Chapter 4

Interval Variational Inequalities

4.1 Introduction

Variational inequalities (VIs) play a major role in the current research in the optimization theory. In the finite dimensional setting a VI can be viewed as the generalization of the necessary optimality condition for minimizing a differentiable function over a convex set. During the last three decades, VIs are used as a tool to solve optimization problems, see, for instance [3, 8, 41, 54, 69, 101], and the references therein. In Chapters 4 and 5, we study variational inequalities for IVFs. Interval variational inequalities are introduced and some brief study is made on them in Chapter 4 which has appeared in the proceeding of SocPros [70]. Further in Chapter 5, we study thoroughly this topic in the context of interval optimization. The results obtained in Chapter 5 are new and different from the results obtained in Chapter 4.

4.1.1 Motivation

It is well-known that the conventional variational inequalities (VIs) are important tool to solve conventional optimization problems. Motivated by this, in this chapter, we propose a study of Stampacchia and Minty VIs for IVFs.

4.1.2 Contribution

In this chapter, we

- introduce the concept of interval variational inequalities,
- discuss the solution sets of these variational inequalities with the help of suitable examples, and
- derive a necessary and sufficient condition to find the efficient solution of an IOP.

4.2 Stampacchia and Minty Interval Variational Inequalities

This section presents IVIs and deduces the relationship among their solution sets. Let X be a topological vector space and $Y = \mathbb{R}_I$ be a set of all closed and bounded intervals. Let K be a non-empty convex subset of X. Let $T : K \to \mathscr{L}(X, Y)$ be a mapping, where $\mathscr{L}(X, Y)$ denotes the space of gH-continuous linear maps from X to Y. For every $l \in \mathscr{L}(X, Y)$, the value of l at x is denoted by $\langle l, x\rangle$.
We define interval variational inequality problems (IVIPs) as follows:

- Stampacchia interval variational inequality problem (SIVIP) is to find $\bar{x} \in K$ satisfying

$$\langle T(\bar{x}), y - \bar{x} \rangle \nprec \mathbf{0}, \ \forall y \in K. \tag{4.1}$$

- Minty interval variational inequality problem (MIVIP) is to find $\bar{x} \in K$ satisfying

$$\langle T(y), y - \bar{x} \rangle \nprec \mathbf{0}, \ \forall y \in K. \tag{4.2}$$

We call equation (4.1) as Stampacchia interval variational inequality (SIVI) and equation (4.2) as Minty interval variational inequality (MIVI).

Note 11. If we take $Y = \mathbb{R}$ in (4.1) and (4.2) then these inequalities reduces to the conventional Stampacchia and Minty VIs respectively.

We denote solution set of (4.1) by Sol(SIVIP) and solution set of (4.2) by Sol(MIVIP). Let us now look at some IVIPs and their solution sets.

Example 4.1. *Let $X = \mathbb{R}$ and $K = [0, 10]$. Let $\underline{f}(x), \overline{f}(x) : \mathbb{R} \to \mathbb{R}$ be defined as $\underline{f}(x) = -x$ and $\overline{f}(x) = x^2$. Let $T : K \to \mathscr{L}(X, Y)$ be defined by $T(x) = [\underline{f}(x), \overline{f}(x)], \ \forall x \in K$.*

$$\begin{aligned} \textit{Sol(SIVIP)} &= \{\bar{x} \in K | \langle T(\bar{x}), y - \bar{x} \rangle \nprec \boldsymbol{0}, \ \forall y \in K\} \\ &= \{\bar{x} \in K | \langle [-\bar{x}, \bar{x}^2], y - \bar{x} \rangle \nprec \boldsymbol{0}, \ \forall y \in K\} \\ &= \{\bar{x} \in K | [-\bar{x}(y - \bar{x}), \bar{x}^2(y - \bar{x})] \nprec \boldsymbol{0}, \ \forall y \in K\} \\ &= \{\bar{x} \in K | [-\bar{x}(y - \bar{x}), \bar{x}^2(y - \bar{x})] \prec \boldsymbol{0} \textit{ is not true for any } y \in K\}. \end{aligned}$$

At $\bar{x} = 0$:

Case *1.* $0(y - 0) = 0 < 0$ *and* $0 \leq 0$*, which is not true for any $y \in K$.*

Case *2.* $0 \leq 0$ *and* $0 < 0$*, which is also not true for any $y \in K$.*

Case *3.* $0 < 0$ *and* $0 < 0$*, which is again not true for any* $y \in K$*.*

Therefore, $[0(y-0), 0(y-0)] \prec \boldsymbol{0}$ *is not true for any* $y \in K$

$\Longrightarrow$ $[0(y-0), 0(y-0)] \not\prec \boldsymbol{0},\ \forall y \in K$

$\Longrightarrow$ 0 *is a solution of SIVIP.*

At $\bar{x} = 10$*:*

Case *1.* $-10(y-10) < 0$ *and* $100(y-10) \leq 0$*, which is not true for any* $y \in K$*.*

Case *2.* $-10(y-10) \leq 0$ *and* $100(y-10) < 0$*, which is also not true for any* $y \in K$*.*

Case *3.* $-10(y-10) < 0$ *and* $100(y-10) < 0$*, which is again not true for any* $y \in K$*.*

Therefore, $[-10(y-10), 100(y-10)] \prec \boldsymbol{0}$ *is not true for any* $y \in K$

$\Longrightarrow$ $[-10(y-10), 100(y-10)] \not\prec \boldsymbol{0},\ \forall y \in K$

$\Longrightarrow$ 10 *is a solution of SIVIP.*

At $\bar{x} = \alpha \in (0, 10)$*:*

Case *1.* $-\alpha(y-\alpha) < 0$ *and* $\alpha^2(y-\alpha) \leq 0$*, which is not true for any* $y \in K$*.*

Case *2.* $-\alpha(y-\alpha) \leq 0$ *and* $\alpha^2(y-\alpha) < 0$*, which is also not true for any* $y \in K$*.*

Case *3.* $-\alpha(y-\alpha) < 0$ *and* $\alpha^2(y-\alpha) < 0$*, which is again not true for any* $y \in K$*.*

Therefore, $[-\alpha(y-\alpha), \alpha^2(y-\alpha)] \prec \boldsymbol{0}$ *is not true for any* $y \in K$

$\Longrightarrow$ $[-\alpha(y-\alpha), \alpha^2(y-\alpha)] \not\prec \boldsymbol{0},\ \forall y \in K.$

$\Longrightarrow$ α *is a solution of SIVIP.*

Thus, Sol(SIVIP)$= [0, 10]$.

Let us now calculate Sol(MIVIP).

$$\begin{aligned}
\textit{Sol(MIVIP)} &= \{\bar{x} \in K | \langle T(y), y - \bar{x}\rangle \not\prec \boldsymbol{0},\ \forall y \in K\} \\
&= \{\bar{x} \in K | \langle [-y, y^2], y - \bar{x}\rangle \not\prec \boldsymbol{0},\ \forall y \in K\} \\
&= \{\bar{x} \in K | [-y(y-\bar{x}), y^2(y-\bar{x})] \not\prec \boldsymbol{0},\ \forall y \in K\} \\
&= \{\bar{x} \in K | [-y(y-\bar{x}), y^2(y-\bar{x})] \prec \boldsymbol{0} \text{ is not true for any } y \in K\}.
\end{aligned}$$

At $\bar{x} = 0$*:*

Case *1.* $-y^2 < 0$ *and* $y^3 \leq 0$*, which is not true for any* $y \in K$*.*

Case *2.* $-y^2 \leq 0$ *and* $y^3 < 0$*, which is also not true for any* $y \in K$*.*

Case *3.* $-y^2 < 0$ *and* $y^3 < 0$*, which is again not true for any* $y \in K$*.*

Therefore, $[-y(y-0), y^2(y-0)] \prec \boldsymbol{0}$ *is not true for any* $y \in K$

$\Longrightarrow$ $[-y(y-0), y^2(y-0)] \not\prec \boldsymbol{0},\ \forall y \in K$

$\Longrightarrow$ 0 *is a solution of MIVIP.*

At $\bar{x} = 10$*:*

Case *1.* $-y(y-10) < 0$ *and* $y^2(y-10) \leq 0$*, which is not true for any* $y \in K$*.*

Case *2.* $-y(y-10) \leq 0$ *and* $y^2(y-10) < 0$*, which is also not true for any* $y \in K$*.*

Case *3.* $-y(y-10) < 0$ *and* $y^2(y-10) < 0$*, which is again not true for any* $y \in K$*.*

Therefore, $[-y(y-10), y^2(y-10)] \prec \boldsymbol{0}$ *is not true for any* $y \in K$

$\Longrightarrow$ $[-y(y-10), y^2(y-10)] \not\prec \boldsymbol{0},\ \forall y \in K$

$\Longrightarrow$ 10 *is a solution of MIVIP.*

At $\bar{x} = \alpha \in (0, 10)$*:*

Case *1.* $-y(y-\alpha) < 0$ *and* $y^2(y-\alpha) \leq 0$*, which is not true for any* $y \in K$*.*

Case *2.* $-y(y-\alpha) \leq 0$ *and* $y^2(y-\alpha) < 0$*, which is also not true for any* $y \in K$*.*

Case *3.* $-y(y-\alpha) < 0$ *and* $y^2(y-\alpha) < 0$*, which is again not true for any* $y \in K$*.*

Therefore, $[-y(y-\alpha), y^2(y-\alpha)] \prec \boldsymbol{0}$ *is not true for any* $y \in K$

$\Longrightarrow$ $[-y(y-\alpha), y^2(y-\alpha)] \not\prec \boldsymbol{0},\ \forall y \in K$

$\Longrightarrow$ α *is a solution of MIVIP.*

Thus, Sol(MIVIP)= $[0, 10]$*.*

In this example Sol(SIVIP)=Sol(MIVIP) but this is not true in general, to see this consider the example below.

Example 4.2. *Let* $X = \mathbb{R}$ *and* $K = [-1, 1]$*. Let* $T : K \to \mathscr{L}(X, Y)$ *be defined as*

$$T(x) = \begin{cases} [x^2, -x], -1 \leq x < 0 \\ [-x, x^2], 0 \leq x \leq 1 \end{cases}$$

Sol(SIVIP)$= \{\bar{x} \in K | \langle T(\bar{x}), y - \bar{x} \rangle \nprec \boldsymbol{0}, \ \forall y \in K\}$.

Case *1.* $-1 \leq \bar{x} < 0$.

$$\begin{aligned} \textit{Sol(SIVIP)} &= \{\bar{x} \in K | \langle [\bar{x}^2, -\bar{x}], y - \bar{x} \rangle \nprec \boldsymbol{0}, \ \forall y \in K\} \\ &= \{\bar{x} \in K | [\bar{x}^2(y - \bar{x}), -\bar{x}(y - \bar{x})] \nprec \boldsymbol{0}, \ \forall y \in K\}. \end{aligned}$$

At $\bar{x} = -1$*:*

Subcase *1.* $y + 1 < 0$ *and* $-(y + 1) \leq 0$*, which is not true for any* $y \in K$*.*

Subcase *2.* $y + 1 \leq 0$ *and* $-(y + 1) < 0$*, which is also not true for any* $y \in K$*.*

Subcase *3.* $y + 1 < 0$ *and* $-(y + 1) < 0$*, which is again not true for any* $y \in K$*.*

Therefore, $[y + 1, -(y + 1)] \prec \boldsymbol{0}$ *is not true for any* $y \in K$*.*

$\implies [y + 1, -(y + 1)] \nprec \boldsymbol{0}, \ \forall y \in K$.

$\implies -1$ *is a solution of SIVIP.*

At $\bar{x} = \alpha \in (-1, 0)$*:*

Subcase *1.* $\alpha^2(y - \alpha) < 0$ *and* $-\alpha(y - \alpha) \leq 0$*, which is true atleast for one value of* y*.*

Therefore, $[\alpha^2(y - \alpha), -\alpha(y - \alpha)] \prec \boldsymbol{0}$ *is true for some* $y \in K$

$\implies [\alpha^2(y - \alpha), -\alpha(y - \alpha)] \nprec \boldsymbol{0}, \ \forall y \in K$ *is not true*

$\implies \alpha$ *is not a solution of SIVIP.*

Case *2.* $0 \leq \bar{x} \leq 1$.

$$\begin{aligned} \textit{Sol(SIVIP)} &= \{\bar{x} \in K | \langle [-\bar{x}, \bar{x}^2], y - \bar{x} \rangle \nprec \boldsymbol{0}, \ \forall y \in K\} \\ &= \{\bar{x} \in K | [-\bar{x}(y - \bar{x}), \bar{x}^2(y - \bar{x})] \nprec \boldsymbol{0}, \ \forall y \in K\} \end{aligned}$$

By following the similar procedure as in the above example it can be seen that $[0,1] \in$ *Sol(SIVIP). Thus, Sol(SIVIP)=* $\{-1\} \cup [0,1]$.

Let us now calculate Sol(MIVIP).

Sol(MIVIP)= $\{\bar{x} \in K | \langle T(y), y-\bar{x}\rangle \nprec \boldsymbol{0},\ \forall y \in K\}$.

Case *1.* $-1 \leq \bar{x} < 0$.

$$\begin{aligned} \textit{Sol(MIVIP)} &= \{\bar{x} \in K | \langle [y^2, -y], y-\bar{x}\rangle \nprec \boldsymbol{0},\ \forall y \in K\} \\ &= \{\bar{x} \in K | [y^2(y-\bar{x}), -y(y-\bar{x})] \nprec \boldsymbol{0},\ \forall y \in K\} \end{aligned}$$

At $\bar{x} = -1$*:*

Subcase *1.* $y^2(y+1) < 0$ *and* $-y(y+1) \leq 0$*, which is not true for any* $y \in K$*.*

Because $y+1 \geq 0,\ \forall y \in K$ *so* $y^2(y+1) \nless 0$ *for any* $y \in K$*.*

Subcase *2.* $y^2(y+1) \leq 0$ *and* $-y(y+1) < 0$*, which is also not true for any* $y \in K$*.*

Subcase *3.* $y^2(y+1) < 0$ *and* $-y(y+1) < 0$*, which is again not true for any* $y \in K$*.*

Therefore, $[y^2(y+1), -y(y+1)] \prec \boldsymbol{0}$ *is not true for any* $y \in K$

$\implies [y^2(y+1), -y(y+1)] \nprec \boldsymbol{0},\ \forall y \in K$

$\implies -1$ *is a solution of MIVIP.*

It can be checked that no other $\bar{x} \in K$ *is a solution of MIVIP. Thus, Sol(MIVIP)=* $\{-1\}$.

For this example, Sol(MIVIP)⊆Sol(SIVIP) however, the next example shows that this is not always true.

Example 4.3. *Let* $X = \mathbb{R}$ *and* $K = [-1,0]$*. Let* $T : K \to \mathscr{L}(X,Y)$ *be defined as* $T(x) = [2x, 1]$*.*

$$\begin{aligned} \textit{Sol(MIVIP)} &= \{\bar{x} \in K | \langle T(y), y-\bar{x}\rangle \nprec \boldsymbol{0},\ \forall y \in K\} \\ &= \{\bar{x} \in K | \langle [2y, 1], y-\bar{x}\rangle \nprec \boldsymbol{0},\ \forall y \in K\} \\ &= \{\bar{x} \in K | [2y(y-\bar{x}), y-\bar{x}] \nprec \boldsymbol{0},\ \forall y \in K\}. \end{aligned}$$

At $\bar{x} = 0$*:*

Case *1.* $2y^2 < 0$ *and* $y \leq 0$*, which is not true for any* $y \in K$*.*

Case *2.* $2y^2 \leq 0$ *and* $y < 0$*, which is also not true for any* $y \in K$*.*

Case *3.* $2y^2 < 0$ *and* $y < 0$*, which is again not true for any* $y \in K$*.*

Therefore, 0 *is a solution of MIVIP.*

$$\begin{aligned}
\textit{Sol(SIVIP)} &= \{\bar{x} \in K | \langle T(\bar{x}), y - \bar{x} \rangle \not< \boldsymbol{0},\ \forall y \in K\} \\
&= \{\bar{x} \in K | \langle [2\bar{x}, 1], y - \bar{x} \rangle \not< \boldsymbol{0},\ \forall y \in K\} \\
&= \{\bar{x} \in K | [2\bar{x}(y - \bar{x}), y - \bar{x}] \not< \boldsymbol{0},\ \forall y \in K\}.
\end{aligned}$$

At $\bar{x} = 0$*:*

Case *1.* $0 < 0$ *and* $y \leq 0$*, which is not true for any* $y \in K$*.*

Case *2.* $0 \leq 0$ *and* $y < 0$*,which is true atleast for one value of* y*, for instance, take* $y = -\frac{1}{2}$*. Therefore,* 0 *is not a solution of SIVIP. Thus, Sol(MIVIP)*$\nsubseteq$*Sol(SIVIP).*

Note from examples (4.1), (4.2), and (4.3) that solution sets of Stampacchia and Minty IVIs are not related to each other in general.

Proposition 4.1 provides the relation between the solution sets of these VIs.

Proposition 4.1. *Let* K *be a non-empty convex subset of a topological vector space* X *and* $T : K \to \mathscr{L}(X, Y)$ *be a mapping. If* T *is pseudomonotone, then Sol(SIVIP)* $\subseteq$ *Sol(MIVIP).*

Proof. Let $\bar{x}$ be a solution of SIVIP.

Then, $\langle T(\bar{x}), y - \bar{x} \rangle \not< \mathbf{0},\ \forall y \in K$

$\implies \langle T(y), y - \bar{x} \rangle \not< \mathbf{0},\ \forall y \in K$ [By pseudomonotonicity of T]

$\implies \bar{x}$ is a solution of MIVIP.

4.3 Relationship Between Interval Variational Inequalities and Interval Optimization

Next, we give a necessary and sufficient condition of optimality for an IOP with the help of Stampacchia interval variational inequality.

Let K be a non-empty convex subset of $\mathbb{R}^n$ and $\mathbf{F} : K \to \mathbb{R}_I$ be an IVF. Consider the IOP:

$$\min_{x \in K} \mathbf{F}(x). \tag{4.3}$$

Proposition 4.2. *Let K be a non-empty convex subset of $\mathbb{R}^n$ and $\boldsymbol{F} : K \to \mathbb{R}_I$ be a gH-Gâteaux differentiable IVF on an open set containing K. If $\bar{x}$ is a solution of the optimization problem (4.3), then $\bar{x}$ is a solution of SIVIP with $T(\bar{x}) = \boldsymbol{F}_{\mathscr{G}}(\bar{x})$.*

Proof. Since every gH-Gâteaux differentiable IVF is gH-directional differentiable, therefore, by Theorem 1.35, $\mathbf{F}_{\mathscr{G}}(\bar{x})(y - \bar{x}) \not\prec \mathbf{0}, \ \forall y \in K$.

, i.e. , $\langle \mathbf{F}_{\mathscr{G}}(\bar{x}), (y - \bar{x}) \rangle \not\prec \mathbf{0}, \ \forall y \in K$

$\implies$ $\bar{x}$ is a solution of SIVIP with $T(\bar{x}) = \mathbf{F}_{\mathscr{G}}(\bar{x})$. $\square$

Example 4.4. *Let $X = \mathbb{R}$ and $K = [1, 10]$. Let $\underline{f}, \overline{f} : \mathbb{R} \to \mathbb{R}$ be defined as $\underline{f}(x) = x$ and $\overline{f}(x) = 10x$. Let $\boldsymbol{F}(x) = [\underline{f}(x), \overline{f}(x)], \ \forall x \in K$.*

In Fig. 4.1, the objective function $\boldsymbol{F}$ is depicted by the shaded region. From the Fig. 4.1, it is clear that $\bar{x} = 1$ is an efficient point of (4.3). Let us now check whether $\bar{x}$ is a solution of SIVIP with $T(1) = \boldsymbol{F}_{\mathscr{G}}(1)$ or not.

$$\begin{aligned}
\boldsymbol{F}_{\mathscr{G}}(1)(h) &= \lim_{\lambda \to 0+} \frac{1}{\lambda} \odot (\boldsymbol{F}(1 + \lambda h) \ominus_{gH} \boldsymbol{F}(1)) \\
&= \lim_{\lambda \to 0+} \frac{1}{\lambda} \odot ([1 + \lambda h, 10 + 10\lambda h] \ominus_{gH} [1, 10]) \\
&= \lim_{\lambda \to 0+} \frac{1}{\lambda} \odot [\lambda h, 10\lambda h]
\end{aligned}$$

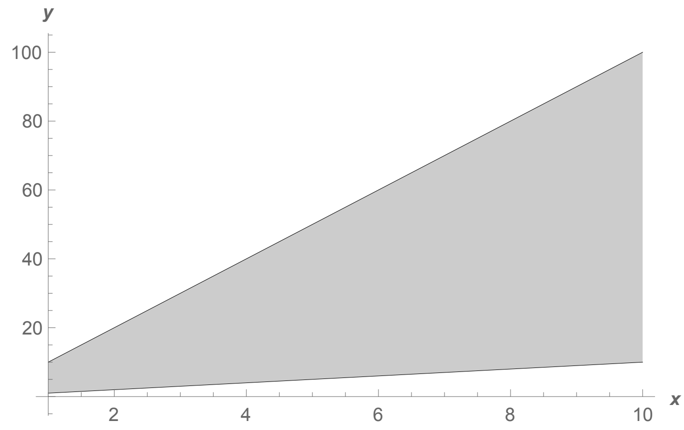

FIGURE 4.1: The objective function $\mathbf{F}$ of Example 4.4

$$= \quad [h, 10h]$$

$$= \quad [1, 10] \odot h$$

Clearly, $\boldsymbol{F}_{\mathscr{G}}(1)$ is linear and gH-continuous on $\mathbb{R}$. Therefore, $\boldsymbol{F}$ is a gH-Gâteaux differentiable function at 0.

Consider SIVIP with $T(1) = \boldsymbol{F}_{\mathscr{G}}(1)$, we have

$$\{\bar{x} \in K | \langle \boldsymbol{F}_{\mathscr{G}}(1), y - \bar{x} \rangle \nprec \boldsymbol{0}, \ \forall y \in K\}$$
$$= \{\bar{x} \in K | [y - \bar{x}, 10(y - \bar{x})] \nprec \boldsymbol{0}, \ \forall y \in K\}.$$

At $\bar{x} = 1$:

Case *1.* $y - 1 < 0$ *and* $10(y-1) \leq 0$, *which is not true for any* $y \in K$.

Case *2.* $y - 1 \leq 0$ *and* $10(y-1) < 0$, *which is also not true for any* $y \in K$.

Case *3.* $y - 1 < 0$ *and* $10(y-1) < 0$, *which is again not true for any* $y \in K$.

Therefore, $[y - 1, 10(y-1)] \prec \boldsymbol{0}$ *is not true for any* $y \in K$

$\implies [y - 1, 10(y-1)] \nprec \boldsymbol{0}, \ \forall y \in K$

$\implies 1$ *is a solution of SIVIP with* $T(1) = \boldsymbol{F}_{\mathscr{G}}(1)$.

Proposition 4.3. *Let K be a non-empty convex subset of $\mathbb{R}^n$ and $\boldsymbol{F} : K \to \mathbb{R}_I$ be a* *pseudoconvex gH-Gâteaux differentiable IVF on an open set containing K. If $\bar{x}$ is a solution of SIVIP with $T(\bar{x}) = \boldsymbol{F}_{\mathscr{G}}(\bar{x})$, then it is an efficient point of the optimization problem (4.3).*

Proof. Suppose that $\bar{x}$ is a solution of SIVIP with $T(\bar{x}) = \mathbf{F}_{\mathscr{G}}(\bar{x})$, but not an efficient point of (4.3). Then, there exists a vector $y \in K$ such that $\mathbf{F}(y) \prec \mathbf{F}(\bar{x})$.
By pseudoconvexity of $\mathbf{F}$, we have $\langle \mathbf{F}_{\mathscr{G}}(\bar{x}), y - \bar{x} \rangle \prec \mathbf{0}$, which is a contradiction to the fact that $\bar{x}$ is a solution of SIVIP. Thus, $\bar{x}$ is also an efficient point of (4.3). □

Example 4.5. *Let $X = \mathbb{R}$ and $K = [0, 1]$. Let $\boldsymbol{F} : K \to \mathbb{R}_I$ be defined as $\boldsymbol{F}(x) = [x^2, x], \ \forall x \in K$.*
Clearly $\boldsymbol{F}(x) \not\prec \boldsymbol{F}(0), \ \forall x \in K$, therefore, $\bar{x} = 0$ is the efficient point of (4.3).
Consider

$$\begin{aligned}
\boldsymbol{F}_{\mathscr{G}}(0)(h) &= \lim_{\lambda \to 0+} \frac{1}{\lambda} \odot (\boldsymbol{F}(0 + \lambda h) \ominus_{gH} \boldsymbol{F}(0)) \\
&= \lim_{\lambda \to 0+} \frac{1}{\lambda} \odot (\boldsymbol{F}(\lambda h) \ominus_{gH} \boldsymbol{F}(0)) \\
&= \lim_{\lambda \to 0+} \frac{1}{\lambda} \odot [\lambda^2 h^2, \lambda h] \\
&= \lim_{\lambda \to 0+} \frac{1}{\lambda} \odot [\lambda h^2, h] \\
&= [0.h] \\
&= [0, 1] \odot h.
\end{aligned}$$

Clearly $\boldsymbol{F}_{\mathscr{G}}(0)$ is linear and gH-continuous on $\mathbb{R}$.
Therefore, $\boldsymbol{F}$ is a gH-Gâteaux differentiable function at 0.
Also note that $\boldsymbol{F}$ is convex and hence, pseudoconvex. Thus, $\boldsymbol{F}$ satisfies hypothesis of above theorem. Let us now check that it satisfies its conclusion as well.

Consider SIVIP with $T(0) = \boldsymbol{F}_{\mathscr{G}}(0)$*, i.e.,*

$$\{\bar{x} \in K | \langle \boldsymbol{F}_{\mathscr{G}}(0), y - \bar{x} \rangle \not< \boldsymbol{0},\ \forall y \in K.\} \tag{4.4}$$

Then, solution set of (4.4) is:

$$\begin{aligned}
&\{\bar{x} \in K | \langle \boldsymbol{F}_{\mathscr{G}}(0), y - \bar{x} \rangle \not< \boldsymbol{0},\ \forall y \in K\} \\
=&\{\bar{x} \in K | \langle [0,1], y - \bar{x} \rangle \not< \boldsymbol{0},\ \forall y \in K\} \\
=&\{\bar{x} \in K | [0, y - \bar{x}] \not< \boldsymbol{0},\ \forall y \in K\}.
\end{aligned}$$

At $\bar{x} = 0$*:*

Case *1.* $0 < 0$ *and* $y \leq 0$*, which is not true for any* $y \in K$*.*

Case *2.* $0 \leq 0$ *and* $y < 0$*, which is also not true for any* $y \in K$*.*

Case *3.* $0 < 0$ *and* $y < 0$*, which is again not true for any* $y \in K$*.*

Therefore, $[0, y - 0] \not< \boldsymbol{0},\ \forall y \in K$

$\implies$ 0 *is a solution of SIVIP with* $T(0) = \boldsymbol{F}_{\mathscr{G}}(0)$*.*

4.4 Concluding Remarks

In this chapter, interval variational inequalities have been introduced, and a relation between their solution sets has been analyzed. In the sequel, interval-valued pseudoconvex and pseudomonotone functions have also been introduced. For an IOP a necessary and sufficient condition of optimality has been given in terms of SIVI. The whole investigation is backed up by relevant illustrative examples.

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Chapter 5

Stampacchia and Minty Variational Inequalities for Interval-valued Functions and Their Relationship with Interval Optimization

5.1 Introduction

The systematic study of finite-dimensional variational inequalities began in the mid-1960s, and the subject has evolved into a fruitful discipline in mathematical programming over the last four decades. A rich mathematical theory, a slew of effective solution algorithms, a slew of intriguing connections to various disciplines, and a wide range of important applications in engineering and economics are among the developments [3, 8, 41, 54, 69, 101]. In this chapter, we study finite dimensional variational inequalities for interval-valued functions.

5.1.1 Motivation

The finite-dimensional conventional variational inequality (VI) provides a broad unifying framework for the study of optimization and equilibrium problems. Inspired by this, in this chapter, we study variational inequalities for IVFs.

5.1.2 Contribution

The major contributions of this chapter are the following:

- Existence and uniqueness results for the proposed IVIs
- Derivation of a necessary and sufficient optimality condition for a constrained IOP in terms of SIVI
- To find a sufficient optimality condition for a constrained IOP in terms of MIVI
- Suitable examples in support of all the given results.

5.2 Interval variational inequalities

In this section, we first introduce VIs for IVFs. Then, we derive relationships between their solution sets (Theorems 5.3 and 5.4). We also provide existence and uniqueness results for these VIs.

Definition 5.1. Let X be a nonempty subset of $\mathbb{R}^n$ and $\mathbf{T} : X \to \mathbb{R}_I^n$ be an interval vector-valued function.

(i) Stampacchia interval variational inequality problem (SIVIP) is a problem to find a vector $\bar{x} \in X$ satisfying

$$\mathbf{0} \preceq (y - \bar{x})^\top \odot \mathbf{T}(\bar{x}) \text{ for all } y \in X. \tag{5.1}$$

We call the inequality (5.1) as Stampacchia interval variational inequality (SIVI).

(ii) Minty interval variational inequality problem (MIVIP) is a problem to find a vector $\bar{x} \in X$ satisfying

$$\mathbf{0} \preceq (y - \bar{x})^\top \odot \mathbf{T}(y) \text{ for all } y \in X. \tag{5.2}$$

We call the inequality (5.2) as Minty interval variational inequality (MIVI).

Note that if instead of interval vector-valued function, we take $\mathbf{T}$ as a real vector-valued function, then the inequalities (5.1) and (5.2) reduce to the conventional Stampacchia and Minty variational inequalities [6, 41] respectively.

Example 5.1. *For the better understanding of Stampacchia and Minty IVIs, we solve these inequalities for* $\boldsymbol{T} : X \to {\mathbb{R}_I}^2$ *given by* $\boldsymbol{T}(x_1, x_2) = ([-2, 1] \odot x_1^2, [3, 4] \odot (x_1^2 + x_2^2))$*, where* $X = [-1, 0] \times [-1, 0]$*.*

We first solve SIVIP. In this direction, we consider the following possible cases.

Case 1. $\bar{x} = (-1, -1)$.

In this case, for all $y \in X$, *we have*

$$\begin{aligned} (y - \bar{x})^\top \odot \boldsymbol{T}(\bar{x}) &= ((y_1 + 1), (y_2 + 1))^\top \odot ([-2, 1], [3, 4] \odot 2) \\ &= [-2, 1] \odot (y_1 + 1) \oplus [6, 8] \odot (y_2 + 1). \end{aligned}$$

This implies

$$(y - \bar{x})^\top \odot \boldsymbol{T}(\bar{x}) = [-2(y_1 + 1) + 6(y_2 + 1), (y_1 + 1) + 8(y_2 + 1)]. \quad (5.3)$$

To see that $(-1, -1)$ *is not a solution of SIVIP, take* $y_1 = 0$ *and* $y_2 = -1$ *in* (5.3)*, we get*

$$(y - \bar{x})^\top \odot \boldsymbol{T}(\bar{x}) = [-2, 1].$$

Thus, for this choice of y_1 *and* y_2*, we have*

$$\boldsymbol{0} \not\preceq (y - \bar{x})^\top \odot \boldsymbol{T}(\bar{x}).$$

Hence, $(-1, -1)$ *is not a solution of SIVIP.*

Case 2. $\bar{x} = (0, 0)$.

We then have $(y - \bar{x})^\top \odot \boldsymbol{T}(\bar{x}) = \boldsymbol{0}$ *for all* $y \in X$. *Thus,* $(0, 0)$ *is a solution SIVIP.*

Case 3. $\bar{x} = (-1, 0)$.

For all $y \in X$, *we have*

$$\begin{aligned} (y - \bar{x})^\top \odot \boldsymbol{T}(\bar{x}) &= ((y_1 + 1), y_2)^\top \odot ([-2, 1], [3, 4]) \\ &= [-2, 1] \odot (y_1 + 1) \oplus [3, 4] \odot y_2. \end{aligned}$$

By choosing y_1 and y_2 both equal to -1, we get

$$(y - \bar{x})^\top \odot \boldsymbol{T}(\bar{x}) = [-4, -3] \prec \boldsymbol{0}.$$

Therefore, $(-1, 0)$ is not a solution of SIVIP.

Case 4. $\bar{x} = (0, -1)$.

In this case, we have $\boldsymbol{0} \preceq (y - \bar{x})^\top \odot \boldsymbol{T}(\bar{x}) = [3, 4] \odot (y_2 + 1)$ for all $y_2 \in [-1, 0]$. Thus, $(0, -1)$ is a solution of SIVIP.

Case 5. $\bar{x} = (-1, x_2)$ *such that* $x_2 \in (-1, 0)$.

In this case, we have

$(y - \bar{x})^\top \odot \boldsymbol{T}(\bar{x}) = [-2, 1] \odot (y_1 + 1) \oplus [3, 4](1 + x_2^2)(y_2 - x_2)$.

By choosing $y_1 = -1$ and $y_2 < x_2$, we get $(y - \bar{x})^\top \odot \boldsymbol{T}(\bar{x}) \prec \boldsymbol{0}$. Thus, $\bar{x}$ is not a solution of SIVIP.

Case 6. $\bar{x} = (x_1, -1)$ *such that* $x_1 \in (-1, 0)$.

For all $y \in X$, we have

$$(y - \bar{x})^\top \odot \boldsymbol{T}(\bar{x}) = [-2, 1] \odot x_1^2(y_1 - x_1) \oplus [3, 4] \odot (x_1^2 + 1)(y_2 + 1).$$

By choosing $y_2 = -1$ and $y_1 < x_1$, we get

$$\boldsymbol{0} \npreceq (y - \bar{x})^\top \odot \boldsymbol{T}(\bar{x}).$$

Hence, $\bar{x}$ is not a solution of SIVIP.

Case 7. $\bar{x} = (0, x_2)$ *such that* $x_2 \in (-1, 0)$.

In this case, we have

$(y - \bar{x})^\top \odot \boldsymbol{T}(\bar{x}) = [3, 4] \odot x_2^2(y_2 - x_2)$. *Note here that for any $x_2 \in (-1, 0)$, we* *can choose* $y_2 \in [-1, 0]$ *such that* $y_2 - x_2 < 0$*, and hence* $[3, 4] \odot x_2^2(y_2 - x_2) \prec \boldsymbol{0}$ *for some* $y_2 \in [-1, 0]$*. Hence,* $\bar{x}$ *is not a solution of SIVIP.*

Case 8. $\bar{x} = (x_1, 0)$ *such that* $x_1 \in (-1, 0)$.

By a similar argument as in Case 7, $\bar{x}$ *is not a solution of SIVIP.*

Case 9. $\bar{x} = (x_1, x_2)$ *such that* $x_1,\ x_2 \in (-1, 0)$.

In this case, we have

$$(y - \bar{x})^\top \odot \boldsymbol{T}(\bar{x}) = [-2, 1] \odot x_1^2(y_1 - x_1) \oplus [3, 4](x_1^2 + x_2^2)(y_2 - x_2). \quad (5.4)$$

For any $x_1, x_2 \in (-1, 0)$*, we can choose* $y_1 = x_1$ *and* $y_2 < x_2$ *in (5.4) such that* $(y - \bar{x})^\top \odot \boldsymbol{T}(\bar{x}) \prec \boldsymbol{0}$*. Thus,* $\bar{x}$ *is not a solution of SIVIP.*

Hence, from all the possible cases, the solution set of SIVIP is equal to

$$\{(0, 0), (0, -1)\}.$$

We now solve MIVIP. To solve this inequality we again consider the following possible cases.

Case 1. $\bar{x} = (-1, -1)$.

In this case, we have

$$\begin{aligned} (y - \bar{x})^\top \odot \boldsymbol{T}(y) &= ((y_1 + 1), (y_2 + 1))^\top \odot ([-2, 1]y_1^2, [3, 4] \odot (y_1^2 + y_2^2)) \\ &= [-2, 1] \odot y_1^2(y_1 + 1) \oplus [3, 4] \odot (y_1^2 + y_2^2)(y_2 + 1). (5.5) \end{aligned}$$

If we take $y_2 = -1$ *and* $y_1 = -\frac{1}{2}$ *in (5.5), then we get*

$$\boldsymbol{0} \not\preceq (y - \bar{x})^\top \odot \boldsymbol{T}(y) = [-2, 1] \odot \tfrac{1}{8}.$$

Thus, $(-1, -1)$ *is not a solution of MIVIP.*

Case 2. $\bar{x} = (0, 0)$.

For all $y \in X$*, we have*

$$(y - \bar{x})^\top \odot \boldsymbol{T}(y) = [-2, 1] \odot y_1^3 \oplus [3, 4](y_1^2 + y_2^2)y_2. \tag{5.6}$$

By choosing $y_1 = 0$ *and* $y_2 = -1$ *in (5.6), we get*

$$(y - \bar{x})^\top \odot \boldsymbol{T}(y) = [-4, -3] \prec \boldsymbol{0}.$$

Thus, $(0, 0)$ *is not a solution of MIVIP.*

Case 3. $\bar{x} = (-1, 0)$.

For all $y \in X$*, we have*

$$(y - \bar{x})^\top \odot \boldsymbol{T}(y) = [-2, 1] \odot y_1^2(y_1 + 1) \oplus [3, 4] \odot (y_1^2 + y_2^2)y_2.$$

By choosing y_1 *and* y_2 *both equal to* -1*,we get*

$$(y - \bar{x})^\top \odot \boldsymbol{T}(y) = [-8, -6] \prec \boldsymbol{0}.$$

Thus, $(-1, 0)$ *is not a solution of MIVIP.*

Case 4. $\bar{x} = (0, -1)$.

In this case, we have

$$(y - \bar{x})^\top \odot \boldsymbol{T}(y) \quad = \quad [-2, 1] \odot y_1^3 \oplus [3, 4] \odot (y_1^2 + y_2^2)(y_2 + 1).$$

By choosing y_1 *and* y_2 *both equal to* -1*, we get* $\boldsymbol{0} \npreceq (y - \bar{x})^\top \odot \boldsymbol{T}(y) = [-1, 2]$*.* *Thus,* $(0, -1)$ *is not a solution of MIVIP.*

Case 5. $\bar{x} = (-1, x_2)$ *such that* $x_2 \in (-1, 0)$*.*

In this case, we have

$(y - \bar{x})^\top \odot \boldsymbol{T}(y) = [1, 2] \odot y_1^2(y_1 + 1) \oplus [3, 4](y_1^2 + y_2^2)(y_2 - x_2)$.

By choosing $y_1 = -1$ *and* $y_2 < x_2$*, we get* $(y - \bar{x})^\top \odot \boldsymbol{T}(y) \prec \boldsymbol{0}$*. Thus,* $\bar{x}$ *is not a solution of MIVIP.*

Case 6. $\bar{x} = (x_1, -1)$ *such that* $x_1 \in (-1, 0)$*.*

By a similar argument as in Case 5, $\bar{x}$ *is not a solution of MIVIP.*

Case 7. $\bar{x} = (0, x_2)$ *such that* $x_2 \in (-1, 0)$*.*

In this case, we have

$$(y - \bar{x})^\top \odot \boldsymbol{T}(y) = [-2, 1] \odot y_1^3 \oplus [3, 4] \odot (y_1^2 + y_2^2)(y_2 - x_2).$$

By choosing $y_1 = 0$ *and* $y_2 < x_2$*, we get*

$$(y - \bar{x})^\top \odot \boldsymbol{T}(y) \prec \boldsymbol{0}.$$

Thus, $\bar{x}$ *is not a solution of MIVIP.*

Case 8. $\bar{x} = (x_1, 0)$ *such that* $x_1 \in (-1, 0)$.

For all $y \in X$, *we have*

$$(y - \bar{x})^\top \odot \boldsymbol{T}(y) = [-2, 1] \odot y_1^2(y_1 - x_1) \oplus [3, 4] \odot (y_1^2 + y_2^2)y_2.$$

By choosing $y_1 = 0$ *and* $y_2 = -1$, *we get*

$$(y - \bar{x})^\top \odot \boldsymbol{T}(y) \prec \boldsymbol{0}.$$

Thus, $\bar{x}$ *is not a solution of MIVIP.*

Case 9. $\bar{x} = (x_1, x_2)$ *such that* $x_1,\ x_2 \in (-1, 0)$.

By a similar argument as in Case 8, $\bar{x}$ *is not a solution of MIVIP.*

Hence, MIVIP has no solution.

Remark 5.2. From Example 5.1, it can be observed that, in general, SIVIP and MIVIP need not have the same solution set. However, in the next (Theorem 5.3), we show that if the mapping $\mathbf{T}$ is gH-hemicontionuous and pseudomonotone, then the solution sets of these VIs are identical.

Theorem 5.3. *Let* X *be a nonempty convex subset of* $\mathbb{R}^n$ *and* $\boldsymbol{T}\colon X \to \mathbb{R}_I^n$ *be* gH*-hemicontinuous and pseudomonotone. Then, the solution set of SIVIP is identical to the solution set of MIVIP.*

Proof. Suppose $\bar{x}$ is a solution of MIVIP. Then, for any $y \in X$ and $\lambda \in (0, 1)$, $z_\lambda = \bar{x} + \lambda(y - \bar{x}) \in X$, and hence $\mathbf{0} \preceq (z_\lambda - \bar{x})^\top \odot \mathbf{T}(z_\lambda)$ for all $\lambda \in (0, 1)$. This implies $\mathbf{0} \preceq (y - \bar{x})^\top \odot \mathbf{T}(\bar{x} + \lambda(y - \bar{x}))$ for all $\lambda \in (0, 1)$. Thus, by gH-hemicontinuity of $\mathbf{T}$, we have

$$\mathbf{0} \preceq (y - \bar{x})^\top \odot \mathbf{T}(\bar{x}) \text{ for all } y \in X.$$

Therefore, $\bar{x}$ is a solution of SIVIP.

Suppose now $\bar{z}$ is a solution of SIVIP. Then,

$$\begin{aligned} & \mathbf{0} \preceq (y - \bar{x})^\top \odot \mathbf{T}(\bar{x}) \text{ for all } y \in X \\ \implies \quad & \mathbf{0} \preceq (y - \bar{x})^\top \odot \mathbf{T}(y) \text{ for all } y \in X \text{ because } \mathbf{T} \text{ is pseudomonotone.} \end{aligned}$$

Thus, $\bar{z}$ is a solution of MIVIP. □

Theorem 5.4. *Let X be a nonempty convex subset of $\mathbb{R}^n$ and $\boldsymbol{T} : X \to \mathbb{R}_I^n$ be gH-hemicontinuous and pseudomonotone. Then, the solution sets of both SIVIP and MIVIP are closed and convex.*

Proof. In view of Theorem 5.3, the solution sets of SIVIP and MIVIP are same. Thus, it is sufficient to prove that the solution set of MIVIP is closed and convex.

Suppose $\bar{x}$ and $\hat{x}$ are two solutions of MIVIP. Then, for any $y \in X$,

$$\mathbf{0} \preceq (y - \bar{x})^\top \odot \mathbf{F}(y) \tag{5.7}$$

and

$$\mathbf{0} \preceq (y - \hat{x})^\top \odot \mathbf{F}(y). \tag{5.8}$$

Multiplying (5.7) by $\lambda \in [0, 1]$ and (5.8) by $1 - \lambda$, and then by adding the resultants, we get

$$\begin{aligned} & \mathbf{0} \preceq \lambda(y - \bar{x})^\top \odot \mathbf{F}(y) \oplus (1 - \lambda)(y - \hat{x})^\top \odot \mathbf{F}(y) \\ \implies \quad & \mathbf{0} \preceq (\lambda(y - \bar{x})^\top + (1 - \lambda)(y - \hat{x})^\top) \odot \mathbf{F}(y) \text{ by Lemma 1.37} \\ \text{or,} \quad & \mathbf{0} \preceq (y - (\lambda\bar{x} + (1 - \lambda)\hat{x}))^\top \odot \mathbf{F}(y). \end{aligned}$$

Hence, $\lambda\bar{x} + (1-\lambda)\hat{x}$ is a solution of MIVIP.

Next, to show that the solution set of MIVIP is closed, let $\{x_m\}$ be a sequence of solutions of MIVIP such that $\{x_m\} \to \bar{x}$ as $m \to \infty$. Therefore, for all $y \in X$,

$$\mathbf{0} \preceq (y - \{x_m\})^\top \odot \mathbf{F}(y) \text{ for all } m.$$

Since $(y-\{x_m\})^\top \odot \mathbf{F}(y) \to (y-\bar{x})^\top \odot \mathbf{F}(y)$ as $m \to \infty$, we have $\mathbf{0} \preceq (y-\bar{x})^\top \odot \mathbf{F}(y)$, and thus $\bar{x}$ is a solution of MIVIP. Hence, the solution of MIVIP is closed, and the proof is complete. □

Next, we derive existence results for the solutions of Stampacchia and Minty IVIs. To prove these results, we need the following definition and theorem related to KKM-maps.

Definition 5.5. (KKM-map [42]). Let X be a nonempty convex subset of $\mathbb{R}^n$. A set-valued map $P : X \to 2^X$ is said to be a KKM-map if for every finite subset $\{x_1, x_2, \ldots, x_m\}$ of X,

$$\text{conv}\{x_1, x_2, \ldots, x_m\} \subseteq \bigcup_{i=1}^{m} P(x_i).$$

Theorem 5.6. (Fan-KKM theorem [42]). *Let X be a nonempty convex subset of $\mathbb{R}^n$ and $P : X \to 2^X$ be a KKM-map such that $P(x)$ is closed for all $x \in X$, and $P(x)$ is compact for at least one $x \in X$. Then,*

$$\bigcap_{x \in X} P(x) \neq \emptyset.$$

Theorem 5.7. *Let X be a nonempty compact convex subset of $\mathbb{R}^n$ and $\boldsymbol{T} : X \to \mathbb{R}^n_I$ be gH-hemicontinuous and pseudomonotone. Then, SIVIP has a solution.*

Proof. Define two set-valued maps $P, Q : X \to 2^X$ by

$$P(y) = \{x \in X : \mathbf{0} \preceq (y - x)^\top \odot \mathbf{T}(x)\}, \text{ for all } y \in X$$

and

$$Q(y) = \{x \in X : \mathbf{0} \preceq (y - x)^\top \odot \mathbf{T}(y)\}, \text{ for all } y \in X.$$

Then, P is a KKM-map. To see this, let $\{y_1, y_2, \ldots, y_m\}$ be a finite subset of X and $\tilde{x} \in \text{conv}(\{y_1, y_2, \ldots, y_m\})$. Then, $\tilde{x} = \sum_{i=1}^m \lambda_i y_i$ for some $\lambda_i \geq 0, i = 1, 2, \ldots, m$ with $\sum_{i=1}^m \lambda_i = 1$. If $\tilde{x} \notin \bigcup_{i=1}^m P(y_i)$, then

$$\mathbf{0} \not\preceq (y_i - \tilde{x})^\top \odot \mathbf{T}(\tilde{x}) \text{ for all } i = 1, 2, \ldots, m,$$

and so,

$$\mathbf{0} \not\preceq \sum_{i=1}^m \lambda_i (y_i - \tilde{x})^\top \odot \mathbf{T}(\tilde{x}). \tag{5.9}$$

Note that

$$\begin{aligned}
\mathbf{0} &= (\tilde{x} - \tilde{x})^\top \odot \mathbf{T}(\tilde{x}) \\
&= \left(\sum_{i=1}^m \lambda_i y_i - \sum_{i=1}^m \lambda_1 \tilde{x} \right)^\top \odot \mathbf{T}(\tilde{x}) \\
&= \left(\sum_{i=1}^m \lambda_i (y_i - \tilde{x}) \right)^\top \odot \mathbf{T}(\tilde{x}) \\
&= \sum_{i=1}^m \lambda_i (y_i - \tilde{x})^\top \odot \mathbf{T}(\tilde{x}),
\end{aligned}$$

which is a contradiction, as by (5.9), we have $\mathbf{0} \not\preceq \sum_{i=1}^m \lambda_i (y_i - \tilde{x})^\top \odot \mathbf{T}(\tilde{x})$. Thus, we must have $\text{conv}(\{y_1, y_2, \ldots, y_m\}) \subseteq \bigcup_{i=1}^m P(y_i)$, and hence P is a KKM-map. Since $\mathbf{T}$ is pseudomonotone, by Theorem 5.3, $P(y) \subseteq Q(y)$ for all $y \in X$, and thus, Q is also a KKM-map.

Next, we show that for each $y \in X, Q(y)$ is closed. For this, let $\{x_m\}$ be a sequence in $Q(y)$ for any fixed $y \in X$ such that $\{x_m\}$ converges to $\hat{x} \in X$. Then,

$$\mathbf{0} \preceq (y - \{x_m\})^\top \odot \mathbf{T}(y) \text{ for all } m.$$

Since $(y-\{x_m\})^\top \odot \mathbf{T}(y) \to (y-\hat{x})^\top \odot \mathbf{T}(y)$ as $m \to \infty$, we have $\mathbf{0} \preceq (y-\hat{x})^\top \odot \mathbf{T}(y)$, and thus $\hat{x} \in Q(y)$. Therefore, $Q(y)$ is a closed subset of the compact set X, and so it is compact. By Theorem 5.6, $\bigcap_{y \in X} Q(y) \neq \emptyset$. Hence, there exists $\bar{x} \in X$ such that

$$\mathbf{0} \preceq (y - \bar{x})^\top \odot \mathbf{T}(y) \text{ for all } y \in X.$$

Thus, by Theorem 5.3, $\bar{x}$ is a solution of SIVIP. □

Example 5.2. *Let $X = [1, \infty]$ and $\boldsymbol{T} : X \to \mathbb{R}_I$ be defined as $\boldsymbol{T}(x) = [x, x^2]$ for all $x \in X$. Since X is not compact, Theorem 5.7 cannot be applied. However, it is easy to see that $\bar{x} = 1 \in X$ is such that $\boldsymbol{0} \preceq (y - \bar{x})^\top \odot \boldsymbol{T}(\bar{x}) = (y - 1)^\top \odot [1, 1] = y - 1$ for all $y \in X$, that is, $\bar{x} = 1$ is a solution of SIVIP.*

When X in Theorem 5.7 is not necessarily compact, we have the following two results (Theorems 5.8 and 5.9).

Theorem 5.8. *Let X be a nonempty convex subset of $\mathbb{R}^n$ and $\boldsymbol{T} : X \to I(\mathbb{R})^n$ be gH-hemicontinuous and pseudomonotone. Also, let there exist a compact subset C of $\mathbb{R}^n$ and $\tilde{y} \in X \cap C$ such that*

$$\boldsymbol{0} \npreceq (\tilde{y} - x)^\top \odot \boldsymbol{T}(x) \textit{ for all } x \in X \setminus C. \tag{5.10}$$

Then, SIVIP has a solution.

Proof. Let the set-valued maps $P, Q : X \to 2^X$ be the same as in the proof of Theorem 5.7. Let $\tilde{y} \in X \cap C$. We first show that $P(\tilde{y})$ is compact. If $P(\tilde{y}) \nsubseteq C$, then there exists $x \in P(\tilde{y})$ such that $x \in X \setminus C$. Then,

$$\mathbf{0} \preceq (\tilde{y} - x)^\top \odot \mathbf{T}(x),$$

which contradicts (5.10). Thus, we have $P(\tilde{y}) \subseteq C$. Then, the closure, $\text{cl}(P(\tilde{y}))$ of $P(\tilde{y})$ is a closed subset of the compact set C, and hence compact. As we have seen in Theorem 5.7, P is a KKM-map. Therefore, by Theorem 5.6, $\bigcap_{y \in X} \text{cl}(P(y)) \neq \emptyset$. Since for each $y \in X$, $Q(y)$ is closed and $P(y) \subseteq Q(y)$, we have

$$\text{cl}(P(y)) \subseteq \text{cl}(Q(y)) = Q(y).$$

Therefore,

$$\emptyset \neq \bigcap_{y \in X} \text{cl}(P(y)) \subseteq \bigcap_{y \in X} Q(y).$$

Thus, there exists $\bar{x} \in X$ such that

$$\mathbf{0} \preceq (y - \bar{x})^\top \odot \mathbf{T}(y) \text{ for all } y \in X.$$

Hence, by Theorem 5.3, $\bar{x}$ is a solution of SIVIP. □

Theorem 5.9. *Let X be a closed convex subset of $\mathbb{R}^n$ and $\boldsymbol{T} : X \to \mathbb{R}^n_I$ be gH-continuous and pseudomonotone. Assume that there exist a compact subset D of $\mathbb{R}^n$ and $\tilde{y} \in X \cap D$ satisfying*

$$\boldsymbol{0} \npreceq (\tilde{y} - x)^\top \odot \boldsymbol{T}(\tilde{y}) \textit{ for all } x \in X \setminus D. \tag{5.11}$$

Then, SIVIP has a solution.

Proof. Let the set-valued maps $P, Q : X \to 2^X$ be the same as in the proof of Theorem 5.7. Let $\tilde{y} \in X \cap D$. By the same argument as in the proof of Theorem 5.7, we derive that Q is a KKM-map and for each $y \in X$, $Q(y)$ is closed.

Next, we show that $Q(\tilde{y}) \subseteq X \cap D$. If $Q(\tilde{y}) \nsubseteq D$, then there exists $x \in Q(\tilde{y})$ such that $x \in X \setminus D$. Then,

$$\mathbf{0} \preceq (\tilde{y} - x)^\top \odot \mathbf{T}(\tilde{y}),$$

which contradicts (5.11). Hence, $Q(\tilde{y}) \subseteq D$, and thus $Q(\tilde{y}) \subseteq X \cap D$. Since X is closed and D is compact, $X \cap D$ is compact. Therefore, $Q(\tilde{y})$ is a closed subset of the compact set $X \cap D$, and hence $Q(\tilde{y})$ is compact. Then, by Theorem 5.6, $\bigcap_{y \in X} Q(y) \neq \emptyset$. The rest of the proof is similar to the proof of Theorem 5.7. □

To prove our next existence result (Theorem 5.11), we need the following Browder-type fixed point theorem for set-valued functions.

Theorem 5.10. (Browder-type fixed point theorem [9]). *Let X be a nonempty convex subset of $\mathbb{R}^n$ and $P, Q : X \to 2^X$ be two set-valued maps. Assume that the following conditions hold:*

(i) For each $x \in X, \operatorname{conv}(P(x)) \subseteq Q(x)$, $P(x)$ is nonempty, and

(ii) for each $y \in X$, $P^{-1}(y) = \{x \in K : y \in P(x)\}$ is open in X.

Then, there exists $\bar{x} \in X$ such that $\bar{x} \in Q(\bar{x})$.

Theorem 5.11. *Let X be a nonempty convex subset of $\mathbb{R}^n$ and $\boldsymbol{T} : X \to \mathbb{R}^n_I$ be gH-hemicontinuous and pseudomonotone. Assume that there exist a nonempty compact convex subset B of X and a nonempty compact subset D of X such that for each $x \in X \setminus D$, there exists $\tilde{y} \in B$ such that $\boldsymbol{0} \npreceq (\tilde{y} - x)^\top \odot \boldsymbol{T}(x)$. then, SIVIP has a solution.*

Proof. For each $x \in X$, define set-valued maps $P, Q : X \to 2^X$ by

$$P(x) = \{y \in X : \mathbf{0} \not\preceq (y - x)^\top \odot \mathbf{T}(y)\}$$

and

$$Q(x) = \{y \in X : \mathbf{0} \not\preceq (y - x)^\top \odot \mathbf{T}(x)\}.$$

It is clear that for each $x \in X$, $Q(x)$ is convex. By pseudomonotonicity of $\mathbf{T}$, we have $P(x) \subseteq Q(x)$, and hence $\operatorname{conv}(P(x)) \subseteq \operatorname{conv}(Q(x)) = Q(x)$ for all $x \in X$. For each $y \in X$, the complement of $P^{-1}(y)$ in X is

$$[P^{-1}(y)]^{\complement} = \{x \in X : \mathbf{0} \preceq (y - x)^\top \odot \mathbf{T}(y)\} \text{ is closed in } X,$$

and thus $P^{-1}(y)$ is open in X.

Assume that for all $x \in X$, $P(x)$ is nonempty. Then, all the conditions of Theorem 5.10 are satisfied, and therefore there exist $\hat{x} \in X$ such that $\hat{x} \in Q(\hat{x})$. It follows that

$$\mathbf{0} \not\preceq (\hat{x} - \hat{x})^\top \odot \mathbf{T}(\hat{x}) = \mathbf{0}, \text{ a contradiction.}$$

Hence, there exists $\bar{x} \in X$ such that $P(\bar{x}) = \emptyset$. This implies that for all $y \in X$,

$$\mathbf{0} \preceq (y - \bar{x})^\top \odot \mathbf{T}(y).$$

That is, $\bar{x} \in X$ is a solution of MIVIP. By Theorem 5.3, $\bar{x} \in X$ is a solution of SIVIP. $\square$

To derive existence result for the solutions of MIVIP, we use the following definition of proper quasimonotonicity.

Definition 5.12. Let X be a nonempty subset of $\mathbb{R}^n$. A mapping $\mathbf{T} : X \to \mathbb{R}^n_I$ is said to be properly quasimonotone if for every $x_1, x_2, \ldots, x_m \in X$ and every $y \in \text{conv}(\{x_1, x_2, \ldots, x_m\})$, there exists $i \in \{1, 2, \ldots, m\}$ such that

$$(y - x_i)^\top \odot \mathbf{T}(x_i) \preceq \mathbf{0}.$$

Theorem 5.13. *Let X be a nonempty compact convex subset of $\mathbb{R}^n$ and $\boldsymbol{T} : K \to \mathbb{R}^n_I$ be properly quasimonotone. Then, MIVIP has a solution.*

Proof. Define the set-valued mapping $Q : X \to 2^X$ as

$$Q(y) = \{x \in X : \mathbf{0} \preceq (y - x)^\top \odot \mathbf{T}(y)\}, \text{ for all } y \in X.$$

For any $y_1, y_2, \ldots, y_m \in X$ and $\tilde{y} \in \text{conv}(\{y_1, y_2, \ldots, y_m\})$, proper quasimonotonicity of $\mathbf{T}$ implies that $\tilde{y} \in \bigcup_{i=1}^m Q(y_i)$. Also, by the proof of Theorem 5.4, it is clear that for each $y \in X$, $Q(y)$ is closed. That is, $Q(y)$ is a closed subset of the compact set X, and hence compact. Therefore, by Theorem 5.6, we have $\bigcap_{y \in X} Q(y) \neq \emptyset$. Hence, any $\bar{x} \in \bigcap_{y \in X} Q(y)$ is a solution of MIVIP. □

5.3 Relationship of IVIs with IOPs

In the direction to solve IOPs, in this section, we provide optimality conditions for the following constrained IOP.

$$\min_{x \in K} \mathbf{F}(x), \tag{5.12}$$

where K is a convex subset of $\mathbb{R}^n$ and $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}_I$ is a gH-differentiable convex IVF.

The following lemma gives a characterization of convex IVFs, which is required to prove Theorem 5.16.

Lemma 5.14. *Let K be a nonempty open convex subset of $\mathbb{R}^n$ and $\boldsymbol{F} : K \to \mathbb{R}_I$ be gH-differentiable at any $x \in K$. If $\boldsymbol{F}$ is convex on K, then*

$$(x_2 - x_1)^\top \odot \nabla \boldsymbol{F}(x_1) \preceq \boldsymbol{F}(x_2) \ominus_{gH} \boldsymbol{F}(x_1) \textit{ for all } x_1, x_2 \in K.$$

Proof. Suppose $\mathbf{F}$ is convex on K, and $x_1,\ x_2 \in K$. Then, for $h = x_2 - x_1$ and $0 < \alpha < 1$,

$$\begin{aligned}
\mathbf{F}(x_1 + \alpha h) &= \mathbf{F}\left((1-\alpha)x_1 + \alpha(x_1 + h)\right) \\
&\preceq (1-\alpha) \odot \mathbf{F}(x_1) \oplus \alpha \odot \mathbf{F}(x_1 + h) \\
&= \mathbf{F}(x_1) \ominus_{gH} \alpha \odot \mathbf{F}(x_1) \oplus \alpha \odot \mathbf{F}(x_1 + h) \text{ by Lemma 1.7.}
\end{aligned}$$

Therefore, by Lemma 1.6,

$$\begin{aligned}
&\mathbf{F}(x_1 + \alpha h) \ominus_{gH} \mathbf{F}(x_1) \preceq \alpha \odot (\mathbf{F}(x_1 + h) \ominus_{gH} \mathbf{F}(x_1)) \\
\text{or, } &\frac{1}{\alpha} \odot (\mathbf{F}(x_1 + \alpha h) \ominus_{gH} \mathbf{F}(x_1)) \preceq \mathbf{F}(x_1 + h) \ominus_{gH} \mathbf{F}(x_1) \\
\text{or, } &\frac{1}{\alpha} \odot (\mathbf{F}(x_1 + \alpha h) \ominus_{gH} \mathbf{F}(x_1)) \preceq \mathbf{F}(x_2) \ominus_{gH} \mathbf{F}(x_1).
\end{aligned}$$

Hence, as $\alpha \to 0+$, with the help of Definition 1.27, we get

$$(x_2 - x_1)^\top \odot \nabla \mathbf{F}(x_1) \preceq \ \mathbf{F}(x_2) \ominus_{gH} \mathbf{F}(x_1).$$

□

Remark 5.15. Lemma 5.14 gives a mathematically correct version of Theorem 2.2 in [47]. To see that Theorem 2.2 in [47] is not correct consider $\mathbf{F} : (1, 4) \to \mathbb{R}_I$ as

$\mathbf{F}(x) = [x, x^2]$. Clearly, $\mathbf{F}$ is convex. However,

$$(3-2) \odot \nabla\mathbf{F}(2) = [1, 4] \not\preceq \mathbf{F}(3) \oplus (-1) \odot \mathbf{F}(2) = [-3, 7].$$

Theorem 5.16. *Let K be a nonempty convex subset of $\mathbb{R}^n$ and $\boldsymbol{F} : \mathbb{R}^n \to \mathbb{R}_I$ be a convex and gH-differentiable IVF. Then, $\bar{x}$ is a weak efficient solution of (5.12) if and only if $\bar{x}$ is a solution of SIVIP with $\boldsymbol{T} = \nabla\boldsymbol{F}$.*

Proof. Let $\bar{x}$ be a weak efficient solution of (5.12).

For any $y \in K$, define a function $\boldsymbol{\psi} : [0, 1] \to \mathbb{R}_I$ by

$$\boldsymbol{\psi}(\lambda) = \mathbf{F}(\bar{x} + \lambda(y - \bar{x})) \text{ for all } \lambda \in [0, 1].$$

Since $\lambda = 0$ is an efficient solution of $\boldsymbol{\psi}$, by Lemma 2.1, we have

$$\begin{aligned} & \mathbf{0} \preceq \nabla\boldsymbol{\psi}(0) \\ \implies \quad & \mathbf{0} \preceq (y - \bar{x})^\top \odot \nabla\mathbf{F}(\bar{x}) \text{ for all } y \in K. \end{aligned}$$

Thus, $\bar{x}$ is a solution of SSIVIP with $\mathbf{T} = \nabla\mathbf{F}$.

Conversely, let $\bar{x}$ be a solution of SIVIP with $\mathbf{T} = \nabla\mathbf{F}$. On contrary, suppose that $\bar{x}$ is not a weak efficient solution of (5.12). Then, there exists $x \in K$ such that $\mathbf{F}(\bar{x}) \not\preceq \mathbf{F}(x)$. Therefore,

$$\mathbf{0} \not\preceq \mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}). \tag{5.13}$$

Also, since $\mathbf{F}$ is convex, by Lemma 5.14, we have

$$(x - \bar{x})^\top \odot \nabla\mathbf{F}(\bar{x}) \preceq \mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \tag{5.14}$$

By (5.13) and (5.14), we get

$$\mathbf{0} \not\preceq (x - \bar{x})^\top \odot \nabla \mathbf{F}(\bar{x}),$$

which is a contradiction as $\bar{x}$ is assumed to be a solution of SIVIP with $\mathbf{T} = \nabla \mathbf{F}$. $\square$

Example 5.3. *In this example, we provide an instance to verify Theorem 5.16. Let $K \subseteq \mathbb{R}^2$ be the set $\{(x_1, x_2) : 1 \leq x_1 \leq 1.5, 1 \leq x_2 \leq 1.5\}$. Consider the IOP*

$$\min_{x \in K} \boldsymbol{F}(x_1, x_2), \tag{5.15}$$

where

$$\begin{aligned}
\boldsymbol{F}(x_1, x_2) &= [\underline{f}(x_1, x_2), \overline{f}(x_1, x_2)] \text{ and} \\
\underline{f}(x_1, x_2) &= -1 + x_1^2 + 2x_2 \\
\text{and } \overline{f}(x_1, x_2) &= 4 + x_1^2 + 2x_2^2.
\end{aligned}$$

The graphs of functions $\underline{f}$ and $\overline{f}$ are shown in Figure 5.1. The red dots in the surfaces of $\underline{f}(x_1, x_2)$ and $\overline{f}(x_1, x_2)$ are the locations of the minima of the functions $\underline{f}$ and $\overline{f}$, respectively. From the figure, it is evident that $\bar{x} = (1, 1)$ is the only efficient solution of (5.15).

Notice that $\underline{f}$ and $\overline{f}$ are convex on K, and therefore $\boldsymbol{F}$ is convex on K. We now see that $\bar{x}$ is also a solution of SIVIP with $\boldsymbol{T} = \nabla \boldsymbol{F}(\bar{x})$. At $\bar{x}$, the partial derivatives of $\boldsymbol{F}$ are

$$\begin{aligned}
D_1 \boldsymbol{F}(\bar{x}) &= \left[\min\left\{\frac{\partial \underline{f}}{\partial x_1}(\bar{x}), \frac{\partial \overline{f}}{\partial x_1}(\bar{x})\right\}, \max\left\{\frac{\partial \underline{f}}{\partial x_1}(\bar{x}), \frac{\partial \overline{f}}{\partial x_1}(\bar{x})\right\}\right] = [2, 2] \\
\text{and } D_2 \boldsymbol{F}(\bar{x}) &= \left[\min\left\{\frac{\partial \underline{f}}{\partial x_2}(\bar{x}), \frac{\partial \overline{f}}{\partial x_2}(\bar{x})\right\}, \max\left\{\frac{\partial \underline{f}}{\partial x_2}(\bar{x}), \frac{\partial \overline{f}}{\partial x_2}(\bar{x})\right\}\right] = [2, 4].
\end{aligned}$$

Therefore,

$$(y-\bar{x})^\top \odot \nabla \boldsymbol{F}(\bar{x}) = ((y_1-1),(y_2-1))^\top \odot ([2,2],[2,4]) = (y_1-1)\odot[2,2]\oplus(y_2-1)[2,4].$$

Clearly, $\boldsymbol{0} \preceq (y_1 - 1) \odot [2,2] \oplus (y_2 - 1) \odot [2,4]$ *for all* $y_1, y_2 \in [1, 1.5]$. *That is,* $\boldsymbol{0} \preceq (y - \bar{x})^\top \odot \nabla \boldsymbol{F}(\bar{x})$ *for all* $y \in K$. *Thus,* $\bar{x}$ *is a solution of SIVIP.*

Further, by following a similar procedure as in Example 5.1, it can be seen that $\bar{x}$ *is the only solution of SIVIP with* $\boldsymbol{T} = \nabla \boldsymbol{F}(\bar{x})$.

In our next theorem (Theorem 5.17), we provide a sufficient optimality condition for IOP (5.12) with the help of MIVI.

Theorem 5.17. *Let* K *be a nonempty convex subset of* $\mathbb{R}^n$ *and* $\boldsymbol{F} : K \to \mathbb{R}_I$ *be gH-differentiable. If* $\bar{x}$ *is a solution of MIVIP with* $\boldsymbol{T} = \nabla \boldsymbol{F}$, *then* $\bar{x}$ *is a weak efficient solution of IOP (5.12).*

Proof. Let $y \in K$ be arbitrary. Consider the IVF $\boldsymbol{\phi}(y) = \mathbf{F}(\bar{x} + \lambda(y - \bar{x}))$ for all $\lambda \in [0, 1]$. Since $\boldsymbol{\phi}'(\lambda) = (y - \bar{x})^\top \odot \nabla\mathbf{F}(\bar{x} + \lambda(y - \bar{x}))$ and $\bar{x}$ is a solution of MIVIP with $\mathbf{T} = \nabla\mathbf{F}$, it follows that for each $\lambda \in 0, 1$

$$\mathbf{0} \preceq (y - \bar{x})^\top \odot \nabla\mathbf{F}(\bar{x} + \lambda(y - \bar{x})) = \boldsymbol{\phi}'(\lambda).$$

Thus, by Lemma 1.33, $\boldsymbol{\phi}$ is monotonic increasing on $[0, 1]$, and therefore

$$\mathbf{F}(\bar{x}) = \boldsymbol{\phi}(0) \preceq \boldsymbol{\phi}(1) = \mathbf{F}(y).$$

Hence, $\bar{x}$ is a weak efficient solution of IOP 5.12. □

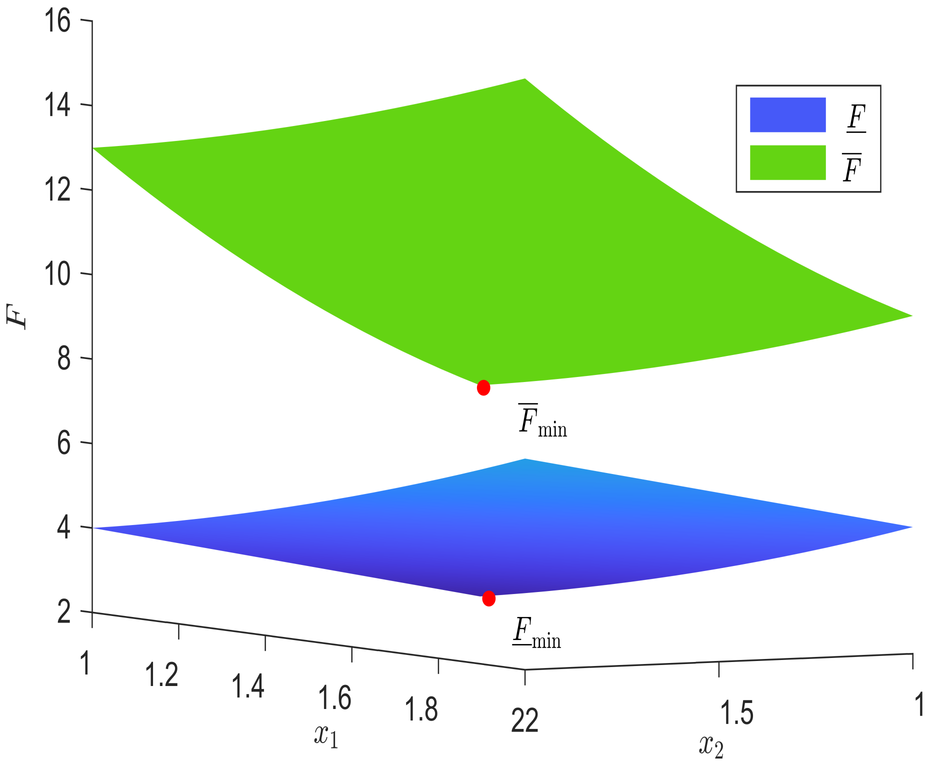

FIGURE 5.1: The objective function $\mathbf{F}(x_1, x_2)$ of Example 5.3

To see that converse of Theorem 5.17 may not be true, consider the IVF $\mathbf{T}$ as given in Example 5.1. Note that $\mathbf{T}(0,0) \preceq \mathbf{T}(x_1, x_2)$ for all $x_1, x_2 \in [-1, 0]$. That is, $(0,0)$ is an efficient point. However, we have seen in Example 5.1 that $(0,0)$ is not a solution of MIVIP.

5.4 Concluding Remarks

In this chapter, the concepts of Stampacchia and Minty IVIs have been introduced. Relationship between the solution sets of these variational inequalities has been explored. Under the assumption of gH-hemicontinuity and pseudomonotonicity, the solution sets of these VIs are shown to be closed and convex. Existence and uniqueness results have been derived for the solutions of the proposed VIs. A necessary optimality condition has been presented for a constrained IOP with the help of gH-differentiability. We have also given a first-order necessary condition for convex IVFs. Further, we have provided a necessary and sufficient optimality condition for a constrained IOP in terms of proposed SIVIs. Finally, a sufficient optimality condition for a constrained IOP is derived in terms of proposed MIVIs.

Based on the proposed research of this chapter, in future, one can work in the following directions.

- There is a huge literature [54, 69, 101] on the iterative methods to solve conventional Stampacchia and Minty VIs. In future, one can try to extend some classical iterative techniques to solve Stampacchia and Minty IVIs.
- Recently several authors proposed many game theoretic solutions to solve IOPs [17, 56, 73]. One can try to explore the relationships between the proposed IVIs and game theoretic problems under interval uncertainties.

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