2)}))} = l_1^{(1)}, \tag{5.3.60}$$

$$\begin{aligned}\lim_{\xi\to\infty} \frac{K^{(1)}(\xi)}{\xi} &= (2(C_{12}^{(1)} - C_{22}^{(1)}\alpha_1^{(1)}\gamma_1^{(1)})(\alpha_2^{(2)}\gamma_1^{(2)} - \alpha_1^{(2)}\gamma_2^{(2)})(-2C_{12}^{(2)} + C_{22}^{(2)}\alpha_1^{(2)}\gamma_1^{(2)} \\ &+ C_{22}^{(2)}\alpha_2^{(2)}\gamma_2^{(2)})\mu_{12}^{(2)})/(\alpha_2^{(2)}((C_{12}^{(1)} - C_{22}^{(1)}\alpha_1^{(1)}\gamma_1^{(1)})(\alpha_2^{(2)}\gamma_1^{(2)} \\ &- \alpha_1^{(2)}\gamma_2^{(2)})\mu_{12}^{(2)} + C_{12}^{(2)}(2\alpha_1^{(2)}\gamma_1^{(1)}\mu_{12}^{(2)} + 2\alpha_1^{(1)}\alpha_1^{(2)}(\mu_{12}^{(1)} - \mu_{12}^{(2)}) \\ &- 2\alpha_1^{(1)}\gamma_1^{(2)}\mu_{12}^{(2)} - 3\alpha_2^{(2)}\gamma_1^{(2)}\mu_{12}^{(2)} + 3\alpha_1^{(2)}\gamma_2^{(2)}\mu_{12}^{(2)}) + C_{22}^{(2)}(2\alpha_2^{(2)2}\gamma_1^{(2)}\gamma_2^{(2)}\mu_{12}^{(2)} \\ &- \alpha_1^{(1)}(\alpha_1^{(2)}\gamma_1^{(2)} + \alpha_2^{(2)}\gamma_2^{(2)})(\alpha_1^{(2)}(\mu_{12}^{(1)} - \mu_{12}^{(2)}) - \gamma_1^{(2)}\mu_{12}^{(2)}) \\ &- \alpha_1^{(2)2}\gamma_1^{(2)}(\gamma_1^{(1)}\mu_{12}^{(1)} + \gamma_2^{(2)}\mu_{12}^{(2)}) + \alpha_1^{(1)}\alpha_1^{(2)}(-\gamma_1^{(1)}\gamma_2^{(2)}\mu_{12}^{(2)} \\ &+ \gamma_1^{(2)2}\mu_{12}^{(2)} - 2\gamma_2^{(2)2}\mu_{12}^{(2)})))) = l_2^{(1)},\end{aligned} \tag{5.3.61}$$

$$\lim_{\xi\to 0} K^{(3)}(\xi) = -(4(\gamma_1^{(3)}(C_{12}^{(3)} - C_{22}^{(3)}\alpha_1^{(3)}\gamma_1^{(3)}) - 2C_{12}^{(3)}\alpha_1^{(3)}\gamma_2^{(3)}
+ C_{22}^{(3)}\alpha_1^{(3)}\alpha_2^{(3)}\gamma_2^{(3)2})(\alpha_2^{(4)}\gamma_1^{(4)} - \alpha_1^{(4)}\gamma_2^{(4)})\mu_{12}^{(4)})/(h\alpha_1^{(4)}\alpha_2^{(4)}(\alpha_1^{(3)}
- 2\alpha_1^{(3)}\alpha_2^{(3)} + \gamma_1^{(3)} - 2\alpha_1^{(3)}\gamma_2^{(3)})\gamma_2^{(4)3}\mu_{12}^{(3)}) = l_1^{(3)}, \tag{5.3.62}$$

$$\lim_{\xi\to\infty} \frac{K^{(1)}(\xi)}{\xi} = -(((\alpha_1^{(4)} - \alpha_2^{(4)})(C_{12}^{(4)}(\alpha_1^{(4)} - \alpha_2^{(4)} + \gamma_1^{(4)} - \gamma_2^{(4)}) - C_{22}^{(4)}\alpha_2^{(4)}\gamma_1^{(4)}\gamma_2^{(4)}
+ C_{22}^{(4)}\alpha_1^{(4)}(\alpha_2^{(4)}(\gamma_1^{(4)} - \gamma_2^{(4)}) + \gamma_1^{(4)}\gamma_2^{(4)}))(C_{12}^{(3)2}((-1 + 2\alpha_2^{(3)})\gamma_1^{(2)}\mu_{12}^{(2)}
+ 4\alpha_2^{(3)2}(\alpha_1^{(3)} - 2\alpha_1^{(3)}\alpha_2^{(3)} + \gamma_1^{(3)} - 2\alpha_1^{(3)}\gamma_2^{(3)})\mu_{12}^{(3)} + \alpha_1^{(2)}(-1 + 2\alpha_2^{(3)})(\mu_{12}^{(2)}
- 2(\alpha_2^{(3)} - \gamma_1^{(3)} + \gamma_2^{(3)} + \alpha_1^{(3)}(-1 + 2\alpha_2^{(3)} + 2\gamma_2^{(3)}))\mu_{12}^{(3)})) - C_{22}^{(3)}(\alpha_1^{(3)}(C_{12}^{(2)}
- C_{22}^{(2)}\alpha_1^{(2)}\gamma_1^{(2)})\gamma_1^{(3)}(-\gamma_1^{(3)} + \alpha_1^{(3)}(-1 + 2\alpha_2^{(3)} + 2\gamma_2^{(3)}))\mu_{12}^{(3)}) + C_{22}^{(3)}(\alpha_1^{(3)2}
\times \gamma_1^{(2)}\gamma_1^{(3)}\mu_{12}^{(2)} - 4\alpha_2^{(3)4}\gamma_1^{(3)}\gamma_2^{(3)2}\mu_{12}^{(3)} + 2\alpha_1^{(3)}\alpha_2^{(3)2}\gamma_2^{(3)}(-\gamma_1^{(2)}\gamma_1^{(3)}\mu_{12}^{(2)} + 2\alpha_2^{(3)2}\gamma_2^{(3)}
\times (-1 + 2\alpha_2^{(3)} + 2\gamma_2^{(3)})\mu_{12}^{(3)}) + \alpha_1^{(2)}(\alpha_1^{(3)}\gamma_1^{(3)} - 2\alpha_2^{(3)2}\gamma_2^{(3)})(2\alpha_2^{(3)}\gamma_1^{(3)}\gamma_2^{(3)}\mu_{12}^{(3)}
- 2\alpha_1^{(3)}\alpha_2^{(3)}\gamma_2^{(3)}(-1 + 2\alpha_2^{(3)} + 2\gamma_2^{(3)})\mu_{12}^{(3)} + \alpha_1^{(3)}\gamma_1^{(3)}(\mu_{12}^{(2)} - 2\alpha_2^{(3)} - 2\gamma_2^{(3)})\mu_{12}^{(3)}))))
+ C_{12}^{(3)}(-(C_{12}^{(2)} - C_{22}^{(2)}\alpha_1^{(2)}\gamma_1^{(2)})(\alpha_1^{(3)} - 2\alpha_1^{(3)}\alpha_2^{(3)} + \gamma_2^{(3)} - 2\alpha_1^{(3)}\gamma_1^{(3)})\mu_{12}^{(3)}
- 2C_{22}^{(3)}(\alpha_1^{(3)}(-1 + \alpha_2^{(3)})\gamma_1^{(2)}\gamma_1^{(3)}\mu_{12}^{(2)} - 4\alpha_1^{(3)}\alpha_2^{(3)3}\gamma_2^{(3)}(-1 + 2\alpha_2^{(3)} + 2\gamma_2^{(3)})\mu_{12}^{(3)}
+ \alpha_2^{(3)2}\gamma_2^{(3)}(\gamma_1^{(2)}\mu_{12}^{(2)} + \alpha_1^{(2)}(\alpha_1^{(3)2}\gamma_1^{(3)}(-1 + 2\alpha_2^{(3)} + 2\gamma_2^{(3)})\mu_{12}^{(3)} + \alpha_2^{(3)}\gamma_2^{(3)}(-2\alpha_1^{(3)2}
\times \mu_{12}^{(3)} - \gamma_1^{(3)}\mu_{12}^{(3)} + \alpha_2^{(3)}(\mu_{12}^{(2)} + 4\gamma_1^{(3)}\mu_{12}^{(3)} - 2\gamma_2^{(3)}\mu_{12}^{(3)})) - \alpha_1^{(3)}(\gamma_1^{(3)2}\mu_{12}^{(3)} + \alpha_2^{(3)}(-1
+ 4\alpha_2^{(3)})\gamma_2^{(3)}(-1 + 2\alpha_2^{(3)} + 2\gamma_2^{(3)})\mu_{12}^{(3)} + (-1 + \alpha_2^{(3)})\gamma_1^{(3)}(-\mu_{12}^{(2)} + 2(\alpha_2^{(3)}
+ \gamma_2^{(3)})\mu_{12}^{(3)})))))\mu_{12}^{(4)})/((-(C_{12}^{(2)}(C_{22}^{(2)}\alpha_1^{(2)}\gamma_1^{(2)})(\alpha_1^{(3)} - 2\alpha_1^{(3)}\alpha_2^{(3)} + \gamma_1^{(3)} - 2\alpha_1^{(3)}\gamma_2^{(3)})\mu_{12}^{(3)}
+ C_{12}^{(3)}((-1 + 2\alpha_2^{(3)})\gamma_1^{(2)}\mu_{12}^{(2)} + 2\alpha_2^{(3)}(\alpha_1^{(3)} - 2\alpha_1^{(3)}\alpha_2^{(3)} + \gamma_1^{(3)} - 2\alpha_1^{(3)}\gamma_2^{(3)})
\times \mu_{12}^{(3)} + \alpha_1^{(2)}(-1 + 2\alpha_2^{(3)})(\mu_{12}^{(2)} - 2(\alpha_2^{(3)} + \gamma_2^{(3)})\mu_{12}^{(3)})) + C_{22}^{(3)}(\alpha_1^{(3)}\gamma_1^{(2)}\gamma_1^{(3)}\mu_{12}^{(2)}
+ 2\alpha_1^{(3)}\alpha_2^{(3)2}\gamma_2^{(3)}(-1 + 2\alpha_2^{(3)} + 2\gamma_2^{(3)})\mu_{12}^{(3)} - 2\alpha_2^{(3)2}\gamma_2^{(3)}(\gamma_1^{(2)}\mu_{12}^{(2)} + \gamma_1^{(3)}\mu_{12}^{(3)})
+ \alpha_1^{(2)}(\alpha_1^{(3)}\gamma_1^{(3)} - 2\alpha_2^{(3)2}\gamma_2^{(3)})(\mu_{12}^{(2)} - 2(\alpha_2^{(3)} + \gamma_2^{(3)})\mu_{12}^{(3)})))(\alpha_2^{(4)}\ (\alpha_1^{(4)}
+ \alpha_2^{(4)})\gamma_1^{(3)}\mu_{12}^{(3)} + \alpha_1^{(3)}(\alpha_2^{(4)2}(\mu_{12}^{(3)} - \mu_{12}^{(4)}) - \alpha_1^{(4)2}\mu_{12}^{(4)} + \alpha_2^{(4)}(\gamma_1^{(4)} - \gamma_2^{(4)})\mu_{12}^{(4)}
+ \alpha_1^{(4)}(-\gamma_1^{(4)} + \gamma_2^{(4)})\mu_{12}^{(4)} + \alpha_1^{(4)}\alpha_2^{(4)}(\mu_{12}^{(3)} + 2\mu_{12}^{(4)}))) - C_{22}^{(4)}(\alpha_1^{(4)}\alpha_2^{(4)}(\alpha_1^{(4)}
+ \alpha_2^{(4)})\gamma_1^{(3)}\gamma_1^{(4)}\mu_{12}^{(3)} + \alpha_1^{(3)}(\alpha_2^{(4)2}\gamma_1^{(4)}\gamma_2^{(4)}\mu_{12}^{(4)} + \alpha_1^{(4)}\alpha_2^{(4)}(\alpha_2^{(4)}\gamma_1^{(4)}(\mu_{12}^{(3)} - \mu_{12}^{(4)})
+ \alpha_2^{(4)}\gamma_2^{(4)}\mu_{12}^{(4)} - 2\gamma_1^{(4)}\gamma_2^{(4)}\mu_{12}^{(4)}) + \alpha_1^{(4)2}(-\alpha_2^{(4)}\gamma_2^{(4)}\mu_{12}^{(4)} + \gamma_1^{(4)}\gamma_2^{(4)}\mu_{12}^{(4)}
+ \alpha_2^{(4)}\gamma_1^{(4)}(\mu_{12}^{(3)} + \mu_{12}^{(4)})))))) = l_2^{(3)}. \tag{5.3.63}$$

Let us rewrite Eqs.(5.3.56)-(5.3.59) for a large value of ξ as

$$\lim_{\xi\to\infty} \overline{m}_{-}^{(1)}(\xi) = \lim_{\xi\to\infty} \frac{\sigma_0^{(1)}}{\sqrt{2\pi}(i\xi)} \left(\frac{k_{-}^{(1)}(\xi)}{k_{-}^{(1)}(0)}\right), \tag{5.3.64}$$

$$\lim_{\xi\to\infty} \overline{n}_{+}^{(1)}(\xi) = \lim_{\xi\to\infty} \frac{\sigma_0^{(1)}}{\sqrt{2\pi}(i\xi)k_{-}^{(1)}(0)k_{+}^{(1)}(\xi)}, \tag{5.3.65}$$

$$\lim_{\xi\to\infty} \overline{m}_{+}^{(3)}(\xi) = \lim_{\xi\to\infty} \frac{-\sigma_0^{(3)}}{\sqrt{2\pi}(i\xi)} \left(\frac{k_{+}^{(3)}(\xi)}{k_{+}^{(3)}(0)}\right), \tag{5.3.66}$$

$$\lim_{\xi\to\infty} \overline{n}_{-}^{(3)}(\xi) = \lim_{\xi\to\infty} \frac{-\sigma_0^{(3)}}{\sqrt{2\pi}(i\xi)k_{+}^{(3)}(0)k_{-}^{(3)}(\xi)}. \tag{5.3.67}$$

The asymptotic expressions of above equations with the help of the Eqs.(5.3.60)-(5.3.63) can be determined as

$$\overline{m}_{-}^{(1)}(\xi) = \frac{\sigma_0^{(1)} e^{5\pi i/4}}{\sqrt{2\pi}} \sqrt{\frac{l_2^{(1)}}{l_1^{(1)}}} \xi^{-1/2}, \quad as \quad \xi \to \infty \tag{5.3.68}$$

$$\overline{n}_{+}^{(1)}(\xi) = \frac{\sigma_0^{(1)} e^{5\pi i/4}}{\sqrt{2\pi l_1^{(1)} l_2^{(1)}}} \xi^{-3/2}, \quad as \quad \xi \to \infty \tag{5.3.69}$$

$$\overline{m}_{+}^{(3)}(\xi) = \frac{-\sigma_0^{(3)} e^{\pi i/4}}{\sqrt{2\pi}} \sqrt{\frac{l_2^{(3)}}{l_1^{(3)}}} \xi^{-1/2}, \quad as \quad \xi \to \infty \tag{5.3.70}$$

$$\overline{n}_{-}^{(3)}(\xi) = \frac{-\sigma_0^{(3)} e^{\pi i/4}}{\sqrt{2\pi l_1^{(3)} l_2^{(3)}}} \xi^{-3/2}, \quad as \quad \xi \to \infty. \tag{5.3.71}$$

Using the inverse Fourier transformation on the expessions of Eqs.(5.3.68)-(5.3.71), we get

$$\lim_{x\to 0} m^{(1)}(x) = \frac{-\sigma_0^{(1)}}{2}\sqrt{\frac{l_2^{(1)}}{\pi l_1^{(1)}}}x^{-1/2}, \tag{5.3.72}$$

$$\lim_{x\to 0} n^{(1)}(x) = \frac{-\sigma_0^{(1)}}{\sqrt{\pi l_1^{(1)} l_2^{(1)}}}x^{1/2}, \tag{5.3.73}$$

$$\lim_{x\to 0} m^{(3)}(x) = \frac{\sigma_0^{(3)}}{2}\sqrt{\frac{l_2^{(3)}}{\pi l_1^{(3)}}}x^{-1/2}, \tag{5.3.74}$$

$$\lim_{x\to 0} n^{(3)}(x) = \frac{\sigma_0^{(3)}}{\sqrt{\pi l_1^{(3)} l_2^{(3)}}}(-x)^{1/2}. \tag{5.3.75}$$

The Eq.(5.3.72) and Eq.(5.3.74) represent the stress components of the first and second crack, respectively. Thus the square root singularities which are present in both equations are very much expected at the crack tip as the physical nature of the problem. The Eq.(5.3.73) and Eq.(5.3.75) represent the displacement components for the cracks, which are required to find the crack opening displacements of the cracks.

5.4 Expressions of Stress intensity factor(SIF) and Crack opening displacement(COD)

The expression for SIF for the first crack is determined as

$$K_I = \lim_{x\to 0}\sqrt{2\pi x}\ \sigma_{yy}^{(1)}(x,0) = -\sigma_0^{(1)}\sqrt{\frac{l_2^{(1)}}{2l_1^{(1)}}}. \tag{5.4.1}$$

The value of normalized SIF(with respect to $\nu_0^{(1)}$) of the original problem by putting back the value of $\sigma_0^{(1)}$ from Eq.(5.2.23) into Eq.(5.4.1) is obtained as

$$SIF = \frac{(C_{22}^{(1)} + \nu_{21}^{(1)} C_{12}^{(1)})}{h} \sqrt{\frac{l_2^{(1)}}{2l_1^{(1)}}}. \quad (5.4.2)$$

The expression for SIF for the second crack is determined as

$$K_I = \lim_{x\to 0} \sqrt{2\pi x}\sigma_{yy}^{(3)}(x,0) = \sigma_0^{(3)} \sqrt{\frac{l_2^{(3)}}{2l_1^{(3)}}}. \quad (5.4.3)$$

The value of normalized SIF(with respect to $\nu_0^{(3)}$) for the second crack of the original problem by putting back the value of $\sigma_0^{(1)}$ from Eq.(5.2.24) into Eq.(5.4.3) is obtained as

$$SIF = -\frac{(C_{22}^{(3)} + \nu_{21}^{(3)} C_{12}^{(3)})}{h} \sqrt{\frac{l_2^{(3)}}{2l_1^{(3)}}}. \quad (5.4.4)$$

The expression for the COD of first crack is given as

$$COD = \frac{-2\sigma_0^{(1)}}{\sqrt{\pi l_1^{(1)} l_2^{(1)}}} x^{1/2} \quad (5.4.5)$$

and the value of normalized COD(with respect to $\nu_0^{(1)}$) of the original problem is obtained by substituting the value of $\sigma_0^{(1)}$ from Eq.(5.2.23) into Eq.(5.4.5) as

$$COD = \frac{2(C_{22}^{(1)} + \nu_{21}^{(1)} C_{12}^{(1)})}{h} \sqrt{\frac{l_2^{(1)}}{2l_1^{(1)}}} x^{1/2}. \quad (5.4.6)$$

The expression for the COD of the second crack is

$$COD = \frac{2\sigma_0^{(3)}}{\sqrt{\pi l_1^{(3)} l_2^{(3)}}} (-x)^{1/2} \quad (5.4.7)$$

and the value of normalized COD(with respect to $\nu_0^{(1)}$) for the original problem is obtained by substituting the value of $\sigma_0^{(1)}$ from Eq.(5.2.23) into Eq.(5.4.7) as

$$COD = -\frac{2(C_{22}^{(3)} + \nu_{21}^{(3)} C_{12}^{(3)})}{h} \sqrt{\frac{l_2^{(3)}}{2 l_1^{(3)}}} (-x)^{1/2}. \tag{5.4.8}$$

5.5 Numerical results and discussions

For the numerical computation of SIFs and CODs, E-glass epoxy is used for the first and fourth semi-infinite strips viz., Medium-1 and 4, Graphite epoxy is used for the second semi-infinite strip representing Medium-2, and Boron epoxy is used for the third semi-infinite strip representing Medium-3. The material properties(in GPA unit) of all the orthotropic materials used for the model are listed in the table given below.

Materials	C_{11}	C_{22}	C_{12}	μ_{12}	ρ
E-glass epoxy	46.09	12.60	2.86	5.50	2.1
Graphite epoxy	155.36	16.31	3.67	7.48	1.6
Boron epoxy	209.09	19.97	5.06	6.4	1.99

Table 5.1: Engineering material elasticity constants

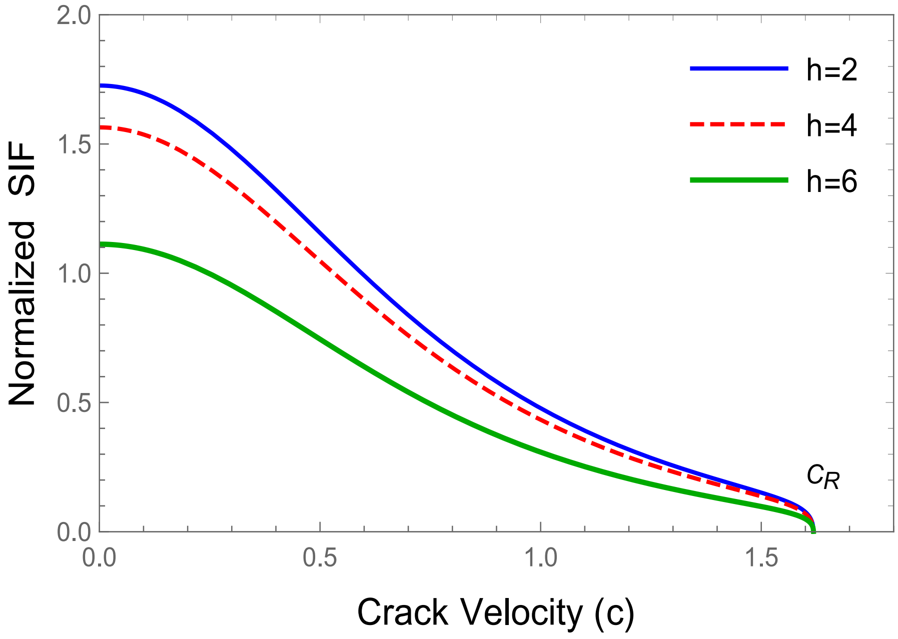

Figure 5.3: Variations of the normalised SIF for first crack vs crack velocity (c) for $h = 2, 4$ and 6 .

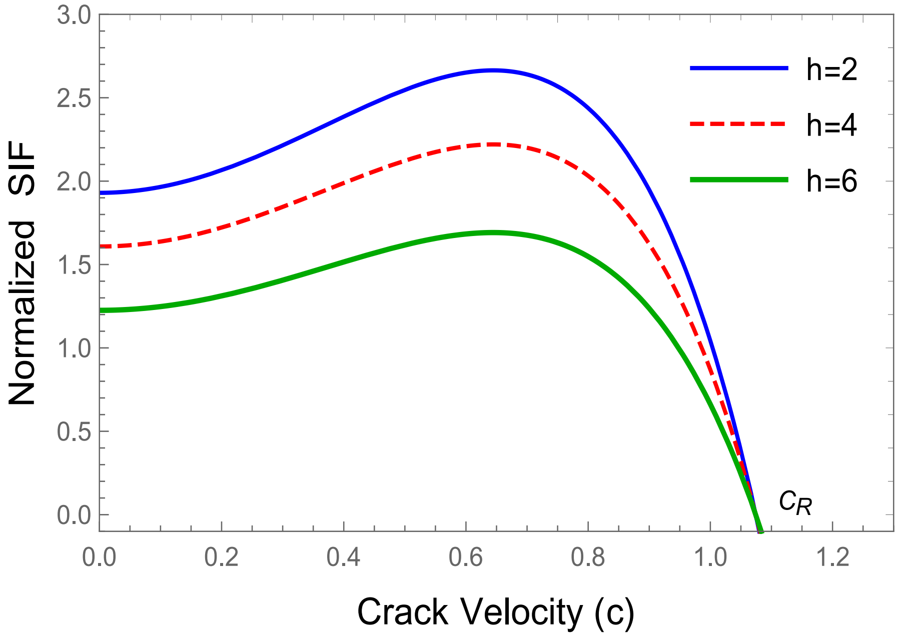

Figure 5.4: Variations of the normalised SIF for second crack vs crack velocity (c) for $h = 2, 4$ and 6.

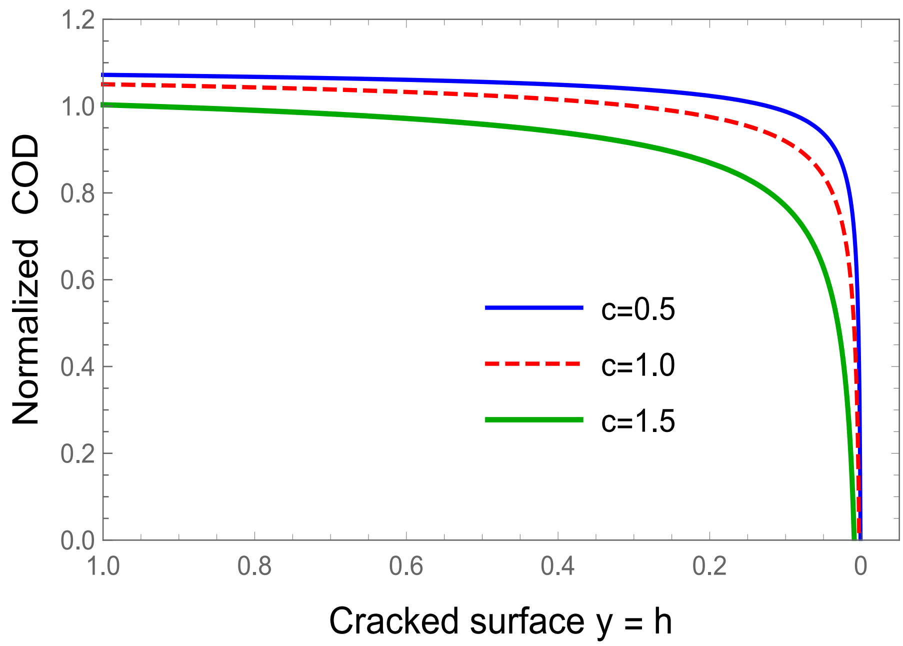

Figure 5.5: Variations of the normalised COD for first crack vs displacement at cracked surface $y = h$ for crack velocity $(c) = 0.5, 1.0$ & 1.5 and $h = 2$.

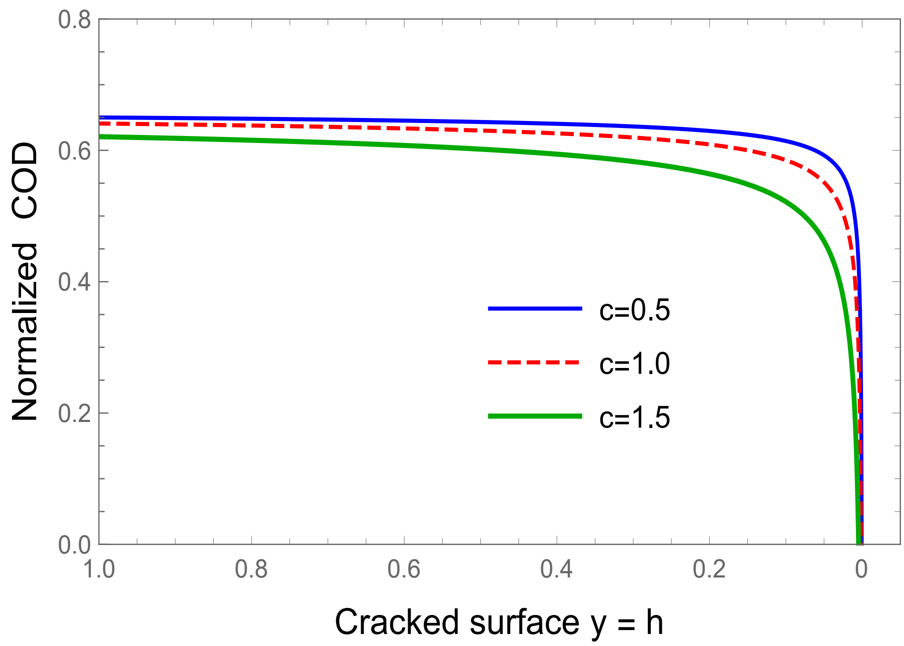

Figure 5.6: Variations of the normalised COD for first crack vs displacement at cracked surface $y = h$ for crack velocity $(c) = 0.5, 1.0$ & 1.5 and $h = 4$.

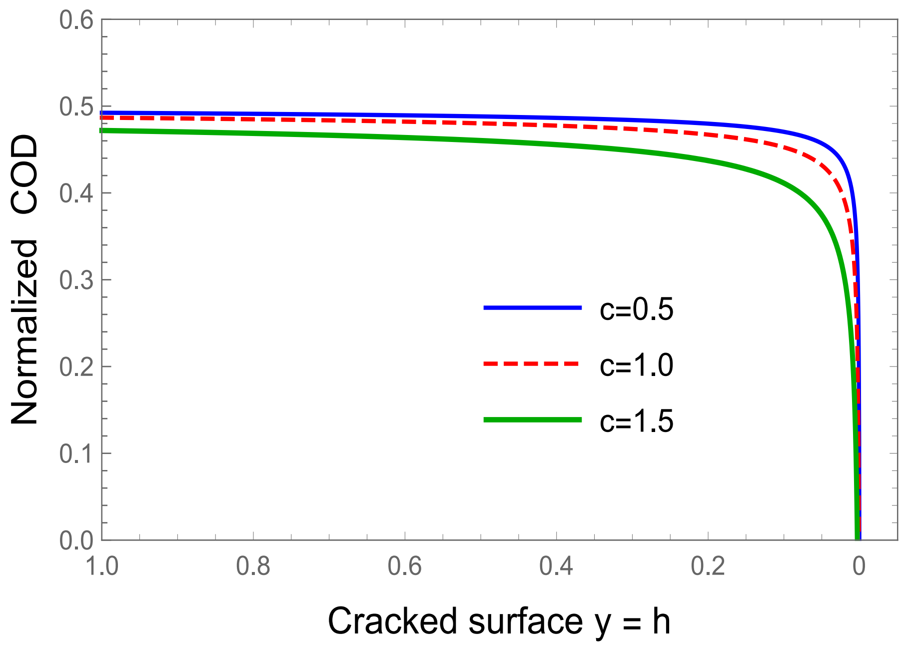

Figure 5.7: Variations of the normalised COD for first crack vs displacement at cracked surface $y = h$ for crack velocity $(c) = 0.5, 1.0$ & 1.5 and $h = 6$.

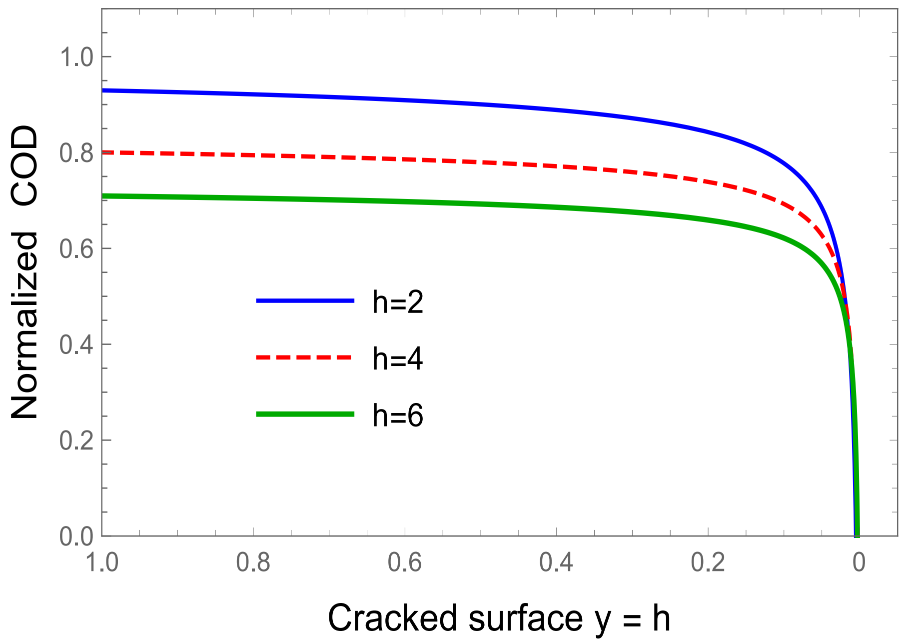

Figure 5.8: Variations of the normalised COD for first crack vs displacement at cracked surface $y = h$ for crack velocity $(c) = 0.5$ and $h = 2, 4$ & 6.

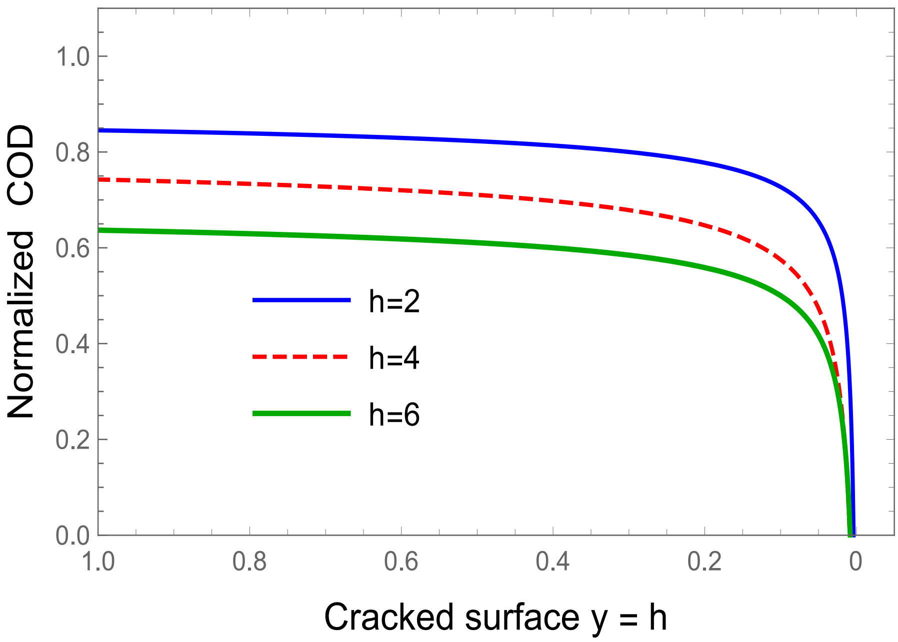

Figure 5.9: Variations of the normalised COD for first crack vs displacement at cracked surface $y = h$ for crack velocity $(c) = 1.0$ and $h = 2, 4$ & 6.

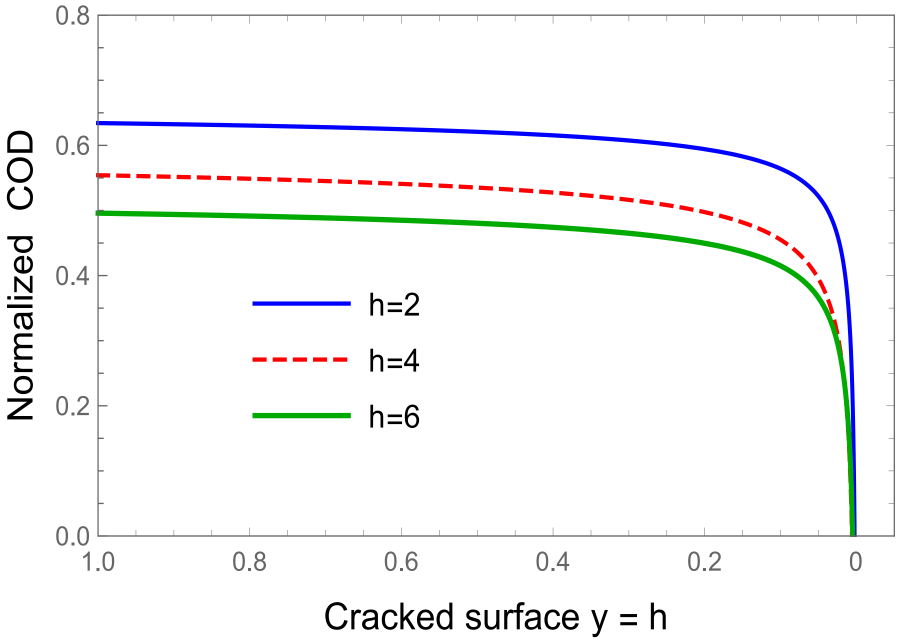

Figure 5.10: Variations of the normalised COD for first crack vs displacement at cracked surface $y = h$ for crack velocity $(c) = 1.5$ and $h = 2, 4$ & 6.

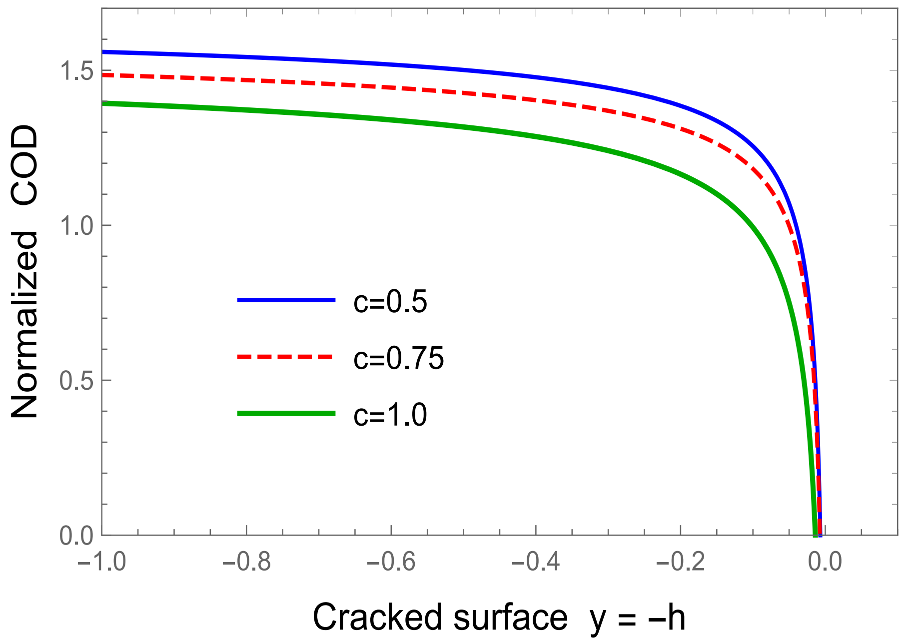

Figure 5.11: Variations of the normalised COD for second crack vs displacement at cracked surface $y = -h$ for crack velocity $(c) = 0.5, 0.75$ & 1.0 and $h = 2$.

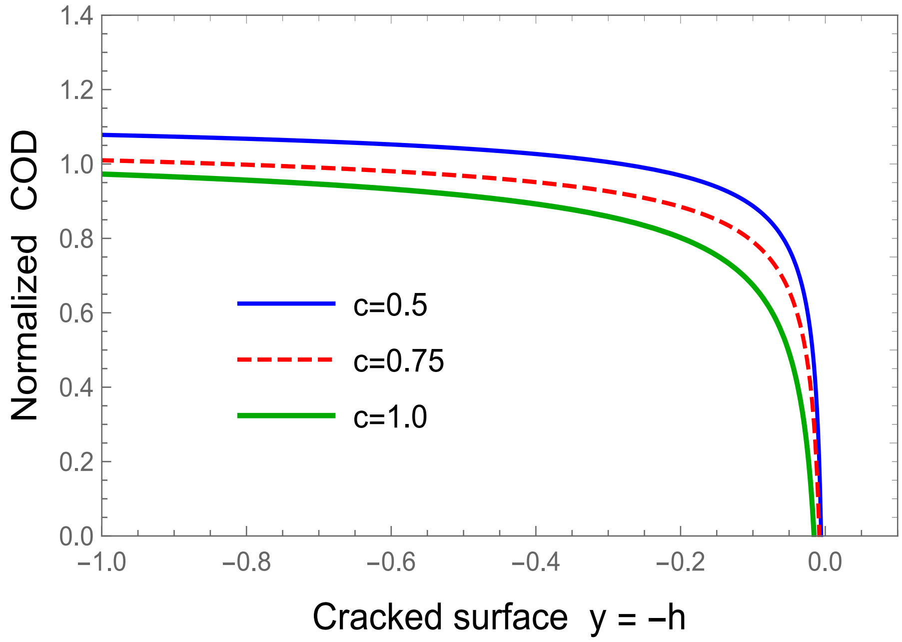

Figure 5.12: Variations of the normalised COD for second crack vs displacement at cracked surface $y = -h$ for crack velocity $(c) = 0.5, 0.75$ & 1.0 and $h = 4$.

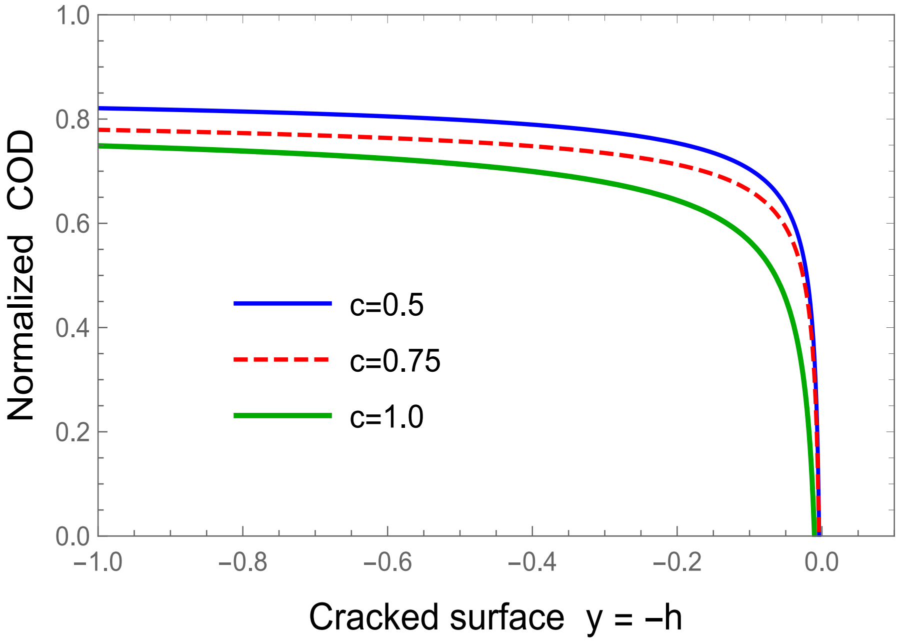

Figure 5.13: Variations of the normalised COD for second crack vs displacement at cracked surface $y = -h$ for crack velocity $(c) = 0.5, 0.75$ & 1.0 and $h = 6$.

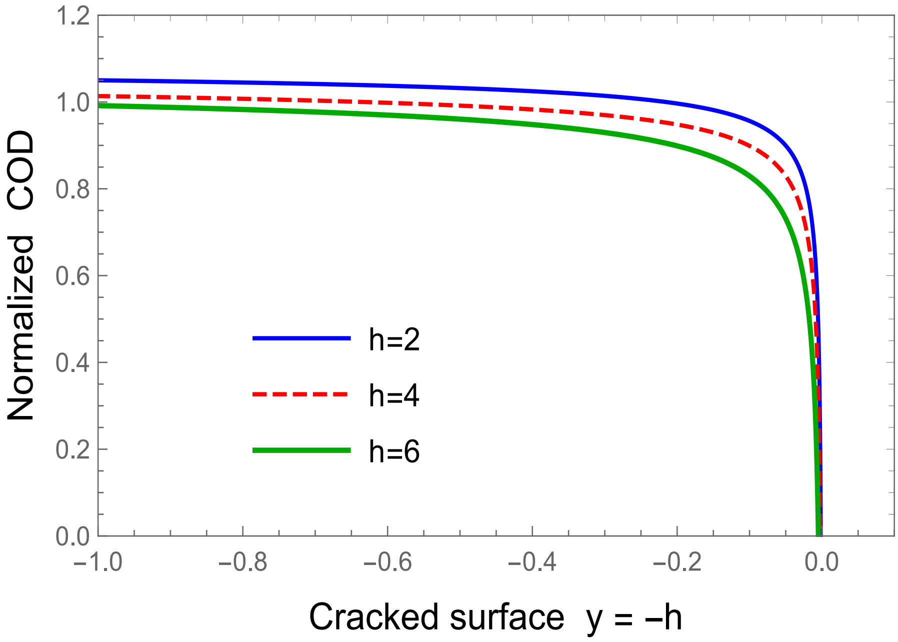

Figure 5.14: Variations of the normalised COD for second crack vs displacement at cracked surface $y = -h$ for crack velocity $(c) = 0.5$ and $h = 2, 4$ & 6.

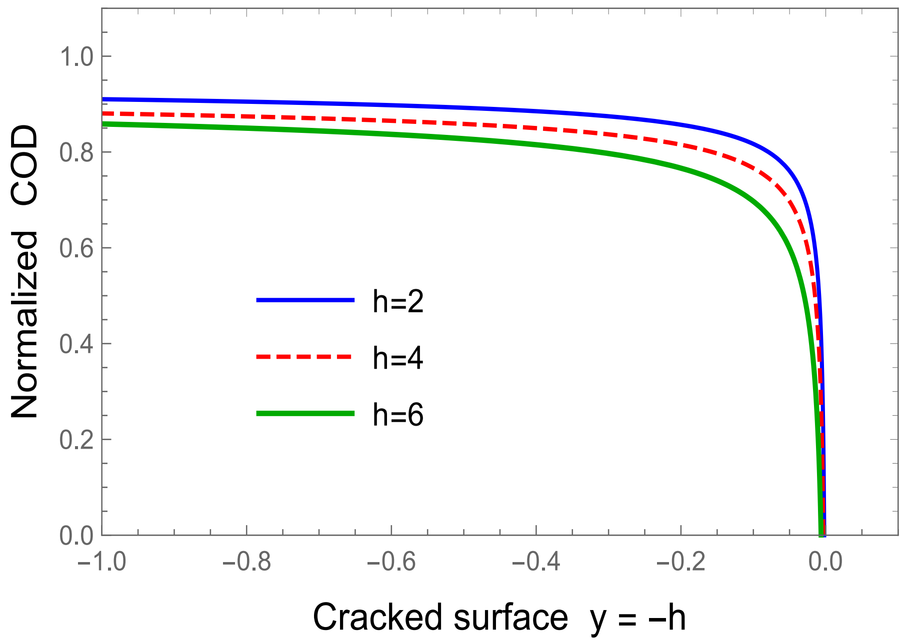

Figure 5.15: Variations of the normalised COD for second crack vs displacement at cracked surface $y = -h$ for crack velocity $(c) = 0.75$ and $h = 2, 4$ & 6.

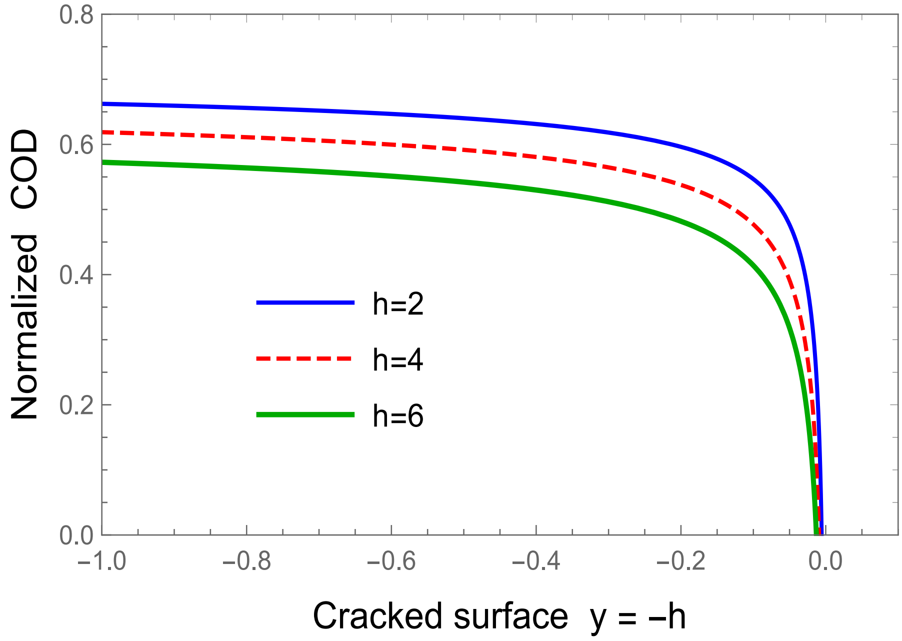

Figure 5.16: Variations of the normalised COD for second crack vs displacement at cracked surface $y = -h$ for crack velocity $(c) = 1$ and $h = 2, 4$ & 6 .

The asymptotic analytical expressions for SIFs and CODs are clearly depending on the material constants, crack velocity and the depth of the strips as seen from the

Eqs.(5.4.2)-(5.4.8). The SIFs for the first and second crack have been depicted graphically in Figs.5.3 and 5.4. It is observed from both the cracks that SIF is decreasing as the crack velocity increasing and it approaches to zero as $c \to C_R$, because, for the Mode-I crack propagation problems, the upper limit of crack velocity c is the rayleigh wave velocity C_R as they are the surface waves. Thus the crack velocity is always lower than the rayleigh wave velocity, i.e., $c < C_R$ and when it approaches to C_R, SIF vanishes, which is entirely reasonable given the physical aspect of the mathematical model. Also, SIF decreases as the depth of the strips increases, which verifies that SIF is inversely proportional to the depths of the strips h as determined by the expressions of SIFs. Also, it is noted that the value of the SIF for the second crack is considerably higher than for the first crack. This implies that increasing the strip depths can prevent the material from failing.

The graphical representation of CODs can be shown from the Figs.5.5-5.10 for the first crack and through Figs.5.11-5.16 for the second crack, respectively. For both cracks, the crack opening displacement becomes zero at the origin of the cracks. Also, it can be seen that through Figs.5.5-5.7 and Figs.5.11-5.13, as the crack velocity increases, COD decreases.

The effect of depth of the strips has been observed through Figs.5.8-5.10 and Figs.5.14-5.16 , as the depth of the strip increases the value of the COD decreases for both cracks. It is also observed, as per the SIF, the values of CODs are also much higher for the second crack than the first crack. Also, SIF and COD will give different values for different combinations of the orthotropic materials.

5.6 Conclusion

The problem of two moving interfacial semi-infinite cracks coming from the opposite direction, situated at different interfaces of four semi-infinite orthotropic strips media in the composite medium, is solved using the W-H technique with the help of Fourier transformation for converting the mixed boundary value problem into standard W-H equation for both the cracks. The approximate asymptotic analytical expressions of the stress intensity factor and crack opening displacement have been obtained. The variation of normalized SIFs and CODs have been shown graphically for various crack velocities and depths of the strips. It is also observed that by taking the suitable orthotropic materials and depths of the strip, SIF and COD can be controlled.

The problem studied here can be applied in real life to improve safety standards in various industries. For example, in the automotive industry, it can help to identify the stress levels in vehicle components, leading to improved designs and manufacturing processes. It can also be used to identify potential fracture locations, allowing for implementing preventive maintenance measures to reduce the potential for vehicle failure. In the aerospace industry, the research in this chapter can be used to ensure the safety of aircraft structures. In civil engineering, it can be used to ensure the safety of bridges and other large structures. Overall, the research from this chapter can be used to improve safety standards in various industries and thus lead to a better quality of life.