# INTERVAL VARIATIONAL ANALYSIS AND ITS APPLICATIONS IN INTERVAL OPTIMIZATION

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# Chapter 6

# Conclusions and future scopes

A compendium of the main conclusions that can be made from this thesis is given in this chapter. It also includes a summary of the directions for future research scopes.

## 6.1 General conclusions

The principal contributions of this thesis are as follows.

- Several inequalities of interval analysis which are helpful to develop the theories of interval-valued functions and interval optimization problems are proved. The concept of sequence of intervals is also introduced, which is useful to develop the calculus of IVFs.

- The concepts of $gH$-semicontinuity and $gH$-subdifferentiability for IVFs have been studied in detail. Several subdifferential calculus results have been derived for IVFs.

- Conventional Ekeland's variational principle has been extended for IVFs.

- Rigorous information about the optimality conditions of unconstraint and constraint IOPs with the help of all proposed concepts have been found. Also, by using variational inequalities for IVFs, necessary and sufficient optimality conditions for constraint IOPs are explored.

## 6.2 Contributions of the thesis

This thesis appertains to the theories on analysis of IVFs and characterization to the solutions of nonsmooth IOPs through the idea of $gH$-semicontinuity and $gH$-Fréchet subderivatives. Chapter wise contributions made of this dissertation are highlighted below.

**Chapter 2** has attempted to propose Ekeland's variational principle for interval-valued functions (IVFs). To develop the variational principle, the concept of sequence of intervals is studied. In the sequel, the idea of $gH$-semicontinuity for IVFs is explored. A necessary and sufficient condition for an IVF to be $gH$-continuous in terms of $gH$-lower and upper semicontinuity is given. Moreover, we have proved a characterization for $gH$-lower semicontinuity by the level sets of the IVF. With the help of this characterization result, it is ensured the existence of a minimum for an extended $gH$-lower semicontinuous, level-bounded and proper IVF. To find an approximate minima of a $gH$-lower semicontinuous and $gH$-Gâteaux differentiable IVF, the proposed Ekeland's variational principle is used.

**Chapter 3** has explained the notion of Fréchet subdifferentiability or $gH$-Fréchet subdifferentiability to deal with nondifferentiable interval-valued functions (IVFs) (not necessarily convex). In the sequel, we have explored its relationship with $gH$-differentiability and developed various calculus results for $gH$-Fréchet subgradients of extended IVFs. By using the proposed notion of subdifferentiability, we have derived new necessary optimality conditions for unconstrained interval optimization problems (IOPs) with nondifferentiable IVFs. We have also provided a necessary condition for unconstrained weak sharp minima of an extended IVF in terms of the proposed notion of subdifferentiability. The whole study is supported with suitable examples.

**Chapter 4** has described the notion of interval variational inequalities and established a relation between their solution sets. We also have defined interval-valued pseudoconvex and pseudomonotone functions. For an interval optimization problem (IOP) a necessary and sufficient condition of optimality is given in terms of Stampacchia interval variational inequality. The entire study is supported with suitable illustrative examples.

**Chapter 5** has been devoted to explore the Stampacchia and Minty variational inequalities for interval-valued functions (IVFs). In the sequel, it is observed that conventional Stampacchia and Minty variational inequalities are special cases of the proposed inequalities. The relation between solution sets of these two variational inequalities has been analyzed. Existence and uniqueness results are provided for the solutions of the proposed variational inequalities. Moreover, a necessary optimality condition is given for a constrained interval optimization problem (IOP) using the generalized Hukuhara differentiability. It is observed that the first-order characterization of convex IVFs given in the literature is not true and a new first-order necessary condition is given for convex IVFs. By using this new first-order necessary condition for convex IVFs, as an application of the proposed study, a necessary and sufficient optimality condition for a constrained IOP is provided in terms of Stampacchia IVI. Further, a sufficient optimality condition for a constrained IOP is provided in terms of Minty IVI.

## 6.3 Future scopes of studies

Most of the concerned research community is engaged in discovering new theories and simultaneously trying to improve existing theories. From the analysis of the work presented in this thesis, there are several scopes for the extension. Some of the opportunities for future research are mentioned below.

- The future research will demand for giving a $gH$-subgradient technique to obtain the whole solution set of a nonsmooth nonconvex IOP.
- There is a huge literature [54, 69, 101] on the iterative methods to solve conventional Stampacchia and Minty VIs. In future, one can try to extend some classical iterative techniques to solve Stampacchia and Minty IVIs.
- Another promising direction of future research can be the analysis of the fuzzy-valued functions (FVFs) as the alpha-cuts of fuzzy numbers are compact intervals. So, in the future, one can attempt to extend the proposed idea of all these derivatives for fuzzy-valued functions.
- A noisy and uncertain environment in the control system or a differential equation appears due to the incomplete information of demand for a product and changes in the environment. In the future, we will try to solve a control problem in a noisy or uncertain environment with the help of proposed concepts of IVFs.

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