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A. Appendix A

A.1 Proof of the Lemma 1.3

Proof.

Proof of (i). Let $\mathbf{A} = [\underline{a}, \overline{a}]$ and $\mathbf{B} = [\underline{b}, \overline{b}]$. Then,

$$\|\mathbf{A} \oplus \mathbf{B}\|_{I(\mathbb{R})} = \|[\underline{a}, \overline{a}] \oplus [\underline{b}, \overline{b}]\|_{I(\mathbb{R})} = \|[\underline{a} + \underline{b}, \overline{a} + \overline{b}]\|_{I(\mathbb{R})} = \max\{|\underline{a} + \underline{b}|, |\overline{a} + \overline{b}|\}.$$

We now have the following two possible cases.

- **Case** 1. $\|\mathbf{A} \oplus \mathbf{B}\|_{I(\mathbb{R})} = |\underline{a} + \underline{b}|$.

  Since $|\underline{a}+\underline{b}| \leq |\underline{a}|+|\underline{b}| \leq \max\{|\underline{a}|, |\overline{a}|\}+\max\{|\underline{b}|, |\overline{b}|\} = \|\mathbf{A}\|_{I(\mathbb{R})}+\|\mathbf{B}\|_{I(\mathbb{R})}$,

  we get $\|\mathbf{A} \oplus \mathbf{B}\|_{I(\mathbb{R})} \leq \|\mathbf{A}\|_{I(\mathbb{R})} + \|\mathbf{B}\|_{I(\mathbb{R})}$.

- **Case** 2. $\|\mathbf{A} \oplus \mathbf{B}\|_{I(\mathbb{R})} = |\overline{a} + \overline{b}|$.

  Since $|\overline{a}+\overline{b}| \leq |\overline{a}|+|\overline{b}| \leq \max\{|\underline{a}|, |\overline{a}|\}+\max\{|\underline{b}|, |\overline{b}|\} = \|\mathbf{A}\|_{I(\mathbb{R})}+\|\mathbf{B}\|_{I(\mathbb{R})}$,

  therefore, $\|\mathbf{A} \oplus \mathbf{B}\|_{I(\mathbb{R})} \leq \|\mathbf{A}\|_{I(\mathbb{R})} + \|\mathbf{B}\|_{I(\mathbb{R})}$.

Hence, $\|\mathbf{A} \oplus \mathbf{B}\|_{I(\mathbb{R})} \leq \|\mathbf{A}\|_{I(\mathbb{R})} + \|\mathbf{B}\|_{I(\mathbb{R})}$ for all $\mathbf{A},\ \mathbf{B} \in I(\mathbb{R})$.

Proof of (ii). Let $\mathbf{A} = [\underline{a}, \overline{a}],\ \mathbf{B} = [\underline{b}, \overline{b}],\ \mathbf{C} = [\underline{c}, \overline{c}]$ and $\mathbf{D} = [\underline{d}, \overline{d}]$.

We note that

$$\mathbf{A} \preceq \mathbf{C} \implies [\underline{a}, \overline{a}] \preceq [\underline{c}, \overline{c}] \implies \underline{a} \leq \underline{c} \text{ and } \overline{a} \leq \overline{c}. \tag{A.1}$$

Also,

$$\mathbf{B} \preceq \mathbf{D} \implies [\underline{b}, \overline{b}] \preceq [\underline{d}, \overline{d}] \implies \underline{b} \leq \underline{d} \text{ and } \overline{b} \leq \overline{d}. \tag{A.2}$$

From (A.1) and (A.2), we have

$$\begin{aligned} & \underline{a} + \underline{b} \leq \underline{c} + \underline{d} \text{ and } \overline{a} + \overline{b} \leq \overline{c} + \overline{d} \\ \implies & [\underline{a} + \underline{b}, \overline{a} + \overline{b}] \preceq [\underline{c} + \underline{d}, \overline{c} + \overline{d}]. \end{aligned}$$

Thus, $\mathbf{A} \oplus \mathbf{B} \preceq \mathbf{C} \oplus \mathbf{D}$.

□

A.2 Proof of the Lemma 1.4

Proof.

Proof of (i). Let $\mathbf{A} = [\underline{a}, \overline{a}]$, $\mathbf{B} = [\underline{b}, \overline{b}]$ and $\epsilon > 0$.

$\mathbf{A} \ominus_{gH} \mathbf{B} = [\underline{a} - \underline{b}, \overline{a} - \overline{b}]$ or $[\overline{a} - \overline{b}, \underline{a} - \underline{b}]$. Let us now consider the following four possible cases.

- **Case** 1. $\mathbf{A} \ominus_{gH} \mathbf{B} = [\underline{a} - \underline{b}, \overline{a} - \overline{b}]$ and $\|\mathbf{A} \ominus_{gH} \mathbf{B}\|_{I(\mathbb{R})} = |\underline{a} - \underline{b}|$.

  So, we have

  $$\underline{a} - \underline{b} \leq \overline{a} - \overline{b} \text{ and } |\overline{a} - \overline{b}| \leq |\underline{a} - \underline{b}|. \tag{A.3}$$

  Let $\|\mathbf{A} \ominus_{gH} \mathbf{B}\|_{I(\mathbb{R})} < \epsilon$. Then,

  $$|\underline{a} - \underline{b}| < \epsilon. \tag{A.4}$$

By equation (A.4), we have $-\epsilon < \underline{a} - \underline{b} < \epsilon$, and hence $\underline{b} - \epsilon < \underline{a}$.

By equations (A.3) and (A.4), we have $|\overline{a} - \overline{b}| < \epsilon$. This implies $\overline{b} - \epsilon < \overline{a}$.

Therefore, $\mathbf{B} \ominus_{gH} [\epsilon, \epsilon] = [\underline{b} - \epsilon, \overline{b} - \epsilon] \prec [\underline{a}, \overline{a}] = \mathbf{A}$.

Note that by equation (A.4), $\underline{a} < \underline{b} + \epsilon$. Also, by equations (A.3) and (A.4), we have $|\overline{a} - \overline{b}| < \epsilon$. This implies $\overline{a} < \overline{b} + \epsilon$. Therefore, $\mathbf{A} = [\underline{a}, \overline{a}] \prec [\underline{b} + \epsilon, \overline{b} + \epsilon] = \mathbf{B} \oplus [\epsilon, \epsilon]$.

- **Case** 2. $\mathbf{A} \ominus_{gH} \mathbf{B} = [\underline{a} - \underline{b}, \overline{a} - \overline{b}]$ and $\|\mathbf{A} \ominus_{gH} \mathbf{B}\|_{I(\mathbb{R})} = |\overline{a} - \overline{b}|$.

  So, we have

$$\underline{a} - \underline{b} \leq \overline{a} - \overline{b} \text{ and } |\underline{a} - \underline{b}| \leq |\overline{a} - \overline{b}|. \tag{A.5}$$

  Consider

$$\begin{aligned} & \|\mathbf{A} \ominus_{gH} \mathbf{B}\|_{I(\mathbb{R})} < \epsilon \\ \implies & |\overline{a} - \overline{b}| < \epsilon \end{aligned} \tag{A.6}$$

  By equation (A.6), we have

$$\overline{b} - \epsilon < \overline{a}.$$

  By equations (A.5) and A.6), we have $|\underline{a} - \underline{b}| < \epsilon$. This implies $\underline{b} - \epsilon < \underline{a}$.

  Therefore, $\mathbf{B} \ominus_{gH} [\epsilon, \epsilon] = [\underline{b} - \epsilon, \overline{b} - \epsilon] \prec [\underline{a}, \overline{a}]$.

  Note that by equation (A.6), $\overline{a} < \overline{b} + \epsilon$. Also, by equations (A.5) and (A.6), we have $|\underline{a} - \underline{b}| < \epsilon$. This implies $\underline{a} < \underline{b} + \epsilon$. Therefore, $\mathbf{A} = [\underline{a}, \overline{a}] \prec [\underline{b} + \epsilon, \overline{b} + \epsilon] = \mathbf{B} \oplus [\epsilon, \epsilon]$.

- **Case** 3. $\mathbf{A} \ominus_{gH} \mathbf{B} = [\overline{a} - \overline{b}, \underline{a} - \underline{b}]$ and $\|\mathbf{A} \ominus_{gH} \mathbf{B}\|_{I(\mathbb{R})} = |\underline{a} - \underline{b}|$.

  This case can be proved by following the steps similar to **Case** 1.

- **Case** 4. $\mathbf{A} \ominus_{gH} \mathbf{B} = [\overline{a} - \overline{b}, \underline{a} - \underline{b}]$ and $\|\mathbf{A} \ominus_{gH} \mathbf{B}\|_{I(\mathbb{R})} = |\overline{a} - \overline{b}|$.

  This case can be proved by following the steps similar to **Case** 2.

Conversely, let $\mathbf{B} \ominus_{gH} [\epsilon, \epsilon] \prec \mathbf{A} \prec \mathbf{B} \oplus [\epsilon, \epsilon]$.

Note that

$$\begin{aligned} \mathbf{B} \ominus_{gH} [\epsilon, \epsilon] \prec \mathbf{A} \quad &\Longrightarrow \quad [\underline{b} - \epsilon, \overline{b} - \epsilon] \prec [\underline{a}, \overline{a}] \\ &\Longrightarrow \quad \underline{b} - \epsilon < \underline{a} \text{ and } \overline{b} - \epsilon < \overline{a}. \end{aligned} \tag{A.7}$$

Also,

$$\begin{aligned} \mathbf{A} \prec \mathbf{B} \oplus [\epsilon, \epsilon] \quad &\Longrightarrow \quad [\underline{a}, \overline{a}] \prec [\underline{b} + \epsilon, \overline{b} + \epsilon] \\ &\Longrightarrow \quad \underline{a} < \underline{b} + \epsilon \text{ and } \overline{a} < \overline{b} + \epsilon. \end{aligned} \tag{A.8}$$

From equations (A.7) and (A.8), we have

$$\begin{aligned} & \underline{b} - \epsilon < \underline{a} < \underline{b} + \epsilon \text{ and } \overline{b} - \epsilon < \overline{a} < \overline{b} + \epsilon \\ \Longrightarrow \quad & |\underline{a} - \underline{b}| < \epsilon \text{ and } |\overline{a} - \overline{b}| < \epsilon \\ \Longrightarrow \quad & \max\{|\underline{a} - \underline{b}|, |\overline{a} - \overline{b}|\} < \epsilon \\ \text{i.e.,} \quad & \|\mathbf{A} \ominus_{gH} \mathbf{B}\|_{I(\mathbb{R})} < \epsilon. \end{aligned}$$

This completes the proof of (i).

Proof of (ii). Let $\mathbf{A} = [\underline{a}, \overline{a}]$, $\mathbf{B} = [\underline{b}, \overline{b}]$ and $\epsilon > 0$.

Consider $\mathbf{A} \ominus_{gH} [\epsilon, \epsilon] \nprec \mathbf{B}$. This implies $[\underline{a} - \epsilon, \overline{a} - \epsilon] \nprec [\underline{b}, \overline{b}]$. Thus, '$\underline{b} \leq \underline{a} - \epsilon$ and $\overline{b} \leq \overline{a} - \epsilon$' or '$\underline{b} < \underline{a} - \epsilon$ and $\overline{b} > \overline{a} - \epsilon$' or '$\underline{b} > \underline{a} - \epsilon$ and $\overline{b} < \overline{a} - \epsilon$'. Let us consider all these three possibilities in the following three cases.

- **Case** 1. $\underline{b} \leq \underline{a} - \epsilon$ and $\overline{b} \leq \overline{a} - \epsilon$.

  So, we have

$$\begin{aligned} & \underline{a} > \underline{b} \text{ and } \overline{a} > \overline{b}, \text{ because } \epsilon > 0 \\ \implies \quad & \mathbf{B} \prec \mathbf{A} \implies \mathbf{A} \not\preceq \mathbf{B}. \end{aligned}$$

- **Case** 2. $\underline{b} < \underline{a} - \epsilon$ and $\overline{b} > \overline{a} - \epsilon$.

  Since $\underline{b} < \underline{a} - \epsilon$, so $\underline{a} > \underline{b}$, and thus $\mathbf{A} \not\preceq \mathbf{B}$.

- **Case** 3. $\underline{b} > \underline{a} - \epsilon$ and $\overline{b} < \overline{a} - \epsilon$.

  Since $\overline{b} < \overline{a} - \epsilon$, so $\overline{a} > \overline{b}$, and thus $\mathbf{A} \not\preceq \mathbf{B}$.

Hence, proof of (ii) is complete. □

A.3 Proof of the Lemma 1.8

Proof.

Proof of (i). Let $\mathbf{A} = [\underline{a}, \overline{a}]$ and $\mathbf{B} = [\underline{b}, \overline{b}]$.

Then, $\mathbf{A} \ominus_{gH} \mathbf{B} = [\underline{a} - \underline{b}, \overline{a} - \overline{b}]$ or $[\overline{a} - \overline{b}, \underline{a} - \underline{b}]$. Let us now consider the following two possible cases.

- Case 1. $\mathbf{A} \ominus_{gH} \mathbf{B} = [\underline{a} - \underline{b}, \overline{a} - \overline{b}]$.

  We see that

$$\begin{aligned} \mathbf{0} \preceq \mathbf{A} \ominus_{gH} \mathbf{B} \quad & \iff \quad \mathbf{0} \preceq [\underline{a} - \underline{b}, \overline{a} - \overline{b}] \iff 0 \leq \underline{a} - \underline{b} \text{ and } 0 \leq \overline{a} - \overline{b} \\ & \iff \quad \underline{b} \leq \underline{a} \text{ and } \overline{b} \leq \overline{a} \iff \mathbf{B} \preceq \mathbf{A}. \end{aligned}$$

• Case 2. $\mathbf{A} \ominus_{gH} \mathbf{B} = [\overline{a} - \overline{b}, \underline{a} - \underline{b}]$.

Note that

$$\begin{aligned} \mathbf{0} \preceq \mathbf{A} \ominus_{gH} \mathbf{B} &\iff \mathbf{0} \preceq [\overline{a} - \overline{b}, \underline{a} - \underline{b}] \iff 0 \leq \overline{a} - \overline{b} \text{ and } 0 \leq \underline{a} - \underline{b} \\ &\iff \overline{b} \leq \overline{a} \text{ and } \underline{b} \leq \underline{a} \iff \mathbf{B} \preceq \mathbf{A}. \end{aligned}$$

Hence, $\mathbf{0} \preceq \mathbf{A} \ominus_{gH} \mathbf{B} \iff \mathbf{B} \preceq \mathbf{A}$.

Proof of (ii). Let $\mathbf{A} = [\underline{a}, \overline{a}]$ and $\mathbf{B} = [\underline{b}, \overline{b}]$.

Then, $\mathbf{A} \ominus_{gH} \mathbf{B} = [\underline{a} - \underline{b}, \overline{a} - \overline{b}]$ or $[\overline{a} - \overline{b}, \underline{a} - \underline{b}]$. Let us now consider the following two possible cases.

• Case 1. $\mathbf{A} \ominus_{gH} \mathbf{B} = [\underline{a} - \underline{b}, \overline{a} - \overline{b}]$.

In this case, we have $\underline{a} - \underline{b} \leq \overline{a} - \overline{b}$. This implies

$$\underline{b} - \underline{a} \geq \overline{b} - \overline{a}. \tag{A.9}$$

Also, in this case, we have $(-1) \odot (\mathbf{A} \ominus_{gH} \mathbf{B}) = [\overline{b} - \overline{a}, \underline{b} - \underline{a}]$. Notice that

$$\begin{aligned} \mathbf{B} \ominus_{gH} \mathbf{A} &= [\min\{\underline{b} - \underline{a}, \overline{b} - \overline{a}\}, \max\{\underline{b} - \underline{a}, \overline{b} - \overline{a}\}] \\ &= [\overline{b} - \overline{a}, \underline{b} - \underline{a}], \text{ by (A.9)} \\ &= (-1) \odot (\mathbf{A} \ominus_{gH} \mathbf{B}). \end{aligned}$$

• Case 2. $\mathbf{A} \ominus_{gH} \mathbf{B} = [\overline{a} - \overline{b}, \underline{a} - \underline{b}]$.

In this case, we have $\overline{a} - \overline{b} \leq \underline{a} - \underline{b}$. Therefore,

$$\overline{b} - \overline{a} \geq \underline{b} - \underline{a}. \tag{A.10}$$

Also, in this case, we have $(-1) \odot (\mathbf{A} \ominus_{gH} \mathbf{B}) = [\underline{b} - \underline{a}, \overline{b} - \overline{a}]$. Note that

$$\begin{aligned} \mathbf{B} \ominus_{gH} \mathbf{A} &= [\min\{\underline{b} - \underline{a}, \overline{b} - \overline{a}\}, \max\{\underline{b} - \underline{a}, \overline{b} - \overline{a}\}] \\ &= [\underline{b} - \underline{a}, \overline{b} - \overline{a}], \text{ by (A.10)} \\ &= (-1) \odot (\mathbf{A} \ominus_{gH} \mathbf{B}). \end{aligned}$$

Hence, from Case 1 and 2, we get

$$(-1) \odot (\mathbf{A} \ominus_{gH} \mathbf{B}) = \mathbf{B} \ominus_{gH} \mathbf{A}.$$

□

B. Appendix B

B.1 Proof of the Lemma 1.31

Proof. • Case 1. $a < \bar{x} < b$.

Since $\nabla\mathbf{F}(\bar{x})$ exists, either $\nabla\mathbf{F}(\bar{x}) \prec \mathbf{0}$ or $\mathbf{0} \prec \nabla\mathbf{F}(\bar{x})$ or $\mathbf{0} \subseteq \nabla\mathbf{F}(\bar{x})$.

If possible, let $\nabla\mathbf{F}(\bar{x}) \prec \mathbf{0}$. Then,

$$\lim_{x\to\bar{x}} \frac{\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x})}{x - \bar{x}} \prec \mathbf{0}.$$

Therefore, by Definition 3.1 of gH-limit, there exists a $\delta_1 > 0$ such that

$$\frac{\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x})}{x - \bar{x}} \prec \mathbf{0}$$

for all x's satisfying '$0 < |x - \bar{x}| < \delta_1$ and $x \in [a, b]$'.

Let $\bar{x} < x < \bar{x} + \delta_1$. Then, $x - \bar{x} > 0$, and therefore $\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \prec \mathbf{0}$ for all $x \in (\bar{x}, \bar{x} + \delta_1) \cap [a, b]$.

Thus, by Lemma 1.6, we get $\mathbf{F}(x) \prec \mathbf{F}(\bar{x})$ for all $x \in (\bar{x}, \bar{x} + \delta_1) \cap [a, b]$.

This contradicts that $\bar{x}$ is a weak efficient solution of (1.4).

Let us now assume that $\mathbf{0} \prec \nabla\mathbf{F}(\bar{x})$. Then,

$$\mathbf{0} \prec \lim_{x\to\bar{x}} \frac{\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x})}{x - \bar{x}}.$$

Therefore, by Definition 3.1 of gH-limit, there exists a $\delta_2 > 0$ such that

$$\mathbf{0} \prec \frac{\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x})}{x - \bar{x}}$$

for all x's satisfying '$0 < |x - \bar{x}| < \delta_2$ and $x \in [a, b]$'.

Let $\bar{x} - \delta_2 < x < \bar{x}$. Then, $x - \bar{x} < 0$, and therefore $\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \prec \mathbf{0}$ for all $x \in (\bar{x} - \delta_2, \bar{x}) \cap [a, b]$.

Thus, by Lemma 1.6, we get $\mathbf{F}(x) \prec \mathbf{F}(\bar{x})$ for all $x \in (\bar{x} - \delta_2, \bar{x}) \cap [a, b]$.

This contradicts that $\bar{x}$ is a weak efficient solution of (1.4).

Hence, $\mathbf{0} \in \nabla\mathbf{F}(\bar{x})$.

- Case 2. $\bar{x} = a$.

  Suppose on contrary that $\mathbf{0} \not\preceq \nabla\mathbf{F}(\bar{x})$. Then,

$$\mathbf{0} \not\preceq \lim_{x \to \bar{x}^+} \frac{\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x})}{x - \bar{x}}.$$

  Thus, similarly as in Case 1, we get $\mathbf{F}(\bar{x}) \not\preceq \mathbf{F}(x)$ for all $x \in (\bar{x}, \bar{x} + \delta_3) \cap [a, b]$, for some $\delta_3 > 0$. This is a contradiction as $\bar{x}$ is a weak efficient solution of (1.4). Hence, $\mathbf{0} \preceq \nabla\mathbf{F}(\bar{x})$.

- Case 3. $\bar{x} = b$.

  Proof contains similar steps as in Case 2.

□

C. List of Publications

[1] Kumar, G., and Ghosh, D. (2023). Ekeland's variational principle for interval-valued functions. *Computational and Applied Mathematics*, 42(1), 28.

[2] G. Kumar and J.C. Yao (2022). Fréchet subdifferential calculus for interval-valued functions and its applications in nonsmooth interval optimization. *Journal of Nonlinear and Variational Analysis*, (Accepted Manuscript).

[3] Kumar, G., and Ghosh, D. (2020). Interval variational inequalities. *In Soft Computing for Problem Solving 2019: Proceedings of SocProS 2019*, Volume 1 (pp. 309-321). Springer Singapore.

[4] Kumar, G., and Som, T. (2023). Interval variational inequalities and their relationship with interval optimization problems. *The Journal of Analysis*, 1-24.

[5] Kumar, K., Ghosh, D., and Kumar, G. (2022). Weak sharp minima for interval-valued functions and its primal-dual characterizations using generalized Hukuhara subdifferentiability. *Soft Computing*, 26(19), 10253-10273.