Chapter 1

Introduction

1.1 Literature review and introduction of the thesis

Crack propagation is a phenomenon that occurs in many different types of materials and structures, such as metals, composites, rocks, and ice. It refers to the gradual expansion of a crack in a material caused by external stresses or loads. The ability to accurately predict and control crack propagation is crucial for ensuring the safety and longevity of structures, as it can have significant implications for their overall strength and durability.

The study of crack initiation and propagation is a highly enriched area of research in various fields related to fracture mechanics, viz. applied mathematics, structural engineering, mechanical engineering, material sciences, civil engineering, and many more. Crack initiation may occur in elastic materials if either the maximum extension or tensile stress exceeds a specific critical value. The various mathematical models and experimental techniques present in the literature assure the safety and durability of various engineering structures. The applications of this area are wide open, from the aerospace industry to household appliances, which made the study of crack propagation exciting and challenging.

There are several types of cracks that can occur in different materials and structures. Here are some common types of cracks:

Impact Cracks: Impact cracks occur due to a sudden and intense force, such as a collision or a heavy object falling on a surface. This type of crack can be seen in glass or ceramics.

Fatigue Cracks: Fatigue cracks occur due to repeated loading and unloading of a structure, which cause the material to weaken and eventually crack. This type of crack is common in metals, especially in parts that undergo cyclic loading.

Thermal Cracks: Thermal cracks occur due to changes in temperature, which cause the material to expand or contract, resulting in cracks. For example, concrete can create cracks when exposed to extreme temperatures.

Shrinkage Cracks: Shrinkage cracks occur when a material dries or cures, causing it to shrink and form cracks. This type of crack is common in concrete or clay.

Structural Cracks: Structural cracks occur due to design or construction defects, such as inadequate support, overloading, or incorrect placement of reinforcement. This type of crack can be dangerous and may require immediate attention.

Settlement Cracks: Settlement cracks occur when the foundation of a building settles, causing the walls or floors to crack. This type of crack can be seen in older buildings or in buildings that are built on unstable soil.

These are just a few examples of the types of cracks that can occur. The severity and cause of a crack will determine the appropriate course of action for repair or prevention. When these cracks propagate and cross the critical point as the material goes from the elastic zone to the plastic zone, it creates permanent deformation, which is called a fracture. Fractures are the result of a material or structure breaking or cracking due to external forces or internal stresses. There are several types of fractures, which are characterized by the type of forces or stresses that cause them. Here are some common types of fractures:

Tensile Fracture: Tensile fracture occurs when a material is stretched or pulled apart, causing it to break. This type of fracture is characterized by separating the material into two or more pieces, with a smooth, flat surface in the direction of the force. Tensile fractures are common in materials that are brittle or have low ductility, such as glass or ceramics.

Compressive Fracture: Compressive fracture occurs when a material is compressed or squeezed together, causing it to break. This type of fracture is characterized by a crushed or flattened material with a concave surface in the direction of the force. Compressive fractures are common in materials that are ductile or have low compressive strength, such as plastics or soft metals.

Shear Fracture: Shear fracture occurs when a material is subjected to forces that cause it to slide or twist in opposite directions, causing it to break. This type of fracture is characterized by a material that is separated along a diagonal or angled plane with a rough, jagged surface. Shear fractures are common in materials that are under torsion or bending stresses, such as metals or composites.

Fatigue Fracture: Fatigue fracture occurs when a material is subjected to repeated or cyclic loading, causing it to weaken and eventually break. This type of fracture is characterized by a material that has a fibrous or granular surface, with small cracks that eventually grow and coalesce. Fatigue fractures are common in materials which are under cyclic or fluctuating stresses, such as metals or composites.

Ductile Fracture: Ductile fracture occurs in materials that can undergo significant plastic deformation before breaking.

Fracture mechanics is a significant aspect of solid mechanics. Fracture in structures is caused by material faults, loading uncertainty, a lack of construction, or maintenance. The study of fracture mechanics started with the work of Griffith [1] on the theory of rupture, which was further studied by many renowned researchers and mathematicians. Irwin [2, 3], the father of modern fracture mechanics, implemented the theories given earlier combined with his theories to make the real-life implementation into the engineering structures.

In 1938, Westergaard [4] developed a semi-inverse technique for stress field and displacement analysis around the crack tip, which was used by Irwin [5] to extend his previous work [6], the energy release rate concept to describe the stresses and displacements near crack tip by a single constant, which is related to the energy release rate of the crack. This crack parametric constant of energy release rate became famous late as the Stress Intensity Factor, which is one of the crucial parameters in the mathematical study of crack propagation.

Afterward, many researchers worked in this interesting field of fracture mechanics like Rice [7] given the integral and approximate analysis approach of strain concentration in cracks. Bueckner [8] studied the novel principal to find the expression of stress intensity factor. Freund [9] studied cracks in four different loading conditions. In the first loading condition, he studied the constant rate of extension of the crack position. In the second loading condition [10], he worked on the non-uniform rate of extension in crack propagation. Under the third loading condition, Freund [11] studied the stress wave loading conditions on the crack surfaces, and in the fourth loading condition, Freund [12] again studied the obliquely incident stress pulse on the crack surfaces. Melvin [13] worked on the advacement fracture mechanics. Georgiadis [14] studied the integral transform techniques in the complex-plane. Hutchinson and Suo [15] introduced the concept of mixed mode cracking, which describes the behavior of cracks that propagate under combined modes of loading. Bazant and Planas [16] presented a comprehensive review of fracture mechanics applied to concrete and other quasi-brittle materials in their book. Erdogan [17] studied the crack problems using the integral technique, and the related fracture mechanics problems with real-life implementation have been done by Cotterell [18]. Meanwhile, Hertzberg [19] demonstrated in his study, which was published in many revised editions, that considering a material's microscopic structure, one may increase its strength.

The existence of cracks has a significant impact on the material's strength while constructing the structure, and therefore the stresses are altered to such an extent that the detailed stress analysis given by the designer was insufficient. The study of fracture mechanics began mostly in the twentieth century, and the progress made in this discipline has had a significant influence on researchers' keen interest in contributing to the field. Theoretical investigation of crack/cracks propagation behaviour is basically based on elastic body concepts.

Stress and strain are fundamental concepts in mechanics that play a critical role in describing the behavior of materials under external forces. Stress refers to the internal forces acting within a material, while strain describes the deformation that occurs due to those forces. The stress tensor is a mathematical representation of stress that describes the distribution of forces within a material. It is a second-order tensor comprising nine components $(C_{ij}, (i, j = 1, 2, 3))$ in three-dimensional space. These components represent the normal and shear stresses acting on the material in each of the three coordinate directions. The stress tensor is conventionally denoted by the symbol $\sigma_{ij}, (i, j = 1, 2, 3)$.

On the other hand, the strain tensor is a mathematical representation of strain that characterizes the deformation of a material caused by applied forces. Like the stress tensor, it is a second-order tensor with nine components in three-dimensional space. These components represent the changes in length and angular distortions in each of the three coordinate directions. The strain tensor is commonly denoted by the symbol $\epsilon_{ij}, (i, j = 1, 2, 3)$. The stress and strain tensors together enable the determination of the mechanical properties of materials and the prediction of their behavior under different conditions.

In an elastic body, the symmetries $\sigma_{ij} = \sigma_{ji}$ and $\epsilon_{ij} = \epsilon_{ji}$ reduce the nine components of both the strain and stress tensors into six independent components. As a result, determining the state of strain and stress in an elastic body only requires six components, one for each strain and stress tensor. The relation between strain and stress components are given as

$$\sigma_{ij} = F_{ij}\epsilon_{ij} \tag{1.1.1}$$

In the elasticity theoey, Hooke's Law is a principle that describes the relationship between stress and strain in a material that behaves elastically (i.e., returns to its original shape after deformation). According to Hooke's Law, the stress applied to a material is directly proportional to the strain that is produced.

In mathematical terms, this relationship can be expressed as

$$\sigma_{ij} = c_{ijkl}\epsilon_{kl}, \quad (i, j, k, l = 1, 2, 3), \tag{1.1.2}$$

where the coefficients $c_{ijkl}'s$ are the elastic modulli and they describe the physical properties of elastic material.

In this thesis, the stress field around the crack tip for different crack problems and modes has been investigated only for two-dimensional stress components. For two-dimensional crack propagation problem, the stress components near the crack tip, σ_{xx} and σ_{yy} denote the normal or tensile stress components in $'x'$ and $'y'$ directions, respectively. For shear loading conditions, τ_{xy} and τ_{yx} are the shear stress components, as shown in Fig.1.1 for visual understanding.

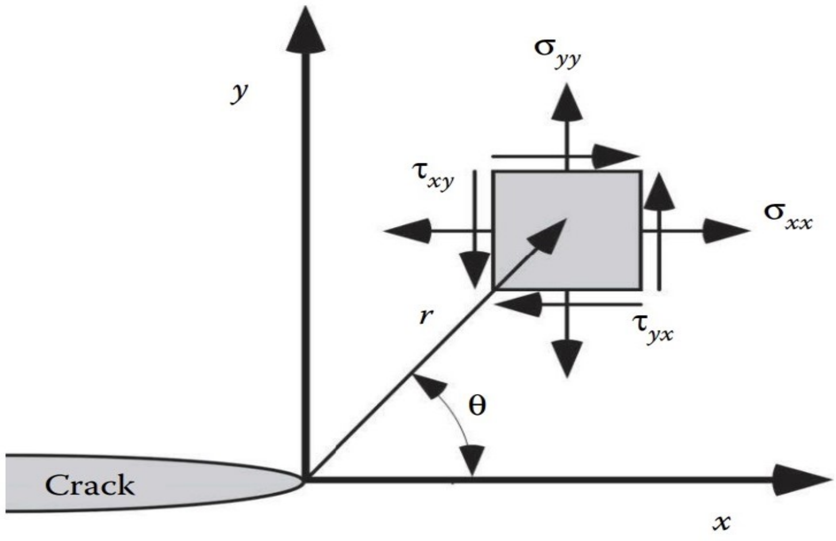

Figure 1.1: Stress field distribution near crack tip

The stress components can be defined in polar co-ordinates as

$$\sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}} cos\left(\frac{\theta}{2}\right)\left[1 - sin\left(\frac{\theta}{2}\right) sin\left(\frac{3\theta}{2}\right)\right], \tag{1.1.3}$$

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} cos\left(\frac{\theta}{2}\right)\left[1 + sin\left(\frac{\theta}{2}\right) sin\left(\frac{3\theta}{2}\right)\right], \tag{1.1.4}$$

$$\tau_{xy} = \frac{K_I}{\sqrt{2\pi r}} cos\left(\frac{\theta}{2}\right) sin\left(\frac{\theta}{2}\right) cos\left(\frac{3\theta}{2}\right), \tag{1.1.5}$$

where K_I is the stress intensity factor at the crack tip. The cartesian form of the two-dimensional stress components for the orthotropic media are given below

$$\sigma_{xx} = \mu_{12}\left(C_{11}\frac{\partial u}{\partial x} + C_{12}\frac{\partial v}{\partial y}\right), \tag{1.1.6}$$

$$\sigma_{yy} = \mu_{12}\left(C_{12}\frac{\partial u}{\partial x} + C_{22}\frac{\partial v}{\partial y}\right), \tag{1.1.7}$$

$$\tau_{xy} = \mu_{12}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \tag{1.1.8}$$

where C_{11}, C_{12}, C_{22} and μ_{12} are the orthotropc composite material constants, u and v are the displacement components in x and y directions, respectively.

In crack propagation studies, mathematicians use a variety of techniques and mathematical models to simulate the behavior of cracks in materials under different loading conditions. This allows them to predict how the crack will grow and how it will affect the overall structural integrity of the material.

The prediction of material strength and structural failure is a crucial task to accomplish in fracture mechanics, and to predict the life of the structure to stay in the elastic region and initiation of the plastic region of the structure needs to be well known so that the loading conditions, and other outer and inner environments can be specified accordingly. It is shown in the Fig.1.2 that as the crack size increases with time, the structure's useful life could be working without any hassle. When the curve goes to the plastic region, it goes to the structural failure, which leads to the material failure.

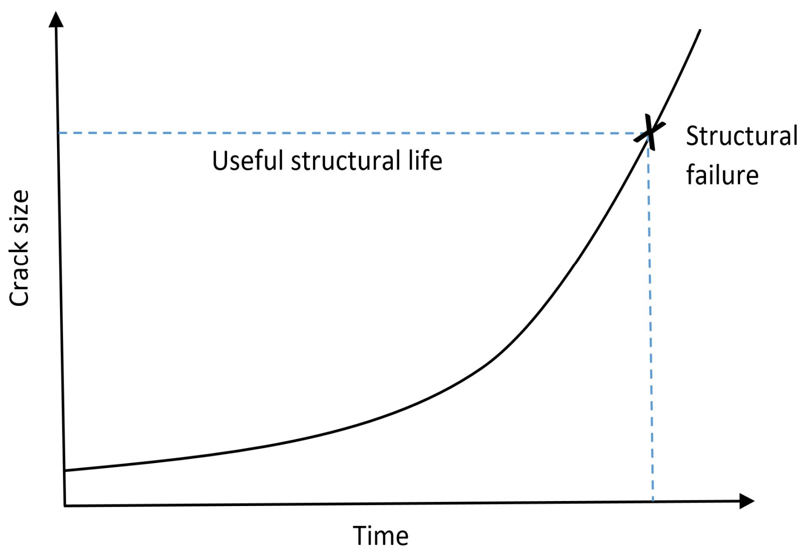

Figure 1.2: loading tolerance lifespan of the structure

One common mathematical model used in crack propagation studies is the Linear Elastic Fracture Mechanics (LEFM) model, which assumes that the material is perfectly elastic and that the crack will propagate when the stress intensity factor (SIF) at the crack tip exceeds a certain critical value. The SIF is a measure of the stress concentration at the crack tip and can be calculated using mathematical techniques such as the Integral transform technique or Wiener-Hopf technique etc.

Mathematicians also use more advanced models, such as the non-linear fracture mechanics model, which takes into account the non-linear behavior of the material and the crack tip, or the cohesive zone model, which models the interaction between the crack surfaces. In addition to modeling crack propagation, mathematicians also work on developing numerical methods to simulate crack growth and to optimize the design of materials and structures to resist the cracking. These methods include optimization algorithms, numerical integration methods, and various simulation techniques.

The mathematical problems concerning crack propagation in elastic media are classified as boundary value problems. A boundary value problem is a problem that involves finding the solution to a given differential equation under specified boundary conditions. There are two different kinds of boundary value problems (BVPs). First BVPs are those in which the stress components are prescribed on the boundary surface, while second BVPs are those in which the displacements are prescribed on the boundary surface. A mixed boundary value problem is one in which the BVP does not fit into either of the two categories mentioned above.

The mathematical theory of fracture mechanics in elasticity mainly contributes to obtaining the requisite stress and strain calculations for the material structures under consideration, which exhibits a slight deviation from Hooke's law within the safe range of alternating stress. Initially, the mathematical expressions of stresses for an elliptic hole at both the end of the major and minor axes as tensile stress and compression stress, respectively, derived by Inglis [20]. Afterward, Griffith [21] gave his criteria for crack energy in elastic media, which was further pursued by many researchers for the theoretical approach of crack study, but for the real-life implication in the engineering problems the work of Irwin [5] is remarkable in which he observed that for an existing crack of length $'a'$ in a sheet subject to linear elasticity, the stress field exhibits singularity at the crack tip of the form $\sigma \propto \frac{K}{\sqrt{a}}$, where $'K'$ is referred to as the SIF. Hence, the linear elastic fracture mechanics base is identified as the crucial SIF. Further, the remarkable concept of the upper limit of crack velocity is the Rayleigh wave speed (C_R), the highest speed at which a wave can propagates through a free surface, has been established by Stroh [22]. The mathematical concepts of path integral and analysis of crack propagation modeling in linear and non-linear elasticity medium have been given by Rice [23] and the crack propagation at the variable velocity has been studied by Kostrov [24]. The study of surface cracks using the stress intensity factor have been done by Li et al. [25] and the virtual crack closure-integral method(VCCM) for cracks to study stress intensity factor have been given by Okada et al. [26]

When a solid material has a crack, it can produce waves that vary sinusoidally in time, called time harmonic waves. These waves are generated on the surface of the crack when a stress is applied to the material, causing the stress to concentrate at the tip of the crack, resulting in a stress singularity. As a result, the crack will open and close, and will generate the time harmonic waves on its surface.

The types of time harmonic waves are produced depend on the orientation and shape of the crack. For instance, a crack that is perpendicular to the applied stress will produce a mode I wave, which is a tensile wave that propagates perpendicular to the surface of the crack. On the other hand, a crack parallel to the applied stress will generate a mode II wave, which is a shear wave that propagates parallel to the surface of the crack. When a time-harmonic wave encounters a crack surface, it can cause a disturbance that results in the reflection, transmission, and diffraction of scattered waves. The nature of this disturbance depends on various factors, including the size and shape of the crack, the frequency of the wave, and the material properties of the surrounding medium. If the wavelength of the incident wave is much larger than the size of the crack, then the scattered waves will be primarily diffracted and transmitted. However, if the wavelength is comparable to or smaller than the crack size, the scattered waves will be reflected and scattered in various directions.

The presence of a crack in the material can affect the propagation of SH waves, causing them to reflect, refract and diffract. These effects can be used to detect and monitor the size, shape and location of the crack, as well as its propagation rate and the stress conditions which are causing it to grow. SH waves are commonly generated when shear stress is applied to the material, causing the crack surfaces to move relative to one another. As a result, the SH wave propagates along the surface of the crack, perpendicular to the direction of particle motion.

In several fields, including materials science, fracture mechanics, civil engineering, mechanical engineering, and many more, wave propagation on the crack surface is an area of great interest. Several kinds of research have already been conducted on this subject in the literature, providing a comprehensive overview of the theories, experiments, and applications of wave propagation in cracked solids and composite materials so far, to name a few; Bedford and Drumheller [27] cover the elastic wave propagation and Nayfeh [28] gives the effect of wave scattering and propagation in the composite media. Ang and Knopoff [29] studied the wave propagation effect on finite cracks. The study of interfacial cracks with the interaction of antiplane shear wave has been done by Srivastava et al. [30]. The study for elastic wave scattering under the antiplane shear loading on interfacial cracks has been given by Bostrom [31]. Some of the key findings from these surveys include the mechanisms of crack initiation, growth, and coalescence under dynamic loading conditions, the effects of material properties and loading conditions on crack behavior, and the techniques for detecting and characterizing cracks in various materials. By analyzing the characteristics of those waves that are generated on the surfaces of various cracks, the researchers can gain insights into the physical properties of the crack and the material in which it exists. This information can be used to develop new non-destructive testing techniques and to improve the design and maintenance of materials which are subject to stresses and strains.

1.2 Modes of fracture

When a material is under stress, it can break in different ways depending on the type of stress and the direction it is applied. There are three main types of fractures that describe how materials break under stress as shown in the Fig.1.3

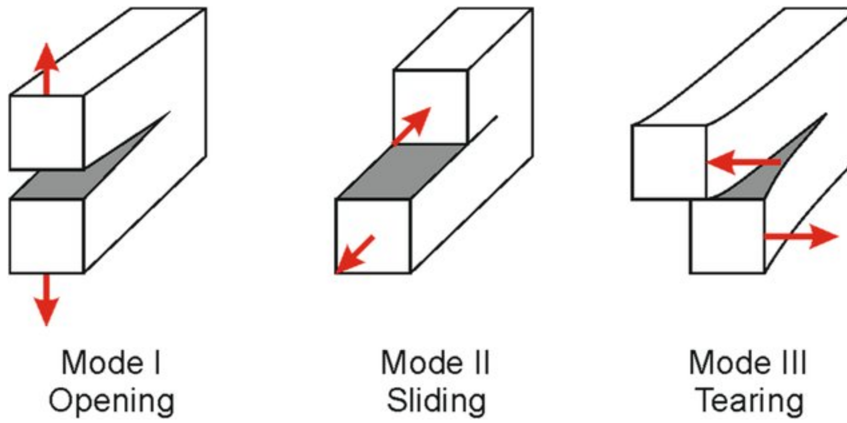

Figure 1.3: Modes of fracture

Mode I fracture is a type of fracture that happens when a material is pulled or stretched. It is also known as an opening or tensile fracture. When a material is under tension, it can develop a crack that spreads through the material in a direction perpendicular to the applied force. This crack can continue to grow until the material finally breaks apart.

Mode II fracture occurs when a material is under the shear stress, which means the material is subjected to a force that causes it to slide or deform in parallel planes. This type of fracture is also known as sliding or in-plane shear fracture. When a material is under shear stress, a crack can propagate through the material in a direction parallel to the applied force. This type of fracture is commonly seen in materials such as metal plates those are bolted together.

Mode III fracture occurs when a material is under anti-plane shear stress, which means that the material is being subjected to a force that causes it to tear or deform in planes perpendicular to the applied force. This type of fracture is also known as tearing or anti-plane shear fracture. When a material is under anti-plane shear stress, a crack can propagate through the material in a direction perpendicular to the applied force. This type of fracture is commonly seen in materials, where the fibers are oriented in a single plane.

Understanding of the three basic modes of fracture is essential in fields such as engineering and materials science, as it can help engineers and scientists to design the stronger and more durable materials those have chances of less failure under different types of stress.

1.3 Crack Propagation Parameters

Crack propagation is a highly complex process that requires interdisciplinary expertise in a variety of fields, including mathematics, materials science, mechanics, and physics. Researchers have made significant strides in understanding the factors and mechanisms that govern crack propagation in recent years. Thanks to advances in both computational methods and experimental techniques. These tools have allowed researchers to investigate crack propagation at various scales, ranging from the atomic to the macroscopic levels.

1.3.1 Stress intensity factor

The stress intensity factor, also denoted by 'K', is a term used in the field of fracture mechanics to describe the strength of the stress field near the tip of a crack in a material. It is calculated by dividing the applied stress by the square root of the size of the crack. The SIF is an important parameter for predicting the growth of a crack under different loading conditions.

SIF has been extensively researched in the field of fracture mechanics since the early work done by Griffith [21]. Over time, many researchers have developed different analytical and numerical methods to calculate SIF for various geometries and loading conditions. One of the early analytical methods for calculating SIFs was the energy method, which was introduced by Irwin [5]. This method uses the energy release rate to determine SIF. It has been commonly used in fracture mechanics to identify the critical crack size at which a material might fail. The initial concept for the formulization for SIF for a finite crack of crack length $'a'$ was defined as

$$K = \sigma\sqrt{\pi a} \tag{1.3.1}$$

Further, the above expression has been classified into the three classical modes of fracture and the expressions for the SIFs for the three different modes of fracture are defined as

$$K_I = \lim_{a\to 0}\sqrt{2\pi a}\ \sigma_{yy}(a,0), \tag{1.3.2}$$

$$K_{II} = \lim_{a\to 0}\sqrt{2\pi a}\ \sigma_{xy}(a,0), \tag{1.3.3}$$

$$K_{III} = \lim_{a\to 0}\sqrt{2\pi a}\ \sigma_{yz}(a,0). \tag{1.3.4}$$

1.3.2 Stress magnification factor

The stress magnification factor is a parameter used to describe the increase in stress near geometric discontinuities such as cracks, holes or notches in a material. It is calculated by dividing the maximum stress at the discontinuity by the nominal stress, which is the stress applied far away from the discontinuity. The stress magnification factor(SMF) is also an important parameter for predicting how a material will behave under different loading conditions.

The SMF shows the shielding (when SMF$<$1) and amplification (when SMF$>$1) effect of the crack.Suppose there is a crack of length a, then the SMF at the crack tip $x = a$ can be determined by

$$M_{Ia} = \frac{K_{Ia}}{K^*_{Ia}} \tag{1.3.5}$$

where K^*_{Ia} is SIF at $x = a$ due to the presence of crack $x = a$ only.

1.3.3 Crack opening displacement

Crack opening displacement (COD) is the amount of opening at a crack's tip when subjected to an external load. It is an important parameter in fracture mechanics, as it indicates the amount of stress required to propagate the crack.

The mathematical expression for the COD, is the distance between the inner surfaces near crack tip. Suppose the vertical displacement components around the crack tip $x = a,\ y = 0$ is $v(a, 0)$, then the COD at the crack tip $x = a$ can be determined by

$$COD = v(a, 0+) - v(a, 0-) \tag{1.3.6}$$

These three parameters: stress intensity factor, stress magnification factor, and crack opening displacement are crucial in fracture mechanics. They help in forecasting how a material will perform under diverse loading circumstances and enable to determine the size of the crack that will lead to the material's failure.

The fundamental concepts of solid mechanics can be applied to address difficulties in the study of stationary and moving cracks in dynamic fracture mechanics. In dynamic fracture mechanics, there are three different kinds of crack problems. The first one is a stationary crack that develops in a solid as a result of dynamic loading. In the second, a moving crack is present in the solid under quasi-static loading, while in the third, a moving fracture is present in the solid under dynamic loading. There are several review articles in the topic of fracture mechanics that deal with stationary and moving cracks, including Kamminen [32], Kasir and Bandopadhyay [33], Achenbach [34] and Freund [35] etc.

The concept of moving crack was first given by Yoffee [36], and the concept of moving semi-infinite crack was initiated by Knauss [37]. An exact solution for two collinear semi-infinite cracks using conformal mapping has been given by Shen and Fan [38]. A study on elastodynamic SIF has been given by Achenbach and gaustesen [39]. The study for SIF for multiple cracks using hypersingular integral equations has been given by Hamzah et al. [40, 41].

The study of edge cracks are more hazardous than interior cracks since they are more prone to damage. Dealing with edge cracks can be tricky since they are more prominent, have a higher inclination to fracture, and have a lesser chance of arrest. Edge cracks have been a challenge for scientists, engineers and researchers for the last few decades. The crack parameters like stress intensity factors are quite important in dealing with edge cracks. Some important works in edge cracks are: Williams [42] studied the stress field distribution on a stationary crack surface, Cruse and Vanburen [43] on three dimensional stress analysis on edge crack model. Abdel et al. [44] studied an important aspect of edge crack identification in the bridges.

Interfacial cracks are a type of cracks that occur at the interface or boundary between two identical or different materials. They can be caused by various factors, including differences in thermal expansion coefficients, mechanical stress, or chemical reactions between the two materials. Interfacial cracks are a common problem in many fields, including aerospace, electronics and biomedical engineering, as they can significantly affect the strength and durability of the materials. Researchers have investigated various methods for preventing or repairing interfacial cracks, including the use of coatings, adhesives, prescribed loading conditions, and other types of surface treatments.

Interfacial edge cracks are cracks that propagate perpendicular to the interface between two materials. They usually occur when the materials have a significant difference in their elastic properties, causing stresses to concentrate at the interface and leading to crack initiation and propagation. On the other hand, interfacial semi-infinite cracks are cracks that extend infinitely parallel to the interface. Interfacial cracks have been studied by many researchers so far. Initially, the concept of interfacial cracks in elastic fracture mechanics in between dissimilar solids has been given by Rice [45], Comninou [46], and many more. A closed-form solution for interfacial edge crack has been given by Wu et al. [47]. The study for multiple interfacial cracks for dissimilar media has been given by Monfared et al. [48]. Recently, Ustinov [49] studied the interfacial semi-infinite cracks for bi-material, and Zhang et al. [50] studied the antiplane mode of an interfacial semi-infinite crack in a bi-material structure.

Investigation of a semi-infinite crack for different materials has been done by Das and Patra [51]. A boundary value problem of a semi-infinite crack in Holder spaces has been studied by Itou and Tani [52]. The effect of mutiple cracks in a sheet using the dislocation method has been studied by Mirhosseini and Fariborz [53]. The closed form solution of a moving semi-infinite crack by determining the dynamic stress intensity factor at the crack tip has been given by Gonzalez and Mason [54].

1.4 Mathematical techniques and methods

This thesis will utilize a combination of theoretical approaches and computational simulations to achieve their objective, employing various mathematical techniques and models. Some of the Mathematical models and techniques which are used in the crack problems of this thesis are given below.

1.4.1 Fourier transformation

The Fourier transform along with is its inverse for a function $f(x, y)$ are defined as

$$\bar{f}(\xi, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x, y)e^{i\xi x}dx, \tag{1.4.1}$$

$$f(x, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{f}(\xi, y)e^{-i\xi x}d\xi, \tag{1.4.2}$$

where $\xi = a + ib$, a and b are real values.

1.4.2 Kronecker delta function

The Kronecker delta (named after Leopold Kronecker) is a function of two variables. The function is 1 if the variables are equal, and 0 otherwise, also defined as

$$\delta_{mn} = \begin{cases} 1, & if\ m = n \\ 0, & if\ m \neq n. \end{cases} \tag{1.4.3}$$

1.4.3 Bessel function

The Bessel function of First kind ($J_n(x)$) is the soluton of the Bessel equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0. \tag{1.4.4}$$

Also, in complex plane the bessel is defined in terms of the integral as

$$J_n(z) = \frac{1}{\pi}\int_0^{\pi}(zcos\theta - n\theta)d\theta. \tag{1.4.5}$$

1.4.4 Schmidt method

In order to find the unknown coefficients in an infinite series, this method is used. Let us suppose we have the system with unknown coefficients $a_n's$ and $b_n's$ as

$$\sum_{n=1}^{\infty} a_n E_n(x) + \sum_{n=1}^{\infty} b_n F_n(x) = p, \tag{1.4.6}$$

$$\sum_{n=1}^{\infty} a_n G_n(x) + \sum_{n=1}^{\infty} b_n H_n(x) = 0, \quad 0 < x < c \tag{1.4.7}$$

where c is a finite number and $E_n(x)$, $F_n(x)$, $G_n(x)$ and $H_n(x)$ are known functions of x.

Now we will construct a set of orthogonal function as $S_n(x)$ using one of the known fuction given above as

$$S_n(x) = \sum_{i=1}^{n} \frac{M_{in}}{M_{nn}} G_i(x), \tag{1.4.8}$$

with $S_1(x) = G_1(x)$ and $S_n(x)$ satisfies the orthogonal properties given as

$$\int_0^1 S_m(x) S_n(x) dx = N_n \delta_{nm}, \tag{1.4.9}$$

$$\int_0^1 [S_n(x)]^2 dx = N_n, \tag{1.4.10}$$

where δ_{mn} is given by Eq.(1.4.3) and M_{in} and the cofactors of the element l_{in} of L_n, defined as

$$L_n = \begin{vmatrix} l_{11} & l_{12} & \dots & l_{1n} \\ l_{21} & l_{22} & \dots & l_{2n} \\ \vdots & \vdots & & \vdots \\ l_{n1} & l_{n2} & \dots & l_{nn} \end{vmatrix}, \tag{1.4.11}$$

where

$$l_{in} = \int_0^1 G_i(x) G_n(x) dx\,. \tag{1.4.12}$$

Now rewritting the series from Eq.(1.4.7) in terms of the orthogonal series generated by $S_n(x)$ with the coefficients c_n as

$$\sum_{n=1}^{\infty} a_n G_n(x) = \sum_{n=1}^{\infty} c_n S_n(x) = -\sum_{n=1}^{\infty} b_n H_n(x). \tag{1.4.13}$$

By using the last two equalities from the Eq.(1.4.13), we get

$$c_n = \sum_{n=1}^{\infty} d_{ni} a_i, \tag{1.4.14}$$

where

$$d_{ni} = -\frac{1}{N_n} \int_0^1 S_n(x) H_i(x) dx.$$

By using the first two equalities and with the help of Eq.(1.4.14), we get

$$b_n = \sum_{i=1}^{\infty} e_{ni} a_i, \tag{1.4.15}$$

where

$$e_{ni} = -\sum_{j=n}^{\infty} \frac{1}{N_j} \frac{M_{jn}}{M_{ij}} \int_0^1 S_j(x) H_i(x) dx.$$

Here in Eq.(1.4.15), the value of one set of unknown has been found in terms of another. Now by putting of value of b_n into Eq.(1.4.6), we get

$$\sum_{n=1}^{\infty} a_n M_n(x) = p, \tag{1.4.16}$$

where

$$M_n(x) = F_n(x) + \sum_{n=1}^{\infty} e_{in} E_n(x).$$

Now using the same proceedure of orthogonalization by constructing anpther set of orthogonal functions say T_n, by using $M_n(x)$, which satisfy the following conditions

$$\int_0^1 T_m(x) T_n(x) dx = K_n \delta_{nm}, \tag{1.4.17}$$

$$\int_0^1 [T_n(x)]^2 dx = K_n. \tag{1.4.18}$$

The coefficients a_n can be determined as

$$a_n = \sum_{j=n}^{\infty} q_j \frac{P_{jn}}{P_{jj}}, \tag{1.4.19}$$

where

$$q_j = \frac{1}{K_j} \int_0^1 p\ T_j(x) dx,$$

and P_{jn} are the cofactors of the element of P_n, given as

$$P_n = \begin{vmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{vmatrix}, \tag{1.4.20}$$

where

$$p_{jn} = \int_0^1 M_j(x) M_n(x) dx\,. \tag{1.4.21}$$

Finally by putting the values of a_n from Eq.(1.4.19) into Eq.(1.4.15), we get the desired values of the coefficients b_n.

1.4.5 Wiener-Hopf technique

The Wiener-Hopf technique is a mathematical technique that has broad applications in applied mathematics. It was first invented by Norbert Wiener and Eberhard Hopf as a way of addressing integral equations. However, it has been used more extensively to solve partial differential equations with mixed boundary conditions and also to study a wide range of materials, including ceramics, composites, and biological tissues.

Several crack propagation problems can be reduced to Wiener-Hopf equation by using the fourier transformation. The equation contains two unknown functions, say $P_+(\xi)$ and $Q_-(\xi)$ in the complex ξ-plane, given as

$$A(\xi)P_+(\xi) + B(\xi)Q_-(\xi) + C(\xi) = 0, \tag{1.4.22}$$

where $\xi = \sigma + i\tau$ is a complex variable and $A(\xi)$, $B(\xi)$ and $C(\xi)$ are known functions, and $P_+(\xi)$ and $Q_+(\xi)$ are non zero analytic functions in the upper half-plane($\tau > \tau_-$) and lower half-plane($\tau > \tau_+$). The other three functions $A(\xi)$, $B(\xi)$ and $C(\xi)$ are known entire functions in the strip ($\tau_- < \tau < \tau_+, -\infty < \sigma < +\infty$) in which the Eq.(1.4.22) holds. It is assumed that $A(\xi) \neq 0$.

Hence we can rewrite Eq.(1.4.22) as

$$P_+(\xi) + \frac{B(\xi)}{A(\xi)}Q_-(\xi) + \frac{C(\xi)}{A(\xi)} = 0. \tag{1.4.23}$$

Now the next step in the Wiener-hopf technique is to find the factorization of the term $B(\xi)/A(\xi)$ of above equation, in terms of nonzero analytic functions $K_+(\xi)$ in upper half plane($\tau > \tau_-$) and $K_-(\xi)$ in lower half-planes($\tau > \tau_+$) as given as

$$\frac{B(\xi)}{A(\xi)} = K_+(\xi)K_-(\xi). \tag{1.4.24}$$

Using the above relation into the Eq.(1.4.23), we get

$$P_+(\xi) + K_+(\xi)K_-(\xi)Q_-(\xi) + \frac{C(\xi)}{A(\xi)} = 0, \tag{1.4.25}$$

or,

$$\frac{P_+(\xi)}{K_+(\xi)} + K_-(\xi)Q_-(\xi) + \frac{C(\xi)}{A(\xi)K_+(\xi)} = 0. \tag{1.4.26}$$

Now the last term of the above equation can be factorized as

$$\frac{C(\xi)}{A(\xi)K_+(\xi)} = L_+(\xi) + L_-(\xi), \tag{1.4.27}$$

where the functions $L_+(\xi)$ and $L_-(\xi)$ are non zero analytic in the above defined upper and lower half planes, respectively. Now using the Eq.(1.4.26) and Eq.(1.4.27) , we get

$$\frac{P_+(\xi)}{K_+(\xi)} + L_+(\xi) = -(K_-(\xi)Q_-(\xi) + L_-(\xi)). \tag{1.4.28}$$

It is clearly observed that the left hand side of above equation is analytic in the defined upper half plane and the right hand side is analytic in the defined lower half plane.

Definition 1.1. Suppose $f(z)$ is analytic on a region A. Suppose also that A is contained in a region B. We say that f can be analytically continued from A to B, if there is a function $g(z)$ such that

(i) $g(z)$ is analytic on B.

(ii) $g(z) = f(z)$ for all z in A.

Now using the analytic continuation by definition(1.1). Both sides of Eq.(1.4.28) are analytic in the whole complex ξ-plane. Suppose both sides of the Eq.(1.4.28) equals to an entire function, say $P(\xi)$ in the following manner

$$\left|\frac{P_+(\xi)}{K_+(\xi)} + L_+(\xi)\right| < |\xi|^p, \quad as\ \xi \to \infty,\ \tau > \tau_-, \tag{1.4.29}$$

$$|K_-(\xi)Q_-(\xi) + L_-(\xi)| < |\xi|^q, \quad as\ \xi \to -\infty,\ \tau < \tau_+. \tag{1.4.30}$$

Using the Extended Liouville theorem, $P(\xi)$ is a polynomial of degree less than or equal to the integral part of $\min(p, q)$. As given by the following equations

$$\frac{P_+(\xi)}{K_+(\xi)} + L_+(\xi) = P(\xi), \tag{1.4.31}$$

$$K_-(\xi)Q_-(\xi) + L_-(\xi) = P(\xi). \tag{1.4.32}$$

Hence, by using the above two equations the values of the unknown functions $P_+(\xi)$ and $Q_-(\xi)$ can be determined.

The crucial part of the Wiener-Hopf technique is the factorization from the Eq.(1.4.24). The theorems given by Noble (1958) can be used to find the factorization.

Theorem 1.4.1. *Let $f(\xi)$ be an analytic function in complex ξ-plane ($\xi = \sigma + i\tau$), analytic in the strip ($\tau_- < \tau < \tau_+, -\infty < \sigma < +\infty$) such that*

$$|f(\tau + i\sigma)| < k|\tau|^{-\lambda}, \quad k, \lambda > 0 \tag{1.4.33}$$

for $|\tau| \to \infty$, the inequality holding uniformly for all σ in the strip $\tau_- + \delta < \tau < \tau_+ - \delta$, for any arbitrary small $\delta > 0$. Then for $\tau_- < a < \tau < b < \tau_+$,

$$f(\xi) = f_+(\xi) + f_-(\xi) \tag{1.4.34}$$

with

$$f_+(\xi) = \frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia} \frac{f(\eta)}{\eta - \xi} d\eta, \tag{1.4.35}$$

$$f_-(\xi) = -\frac{1}{2\pi i}\int_{-\infty+ib}^{\infty+ib} \frac{f(\eta)}{\eta - \xi} d\eta, \tag{1.4.36}$$

where $f_+(\xi)$ and $f_-(\xi)$ are analytic in $(\tau > \tau_-)$ and $(\tau > \tau_+)$, respectively.

Theorem 1.4.2. *Let $lnK(\xi)$ is an analytic function in the strip $\tau_- < \tau < \tau_+$ and $K(\xi) \rightarrow 1$, as $\sigma \rightarrow \pm\infty$ for $\xi = \tau + i\sigma$, then $K(\xi) = K_+(\xi)K_-(\xi)$, where $K_+(\xi)$ and $K_-(\xi)$ are non- zero bounded analytic functions in $\tau > \tau_-$ and $\tau < \tau_+$, respectively.*

By using both the theorems stated above, the decomosition of the Eq.(1.4.24) can be found.