Chapter 3

Fréchet Subdifferential Calculus for Interval-valued Functions and its Applications in Nonsmooth Interval Optimization

3.1 Introduction

It is well-known that the presence of nonsmoothness is inevitable in modern optimization and variational analysis. Nonsmoothness naturally enters not only through initial data of optimization problems but largely via variational principles and perturbation techniques applied to problems with smooth data. In convex optimization, subgradient acts as an essential tool to deal with nonsmooth convex objective functions. Applications of subgradient–based methods in convex optimization is now vastly well-known [16, 78, 83]. Bazaraa et al. [14] extended the notion of subgradients for nonconvex functions under the name of Fréchet subgradients. Fréchet subgradients, introduced in [14], proved a striking tool to deal with nonsmooth optimization problems. Apart from optimization problems, Fréchet subgradient has played a prominent role in nonsmooth analysis, stochastic control, differential games, etc. In this chapter, we extend the notion of Fréchet subgradients for IVFs and with the help of proposed notion of Fréchet subgradients we derive necessary optimality conditions for nonsmooth IOPs.

3.1.1 Motivation

IOPs with nonsmooth and nonconvex IVFs are not extensively studied yet. Recently the authors [1, 4, 5] have presented several optimality conditions for nonsmooth convex IOPs by assuming that the lower function $\underline{f}$ and the upper function $\overline{f}$ are explicitly known for the objective function $\mathbf{F}(x) = [\underline{f}(x), \overline{f}(x)]$. It is to observe that even for a very simple IVF $\mathbf{F}$, it is not always an easy task to find the expressions of $\underline{f}(x)$ and $\overline{f}(x)$, for instance, take

$$\mathbf{F}(x_1, x_2) = \frac{[-2, 3] \odot \cos x_1 + [-1, 2] \odot x_2}{[-1, 2] \odot \sin x_2 + [-1, 2] \odot x_1}.$$

In [19], optimality conditions and duality results for nonsmooth convex IOPs using the parametric representation of its objective and constraint functions are found. However, the parametric process is also practically difficult, because in the parametric process, the number of variables increases with the number of intervals involved in the IVFs, and to verify any property of an IVF one has to verify it for an infinite number of its corresponding real-valued functions (for instance, see Definition 9 in [19]). To overcome these drawbacks, Chauhan et al. [50] coined a new notion of gH-subgradients for convex IVFs. This new notion of gH-subgradients neither requires the parametric form nor the explicit form of the objective function $\mathbf{F}$. However, the optimality results given in [50] assume that the objective function of the IOP is convex, which is also not a mild condition. For instance, a very simple IVF, $\mathbf{F}(x) = [-1, 2] \odot x^2$ is not convex. So, the optimization problems with even such a simple objective function cannot be analyzed using the available techniques of IOPs. Surprisingly, till date there are no methods available to solve nonsmooth IOPs with *nonconvex* objective function.

3.1.2 Contribution

The major contributions of this chapter are the following:

- The notion of Fréchet subgradients is introduced for general IVFs
- various Fréchet subdifferential calculus results are developed for nonconvex IVFs
- necessary optimality conditions for unconstrained nonconvex IOPs are derived
- a necessary condition for unconstrained weak sharp minima is given.

It is to be mentioned that for the results derived in this chapter we do not consider any of the following assumptions.

(i) Parametric form of $\mathbf{F}$ (see [19]),

(ii) the explicit form of $\underline{f}$ and $\overline{f}$ of $\mathbf{F}$ (see [1, 4, 5]), and

(iii) convexity of $\mathbf{F}$ (see [50]).

Hence, the results of this chapter are applicable for general IVFs.

3.2 Calculus of gH-Fréchet subgradients

In this section, we introduce the notion of gH-Fréchet subgradients of IVFs and derive exact calculus results for these subgradients.

Definition 3.1. (*gH-Fréchet subdifferentiability*). Let $\mathbf{F} : \mathbb{R}^n \to \overline{\mathbb{R}_I}$ be a proper extended IVF that is finite at $\bar{x} \in \mathbb{R}^n$. Then, the gH-Fréchet subdifferential set of $\mathbf{F}$ at $\bar{x}$, denoted as $\partial_f \mathbf{F}(\bar{x})$, is defined by

$$\partial_f \mathbf{F}(\bar{x}) = \left\{ \widehat{\mathbf{G}} \in \mathbb{R}_I^n : \mathbf{0} \preceq \liminf_{x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \odot \left(\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{G}} \right) \right\}. \tag{3.1}$$

We call elements of $\partial_f \mathbf{F}(\bar{x})$ as gH-Fréchet subgradients of $\mathbf{F}$ at $\bar{x}$. Further, if $\partial_f \mathbf{F}(\bar{x}) \neq \emptyset$, we say that $\mathbf{F}$ is gH-Fréchet subdifferentiable at $\bar{x}$. If $\mathbf{F}(\bar{x})$ is not finite, we define $\partial_f \mathbf{F}(\bar{x}) = \emptyset$.

Remark 3.2. (Geometrical interpretation of gH-Fréchet subdifferentiability). From Definition 3.1, $\widehat{\mathbf{G}} \in \partial_f \mathbf{F}(\bar{x})$ if and only if

$$\mathbf{0} \preceq \liminf_{x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \odot \left(\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{G}} \right).$$

Therefore, for any $\epsilon > 0$, we get a $\delta > 0$ such that whenever $0 < \|x - \bar{x}\| < \delta$, we have

$$\begin{aligned} & \mathbf{0} \preceq \mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{G}} \oplus \epsilon \|x - \bar{x}\| \\ \Longrightarrow \quad & \mathbf{F}(\bar{x}) \oplus (x - \bar{x})^\top \odot \widehat{\mathbf{G}} \preceq \mathbf{F}(x) \oplus \epsilon \|x - \bar{x}\|. \end{aligned} \tag{3.2}$$

Since (3.2) is true for any ϵ arbitrarily close to 0, $\mathbf{F}(\bar{x}) \oplus (x - \bar{x})^\top \odot \widehat{\mathbf{G}}$ is a supporting function from below to the epigraph of $\mathbf{F}$ at $(\bar{x}, \mathbf{F}(\bar{x}))$. Infact, at the point of nondifferentiability, there can be an infinite number of such supporting IVFs $\mathbf{F}(\bar{x}) \oplus$

$(x - \bar{x})^\top \odot \widehat{\mathbf{G}}$ and the collection of all such $\widehat{\mathbf{G}}$'s form the set $\partial_f \mathbf{F}(\bar{x})$. In other words, there always exists a neighbourhood of the point $\bar{x}$ such that the graph of $\mathbf{F}(\bar{x}) \oplus (x - \bar{x})^\top \odot \widehat{\mathbf{G}}$ does not completely lie above the graph of $\mathbf{F}$. To have a better understanding of this idea, we consider the following example.

Example 3.1. *Consider an IVF* $\boldsymbol{F} : \mathbb{R} \to \mathbb{R}_I$ *given by* $\boldsymbol{F}(x) = [|x|, k|x|]$, *where* $k > 1$ *is a real number.*

Let us apply Definition 3.1 to check gH-Fréchet subdifferentiability of $\boldsymbol{F}$ *at* 0*.*

$$\begin{aligned} \partial_f \boldsymbol{F}(0) &= \left\{ \boldsymbol{G} \in \mathbb{R}_I : \boldsymbol{0} \preceq \liminf_{x \to 0} \frac{1}{|x - 0|} \odot (\boldsymbol{F}(x) \ominus_{gH} \boldsymbol{F}(0) \ominus_{gH} (x - 0) \odot \boldsymbol{G}) \right\} \\ &= \left\{ \boldsymbol{G} : \boldsymbol{0} \preceq \liminf_{x \to 0} \frac{1}{|x|} \odot ([|x|, k|x|] \ominus_{gH} x \odot \boldsymbol{G}) \right\} \end{aligned}$$

Since $\boldsymbol{G} \in \mathbb{R}_I$, *let* $\boldsymbol{G} = [a, b]$ *for some* $a, \ b \in \mathbb{R}$ *with* $a \leq b$*. Therefore,*

$$\partial_f \boldsymbol{F}(0) = \left\{ [a, b] : \boldsymbol{0} \preceq \liminf_{x \to 0} \frac{1}{|x|} \odot ([1, k] \odot |x| \ominus_{gH} x \odot [a, b]) \right\}.$$

Let us now consider the following two possible cases.

Case 1. $x \geq 0$.

Note that

$$\begin{aligned} & \boldsymbol{0} \preceq \liminf_{x \to 0} \frac{1}{|x|} \odot ([1, k] \odot |x| \ominus_{gH} x \odot [a, b]) \\ \implies \quad & \boldsymbol{0} \preceq \liminf_{x \to 0} \frac{1}{x} \odot ([1, k] \odot x \ominus_{gH} x \odot [a, b]) \\ \implies \quad & \boldsymbol{0} \preceq [1, k] \ominus_{gH} [a, b] \\ \implies \quad & [a, b] \preceq [1, k], \ \text{by (i) of Lemma 1.8} \\ \implies \quad & a \leq 1 \text{ and } b \leq k. \end{aligned}$$

Case 2. $x < 0$.

Note that

$$\begin{aligned}
& \boldsymbol{0} \preceq \liminf_{x \to 0} \frac{1}{|x|} \odot ([1, k] \odot |x| \ominus_{gH} x \odot [a, b]) \\
\implies \quad & \boldsymbol{0} \preceq \liminf_{x \to 0} \frac{1}{-x} \odot ([1, k] \odot (-x) \ominus_{gH} x \odot [a, b]) \\
\implies \quad & \boldsymbol{0} \preceq [1, k] \ominus_{gH} (-1) \odot [a, b] \\
\implies \quad & \boldsymbol{0} \preceq [1, k] \ominus_{gH} [-b, -a] \\
\implies \quad & [-b, -a] \preceq [1, k], \text{ by (i) of Lemma 1.8} \\
\implies \quad & b \geq -1 \text{ and } a \geq -k.
\end{aligned}$$

Therefore, from Case 1 and Case 2, we have

$$-k \leq a \leq 1 \text{ and } -1 \leq b \leq k.$$

Hence,

$$\partial_f \boldsymbol{F}(0) = \{[a, b] : -k \leq a \leq 1 \text{ and } -1 \leq b \leq k\}. \tag{3.3}$$

The function **F** with $k = 2$ is depicted by the grey shaded region in Figure 3.1. We also figure out gH-Fréchet subgradient of **F** at 0 namely $\mathbf{G}'$, where $\mathbf{G}'(x) = [-0.5, 1.5] \odot x$. Since $\mathbf{G}'(x)$ belongs to the set (3.3), $\mathbf{G}'(x)$ is a gH-Fréchet subgradient of **F** at 0. $\mathbf{G}'(x)$ is depicted by the dotted region in Figure 3.1. Observe from Figure 3.1 that the graph of $\mathbf{G}'$ does not completely lie above the graph of **F** as reflected in Remark 3.2.

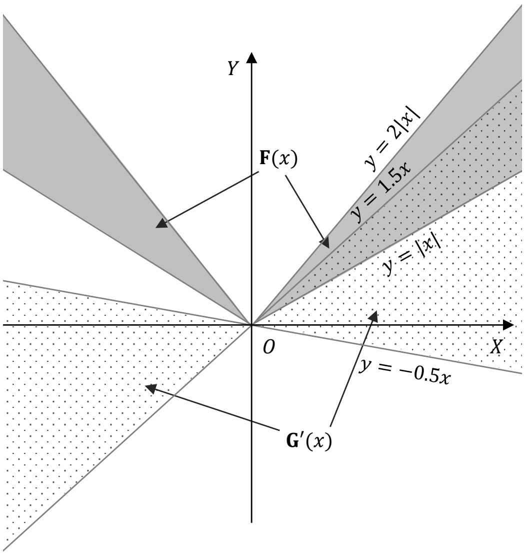

FIGURE 3.1: Geometrical view of gH-Fréchet subdifferentiability of $\mathbf{F}$ of Example 3.1

Note 7. If we take $k = 1$ in Example 3.1, then the IVF $\mathbf{F}$ reduces to a real-valued function given as $f(x) = |x|$. We now apply Definition 3.1 to find gH-Fréchet subgradients of f at 0.

$$\partial_f f(0) = \left\{ a : 0 \leq \liminf_{x \to 0} \frac{|x| - ax}{|x|}, \text{ where } a \in \mathbb{R} \right\}.$$

Similar to Example 3.1, let us now consider the following two cases.

- Case 1. $x \geq 0$.

  In this case, we get

$$0 \leq \liminf_{x \to 0} \frac{x - ax}{x}, \text{ i.e., } a \leq 1.$$

- Case 2. $x < 0$.

  In this case, we get $a \geq -1$.

Therefore, from both the cases, we get

$$\partial_f f(0) = [-1, 1].$$

Remark 3.3. It is to observe that set (3.1) can be empty. For instance, consider an IVF $\mathbf{F} : \mathbb{R} \to \mathbb{R}_I$ given by $\mathbf{F}(x) = [-k|x|, -|x|]$, where $k > 1$ is a real number. Then, by following similar steps as in Example 3.1, it can be seen that $\partial_f \mathbf{F}(0) = \emptyset$.

Next, in Note 8, we show that the notion of gH-subgradients (Definition 1.29) introduced in [50] is a special case of Definition 3.1.

Note 8. If $\mathbf{F} : \mathbb{R}^n \to \overline{\mathbb{R}_I}$ is convex, then $\widehat{\mathbf{G}} \in \mathbb{R}_I^n$ is a gH-subgradient of $\mathbf{F}$ at $\bar{x} \in \mathbb{R}^n$ according to Definition 3.1 if and only if $\widehat{\mathbf{G}}$ is a subgradient of $\mathbf{F}$ at $\bar{x}$ according to Definition 1.29. The reason is as follows.

Let $\widehat{\mathbf{G}} \in \partial_f \mathbf{F}(\bar{x})$. Then,

$$\mathbf{0} \preceq \liminf_{x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \odot \left(\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{G}} \right).$$

Therefore, for any $\epsilon > 0$, we get a $\delta > 0$ such that whenever $0 < \|x - \bar{x}\| < \delta$, we have

$$\mathbf{0} \preceq \mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{G}} \oplus \epsilon \|x - \bar{x}\|.$$

By taking $x = \bar{x} + \lambda d, \ \lambda \downarrow 0$, we get

$$\mathbf{0} \preceq \mathbf{F}(\bar{x} + \lambda d) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} \lambda d^\top \odot \widehat{\mathbf{G}} \oplus \epsilon \|\lambda d\|.$$

In particular, by taking $d = x - \bar{x}$, we get

$$\begin{aligned} & \mathbf{0} \preceq \mathbf{F}(\bar{x} + \lambda(x - \bar{x})) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} \lambda(x - \bar{x})^\top \odot \widehat{\mathbf{G}} \oplus \epsilon\lambda \|x - \bar{x}\| \\ \implies \quad & \mathbf{0} \preceq \mathbf{F}(\lambda x + (1 - \lambda)\bar{x}) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} \lambda(x - \bar{x})^\top \odot \widehat{\mathbf{G}} \oplus \epsilon\lambda \|x - \bar{x}\| \end{aligned}$$

$$\implies \mathbf{0} \preceq \lambda \odot \mathbf{F}(x) \oplus (1-\lambda) \odot \mathbf{F}(\bar{x}) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} \lambda(x-\bar{x})^\top \odot \widehat{\mathbf{G}}$$
$$\oplus\, \epsilon\lambda\|x-\bar{x}\|, \text{ because } \mathbf{F} \text{ is convex}$$
$$\implies \mathbf{0} \preceq \lambda \odot \mathbf{F}(x) \oplus \mathbf{F}(\bar{x}) \ominus_{gH} \lambda \odot \mathbf{F}(\bar{x}) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} \lambda(x-\bar{x})^\top \odot \widehat{\mathbf{G}}$$
$$\oplus\, \epsilon\lambda\|x-\bar{x}\|, \text{ by using Lemma 1.7}$$
$$\implies \mathbf{0} \preceq \mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x-\bar{x})^\top \odot \widehat{\mathbf{G}} \oplus \epsilon\|x-\bar{x}\|.$$

Therefore, by letting $\epsilon \to 0$, we get

$$\mathbf{0} \preceq \mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x-\bar{x})^\top \odot \widehat{\mathbf{G}} \text{ for all } x \in \mathbb{R}^n$$
$$\implies (x-\bar{x})^\top \odot \widehat{\mathbf{G}} \preceq \mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \text{ for all } x \in \mathbb{R}^n, \text{ by (i) of Lemma 1.8.}$$

Thus, $\widehat{\mathbf{G}}$ is a subgradient of $\mathbf{F}$ at $\bar{x}$ according to Definition 1.29.

Conversely, let $\widehat{\mathbf{G}}$ be a subgradient of $\mathbf{F}$ at $\bar{x}$ according to Definition 1.29. Then,

$$(x-\bar{x})^\top \odot \widehat{\mathbf{G}} \preceq \mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \text{ for all } x \in \mathbb{R}^n$$
$$\implies \mathbf{0} \preceq \mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x-\bar{x})^\top \odot \widehat{\mathbf{G}} \text{ for all } x \in \mathbb{R}^n,$$
$$\text{by (i) of Lemma 1.8}$$
$$\implies \mathbf{0} \preceq \liminf_{x \to \bar{x}} \frac{1}{\|x-\bar{x}\|} \odot \left(\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x-\bar{x})^\top \odot \widehat{\mathbf{G}}\right).$$

That is, $\widehat{\mathbf{G}}$ is a subgradient of $\mathbf{F}$ at $\bar{x}$ according to Definition 3.1.

Remark 3.4. It is to mention that Definition 1.29 is applicable only for convex IVFs. However, Definition 3.1 is applicable to more general IVFs, which may not be convex.

Theorem 3.5. *The set (3.1) of gH-Fréchet subgradients is convex.*

Proof. If $\partial_f \mathbf{F}(\bar{x}) = \emptyset$, then the set $\partial_f \mathbf{F}(\bar{x})$ is vacuously convex. So, let $\partial_f \mathbf{F}(\bar{x}) \neq \emptyset$. Consider $\widehat{\mathbf{G}},\ \widehat{\mathbf{H}} \in \partial_f \mathbf{F}(\bar{x})$. Then,

$$\mathbf{0} \preceq \liminf_{x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \odot \left(\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{G}}\right) \tag{3.4}$$

and

$$\mathbf{0} \preceq \liminf_{x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \odot \left(\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{H}}\right). \tag{3.5}$$

On multiplying (3.4) by λ and (3.5) by μ, where $\lambda,\ \mu \in [0, 1]$ with $\lambda + \mu = 1$, and adding the resultant, we get

$$\begin{aligned} \mathbf{0} \ \preceq\ & \lambda \odot \left(\liminf_{x \to \bar{x}} \tfrac{1}{\|x-\bar{x}\|} \odot (\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{G}})\right) \oplus \\ & \mu \odot \left(\liminf_{x \to \bar{x}} \tfrac{1}{\|x-\bar{x}\|} \odot (\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{H}})\right). \end{aligned}$$

Therefore, by (i) of Theorem 2.10, we get

$$\begin{aligned} \mathbf{0} \preceq \liminf_{x \to \bar{x}} \Big(& \tfrac{1}{\|x-\bar{x}\|} \odot (\lambda \odot (\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{G}})) \oplus \\ & \tfrac{1}{\|x-\bar{x}\|} \odot (\mu \odot (\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{H}}))\Big). \end{aligned} \tag{3.6}$$

Notice that the numerator of the right hand side of (3.6) is equal to

$$\begin{aligned} & \Big(\lambda \odot \mathbf{F}(x) \ominus_{gH} \lambda \odot \mathbf{F}(\bar{x}) \ominus_{gH} \lambda(x - \bar{x})^\top \odot \widehat{\mathbf{G}} \oplus \\ & (1 - \lambda) \odot \mathbf{F}(x) \ominus_{gH} (1 - \lambda) \odot \mathbf{F}(\bar{x}) \ominus_{gH} \mu(x - \bar{x})^\top \odot \widehat{\mathbf{H}}\Big) \\ = \ & \Big(\lambda \odot \mathbf{F}(x) \ominus_{gH} \lambda \odot \mathbf{F}(\bar{x}) \ominus_{gH} \lambda(x - \bar{x})^\top \odot \widehat{\mathbf{G}} \oplus \\ & \mathbf{F}(x) \ominus_{gH} \lambda \odot \mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \oplus \lambda \odot \mathbf{F}(\bar{x}) \ominus_{gH} \mu(x - \bar{x})^\top \odot \widehat{\mathbf{H}}\Big), \\ & \quad \text{by using Lemma 1.7} \\ = \ & \mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot (\lambda \odot \widehat{\mathbf{G}} \oplus \mu \odot \widehat{\mathbf{H}}). \end{aligned}$$

Therefore, by (3.6), we get

$$\mathbf{0} \preceq \liminf_{x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \odot \left(\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot (\lambda \odot \widehat{\mathbf{G}} \oplus \mu \odot \widehat{\mathbf{H}}) \right).$$

This implies

$$\left(\lambda \odot \widehat{\mathbf{G}} \oplus \mu \odot \widehat{\mathbf{H}} \right) \in \partial_f \mathbf{F}(\bar{x}),$$

and hence the set (3.1) is convex. □

Next, in Theorem 3.6, we show that a gH-differentiable IVF has only one gH-Fréchet subgradient, which is the gH-gradient of the IVF.

Theorem 3.6. *Let $\boldsymbol{F} : \mathbb{R}^n \to \overline{\mathbb{R}_I}$ be a proper extended IVF. If $\boldsymbol{F}$ is gH-differentiable at $\bar{x} \in \mathbb{R}^n$, then $\boldsymbol{F}$ is also gH-Fréchet subdifferentiable at $\bar{x}$. Moreover, $\partial_f \boldsymbol{F}(\bar{x}) = \{\nabla \boldsymbol{F}(\bar{x})\}$.*

Proof. Since $\mathbf{F}$ is gH-differentiable at $\bar{x}$, we have

$$\begin{aligned}
&\lim_{x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \odot \left(\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \nabla \mathbf{F}(\bar{x}) \right) = \mathbf{0} \\
\implies &\liminf_{x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \odot \left(\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \nabla \mathbf{F}(\bar{x}) \right) = \mathbf{0}. \quad (3.7)
\end{aligned}$$

Therefore, $\nabla \mathbf{F}(\bar{x}) \in \partial_f \mathbf{F}(\bar{x})$, and hence $\mathbf{F}$ is gH-Fréchet subdifferentiable at $\bar{x}$. Consider $\widehat{\mathbf{G}} \in \partial_f \mathbf{F}(\bar{x})$. Then,

$$\begin{aligned}
&\mathbf{0} \preceq \liminf_{x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \odot \left(\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{G}} \right) \\
\implies &\liminf_{x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \odot \left(\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \nabla \mathbf{F}(\bar{x}) \right) \preceq \\
&\liminf_{x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \odot \left(\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{G}} \right), \text{ by (3.7)} \\
\implies &\mathbf{0} \preceq \left(\liminf_{x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \odot \left(\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{G}} \right) \ominus_{gH} \right.
\end{aligned}$$

$$\liminf_{x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \odot \left(\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \nabla \mathbf{F}(\bar{x})\right)\Bigg),$$

by (i) of Lemma 1.8.

Therefore, by using Remark 2.6 and (ii) of Lemma 1.8, we get

$$\mathbf{0} \preceq \Bigg(\liminf_{x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \odot \left(\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{G}}\right) \oplus$$
$$\limsup_{x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \odot \left(\mathbf{F}(\bar{x}) \ominus_{gH} \mathbf{F}(x) \oplus (x - \bar{x})^\top \odot \nabla \mathbf{F}(\bar{x})\right)\Bigg).$$

This implies

$$\mathbf{0} \preceq \limsup_{x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \odot (\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{G}} \ominus_{gH} \mathbf{F}(x) \oplus$$
$$\mathbf{F}(\bar{x}) \oplus (x - \bar{x})^\top \odot \nabla \mathbf{F}(\bar{x}))$$

$$\implies \quad \mathbf{0} \preceq \limsup_{x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} (x - \bar{x})^\top \odot (\nabla \mathbf{F}(\bar{x}) \ominus_{gH} \widehat{\mathbf{G}})$$

$$\implies \quad \mathbf{0} \preceq \limsup_{\lambda \to 0} \frac{1}{\|\lambda d\|} (\lambda d)^\top \odot (\nabla \mathbf{F}(\bar{x}) \ominus_{gH} \widehat{\mathbf{G}}), \text{ where } x = \bar{x} + \lambda d \text{ for any } d \in \mathbb{R}^n \text{ and } \lambda > 0$$

$$\implies \quad \mathbf{0} \preceq \limsup_{\lambda \to 0} d^\top \odot (\nabla \mathbf{F}(\bar{x}) \ominus_{gH} \widehat{\mathbf{G}}) \text{ for any } d \in \mathbb{R}^n$$

$$\implies \quad \nabla \mathbf{F}(\bar{x}) \ominus_{gH} \widehat{\mathbf{G}} = \mathbf{0}$$

$$\implies \quad \nabla \mathbf{F}(\bar{x}) = \widehat{\mathbf{G}}.$$

Since $\widehat{\mathbf{G}}$ is an arbitrarily chosen element of $\partial_f \mathbf{F}(\bar{x})$, the result follows. □

Note 9. Converse of Theorem 3.6 is not true. For instance, consider the $\mathbf{F}$ as in Example 3.1. We have seen that $\mathbf{F}$ is gH-Fréchet subdifferentiable at 0. Let us now find the following limit:

$$\begin{aligned}&\lim_{h\to 0}\tfrac{1}{h}\odot(\mathbf{F}(0+h)\ominus_{gH}\mathbf{F}(0))\\ =\ &\lim_{h\to 0}\tfrac{1}{h}\odot(\mathbf{F}(h)\ominus_{gH}\mathbf{0})=\lim_{h\to 0}\tfrac{1}{h}\odot[|h|,k|h|],\end{aligned}$$

which does not exist. Therefore, $\mathbf{F}$ is not gH-differentiable at 0.

In Theorems 3.7 and 3.8 below, we show that the scalar multiplication of a gH-Fréchet subdifferentiable IVF with any $\lambda > 0$ and the sum of two Fréchet subdifferentiable IVFs are again gH-Fréchet subdifferentiable.

Theorem 3.7. *Let $\boldsymbol{F}:\mathbb{R}^n\to\overline{\mathbb{R}_I}$ be gH-Fréchet subdifferentiable at $\bar{x}\in\mathbb{R}^n$. Then,*

$$\partial_f(\lambda\odot\boldsymbol{F})(\bar{x})=\lambda\odot\partial_f\boldsymbol{F}(\bar{x})\ \textit{for any}\ \lambda>0.$$

Proof. Proof follows directly from Definitions 2.3 and 3.1. □

Theorem 3.8. *Let $\boldsymbol{F}_1$, $\boldsymbol{F}_2:\mathbb{R}^n\to\overline{\mathbb{R}_I}$ be gH-Fréchet subdifferentiable IVFs at $\bar{x}\in\mathbb{R}^n$. Then, $\boldsymbol{F}_1\oplus\boldsymbol{F}_2$ is gH-Fréchet subdifferentiable at $\bar{x}\in\mathbb{R}^n$, and*

$$\partial_f\boldsymbol{F}_1(\bar{x})\oplus\partial_f\boldsymbol{F}_2(\bar{x})\subseteq\partial_f(\boldsymbol{F}_1\oplus\boldsymbol{F}_2)(\bar{x}).$$

Proof. Let $\widehat{\mathbf{G}}\in(\partial_f\mathbf{F}_1(\bar{x})\oplus\partial_f\mathbf{F}_2(\bar{x}))$. Then, $\widehat{\mathbf{G}}=\widehat{\mathbf{H}}\oplus\widehat{\mathbf{K}}$ for some $\widehat{\mathbf{H}}\in\partial_f\mathbf{F}_1(\bar{x})$ and $\widehat{\mathbf{K}}\in\partial_f\mathbf{F}_2(\bar{x})$. Therefore,

$$\mathbf{0}\preceq\liminf_{x\to\bar{x}}\frac{1}{\|x-\bar{x}\|}\odot(\mathbf{F}_1(x)\ominus_{gH}\mathbf{F}_1(\bar{x})\ominus_{gH}(x-\bar{x})^\top\odot\widehat{\mathbf{H}})\tag{3.8}$$

and

$$\mathbf{0}\preceq\liminf_{x\to\bar{x}}\frac{1}{\|x-\bar{x}\|}\odot(\mathbf{F}_2(x)\ominus_{gH}\mathbf{F}_2(\bar{x})\ominus_{gH}(x-\bar{x})^\top\odot\widehat{\mathbf{K}}).\tag{3.9}$$

By adding (3.8) and (3.9), we get

$$\begin{aligned}\mathbf{0} \ &\preceq\ \liminf_{x\to\bar{x}} \frac{1}{\|x-\bar{x}\|} \odot (\mathbf{F}_1(x) \ominus_{gH} \mathbf{F}_1(\bar{x}) \ominus_{gH} (x-\bar{x})^\top \odot \widehat{\mathbf{H}}) \oplus \\ &\quad\ \liminf_{x\to\bar{x}} \frac{1}{\|x-\bar{x}\|} \odot (\mathbf{F}_2(x) \ominus_{gH} \mathbf{F}_2(\bar{x}) \ominus_{gH} (x-\bar{x})^\top \odot \widehat{\mathbf{K}}) \\ &\preceq\ \liminf_{x\to\bar{x}} \frac{1}{\|x-\bar{x}\|} \odot ((\mathbf{F}_1 \oplus \mathbf{F}_2)(x) \ominus_{gH} (\mathbf{F}_1 \oplus \mathbf{F}_2)(\bar{x}) \ominus_{gH} (x-\bar{x})^\top \odot (\widehat{\mathbf{H}} \oplus \widehat{\mathbf{K}})), \\ &\quad\ \text{by (i) of Theorem 2.10.}\end{aligned}$$

This implies

$$\widehat{\mathbf{H}} \oplus \widehat{\mathbf{K}} = \widehat{\mathbf{G}} \in \partial_f(\mathbf{F}_1 \oplus \mathbf{F}_2)(\bar{x}),$$

and hence $\partial_f \mathbf{F}_1(\bar{x}) \oplus \partial_f \mathbf{F}_2(\bar{x}) \subseteq \partial_f(\mathbf{F}_1 \oplus \mathbf{F}_2)(\bar{x})$. □

In the next theorem (Theorem 3.9), we show that every gH-Fréchet subgradient $\widehat{\mathbf{G}}$ of an arbitrary IVF $\mathbf{F} : \mathbb{R}^n \to \overline{\mathbb{R}_I}$ at $\bar{x} \in \mathbb{R}^n$ can be equivalently described via the gH-derivative of another IVF $\mathbf{H}$ such that the difference $\mathbf{F} \ominus_{gH} \mathbf{H}$ attains its local minimum at $\bar{x}$. This property of gH-Fréchet subgradients of $\mathbf{F}$ is used to prove the difference rule for gH-Fréchet subgradients (Theorem 3.11).

Theorem 3.9. *Let $\boldsymbol{F} : \mathbb{R}^n \to \overline{\mathbb{R}_I}$ be a proper extended IVF and $\bar{x} \in \mathbb{R}^n$. Then, $\widehat{\boldsymbol{G}} \in \partial_f \boldsymbol{F}(\bar{x})$ if and only if there exists a function $\boldsymbol{H} : X \to \mathbb{R}_I$ such that*

(i) *$\boldsymbol{H}(x) \preceq \boldsymbol{F}(x)$ for any $x \in \mathbb{R}^n$, $\boldsymbol{H}(\bar{x}) = \boldsymbol{F}(\bar{x})$, and*

(ii) *$\boldsymbol{H}$ is gH-differentiable at $\bar{x}$ with $\boldsymbol{H}'(\bar{x}) = \widehat{\boldsymbol{G}}$.*

Proof. Let us first prove the sufficient part. Since $\mathbf{H}$ is gH-differentiable at $\bar{x}$ and $\mathbf{H}'(\bar{x}) = \widehat{\mathbf{G}}$, we have

$$\begin{aligned}\mathbf{0} &= \lim_{x\to\bar{x}} \frac{1}{\|x-\bar{x}\|} \odot (\mathbf{H}(x) \ominus_{gH} \mathbf{H}(\bar{x}) \ominus_{gH} (x-\bar{x})^\top \odot \widehat{\mathbf{G}}) \\ &= \liminf_{x\to\bar{x}} \frac{1}{\|x-\bar{x}\|} \odot (\mathbf{H}(x) \ominus_{gH} \mathbf{H}(\bar{x}) \ominus_{gH} (x-\bar{x})^\top \odot \widehat{\mathbf{G}}).\end{aligned} \tag{3.10}$$

Notice the following two points.

(a) By (i), we have

$$\mathbf{H}(x) \preceq \mathbf{F}(x) \text{ for any } x \in \mathbb{R}^n,\ \mathbf{H}(\bar{x}) = \mathbf{F}(\bar{x}),\ \text{and}$$

(b) for all $\mathbf{A},\ \mathbf{B},\ \mathbf{C},\ \mathbf{D} \in \mathbb{R}_I$, we have

$$\mathbf{A} \ominus_{gH} \mathbf{B} \ominus_{gH} \mathbf{C} \preceq \mathbf{D} \ominus_{gH} \mathbf{B} \ominus_{gH} \mathbf{C} \text{ whenever } \mathbf{A} \preceq \mathbf{D}.$$

Therefore, (3.10) gives

$$\mathbf{0} \preceq \liminf_{x\to\bar{x}} \frac{1}{\|x-\bar{x}\|} \odot (\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x-\bar{x})^\top \odot \widehat{\mathbf{G}}),$$

and hence $\widehat{\mathbf{G}} \in \partial_f \mathbf{F}(\bar{x})$.

To prove the necessary part, consider $\mathbf{H}(x) = \inf\{\mathbf{F}(x), \mathbf{F}(\bar{x}) \oplus (x-\bar{x})^\top \odot \widehat{\mathbf{G}}\}$ for all $x \in \mathbb{R}^n$.

Clearly, $\mathbf{H}(x) \preceq \mathbf{F}(x)$ for any $x \in \mathbb{R}^n$ and $\mathbf{H}(\bar{x}) = \mathbf{F}(\bar{x})$. Next, to see that $\mathbf{H}$ is gH-differentiable at $\bar{x}$ and $\mathbf{H}'(\bar{x}) = \widehat{\mathbf{G}}$, we evaluate the following limit:

$$\lim_{x\to\bar{x}} \frac{1}{\|x-\bar{x}\|} \odot (\mathbf{H}(x) \ominus_{gH} \mathbf{H}(\bar{x}) \ominus_{gH} (x-\bar{x})^\top \odot \widehat{\mathbf{G}}).$$

- Case 1. If $\mathbf{H}(x) = \mathbf{F}(\bar{x}) \oplus (x - \bar{x})^\top \odot \widehat{\mathbf{G}}$. Then,

$$\begin{aligned}
& \lim_{x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \odot (\mathbf{H}(x) \ominus_{gH} \mathbf{H}(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{G}}) \\
= & \lim_{x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \odot (\mathbf{F}(\bar{x}) \oplus \mathbf{G}(x - \bar{x}) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{G}}) \\
= & \ \mathbf{0}.
\end{aligned}$$

- Case 2. If $\mathbf{H}(x) = \mathbf{F}(x)$.

  Then, since $\widehat{\mathbf{G}} \in \partial_f \mathbf{F}(\bar{x})$, we have

$$\begin{aligned}
\mathbf{0} \ \preceq & \ \liminf_{x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \odot (\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{G}}) \\
= & \ \liminf_{x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \odot (\mathbf{H}(x) \ominus_{gH} \mathbf{H}(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{G}}). \qquad (3.11)
\end{aligned}$$

  Observe that

$$\begin{aligned}
& \liminf_{x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \odot (\mathbf{H}(x) \ominus_{gH} \mathbf{H}(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{G}}) \\
= & \liminf_{x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \odot (\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{G}}) \\
\preceq & \lim_{x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \odot (\mathbf{F}(\bar{x}) \oplus \mathbf{G}(x - \bar{x}) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{G}}) \\
& \text{ because } \mathbf{F}(x) \preceq \mathbf{F}(\bar{x}) \oplus (x - \bar{x})^\top \odot \widehat{\mathbf{G}} \\
= & \ \mathbf{0}.
\end{aligned}$$

  Thus,

$$\liminf_{x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \odot (\mathbf{H}(x) \ominus_{gH} \mathbf{H}(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{G}}) \preceq \mathbf{0}. \qquad (3.12)$$

Therefore, by (3.11) and (3.12), we get

$$\liminf_{x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \odot (\mathbf{H}(x) \ominus_{gH} \mathbf{H}(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{G}}) = \mathbf{0}. \tag{3.13}$$

It is clear from (3.13) that

$$\mathbf{0} \preceq \limsup_{x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \odot (\mathbf{H}(x) \ominus_{gH} \mathbf{H}(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{G}}). \tag{3.14}$$

Again, since

$$\begin{aligned}
& \limsup_{x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \odot (\mathbf{H}(x) \ominus_{gH} \mathbf{H}(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{G}}) \\
= \; & \limsup_{x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \odot (\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{G}}) \\
\preceq \; & \lim_{x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \odot (\mathbf{F}(\bar{x}) \oplus \mathbf{G}(x - \bar{x}) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{G}}) \\
= \; & \mathbf{0},
\end{aligned}$$

we get

$$\limsup_{x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \odot (\mathbf{H}(x) \ominus_{gH} \mathbf{H}(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{G}}) \preceq \mathbf{0}. \tag{3.15}$$

Therefore, by (3.14) and (3.15), we get

$$\limsup_{x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \odot (\mathbf{H}(x) \ominus_{gH} \mathbf{H}(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{G}}) = \mathbf{0}. \tag{3.16}$$

Hence, from (3.13) and (3.16), we have

$$\lim_{x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \odot (\mathbf{H}(x) \ominus_{gH} \mathbf{H}(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{G}} = \mathbf{0}.$$

That is, $\mathbf{H}'(\bar{x}) = \widehat{\mathbf{G}}$. Hence, from Case 1 and Case 2, $\mathbf{H}'(\bar{x}) = \widehat{\mathbf{G}}$, which completes the proof. □

Apart from Theorem 3.9, we also need the following lemma to prove Theorem 3.11.

Lemma 3.10. *Let $\boldsymbol{F}_1,\ \boldsymbol{F}_2 : \mathbb{R}^n \to \overline{\mathbb{R}_I}$ be two proper extended IVFs, which are finite at $\bar{x} \in \mathbb{R}^n$. Further, let $\boldsymbol{F}_1 \ominus_{gH} \boldsymbol{F}_2$ and $\boldsymbol{F}_2$ be gH-Fréchet subdifferentiable at $\bar{x}$. Then, $\boldsymbol{F}_1$ is gH-Fréchet subdifferentiable at $\bar{x}$, and*

$$\partial_f(\boldsymbol{F}_1 \ominus_{gH} \boldsymbol{F}_2)(\bar{x}) \oplus \partial_f \boldsymbol{F}_2(\bar{x}) \subseteq \partial_f \boldsymbol{F}_1(\bar{x}).$$

Proof. We are given that $\mathbf{F}_1 \ominus_{gH} \mathbf{F}_2$ and $\mathbf{F}_2$ are gH-Fréchet subdifferentiable at $\bar{x}$. So, by Theorem 3.8, their sum is gH-Fréchet subdifferentiable at $\bar{x}$. That is, $\mathbf{F}_1$ is gH-Fréchet subdifferentiable at $\bar{x}$. Also, by applying Theorem 3.8 to $\mathbf{F}_1 \ominus_{gH} \mathbf{F}_2$ and $\mathbf{F}_2$, we get

$$\partial_f(\mathbf{F}_1 \ominus_{gH} \mathbf{F}_2)(\bar{x}) \oplus \partial \mathbf{F}_2(\bar{x}) \subseteq \partial_f(\mathbf{F}_1 \ominus_{gH} \mathbf{F}_2 \oplus \mathbf{F}_2)(\bar{x}) = \partial_f \mathbf{F}_1(\bar{x}).$$

□

Theorem 3.11. (Difference rule for gH-Fréchet subgradients). *Let $\boldsymbol{F}_1,\ \boldsymbol{F}_2 : \mathbb{R}^n \to \overline{\mathbb{R}_I}$ be two proper extended IVFs, finite at $\bar{x} \in \mathbb{R}^n$. Assume that $\partial_f \boldsymbol{F}_2(\bar{x}) \neq \emptyset$. Then,*

$$\partial_f(\boldsymbol{F}_1 \ominus_{gH} \boldsymbol{F}_2)(\bar{x}) \subseteq \bigcap_{\widehat{\boldsymbol{G}} \in \partial_f \boldsymbol{F}_2(\bar{x})} (\partial_f \boldsymbol{F}_1(\bar{x}) \ominus_{gH} \widehat{\boldsymbol{G}}) \subseteq \partial_f \boldsymbol{F}_1(\bar{x}) \ominus_{gH} \partial_f \boldsymbol{F}_2(\bar{x}). \tag{3.17}$$

Proof. To prove (3.17), fix any $\widehat{\mathbf{H}} \in \partial_f(\mathbf{F}_1 \ominus_{gH} \mathbf{F}_2)(\bar{x})$ and $\widehat{\mathbf{K}} \in \partial_f \mathbf{F}_2(\bar{x})$. By applying Theorem 3.9, for gH-Fréchet subgradient $\widehat{\mathbf{K}} \in \partial_f \mathbf{F}_2(\bar{x})$, we get an IVF $\mathbf{H}$ such that

$$\mathbf{H}(x) \preceq \mathbf{F}_2(x) \text{ for any } x \in \mathbb{R}^n,\ \mathbf{H}(\bar{x}) = \mathbf{F}_2(\bar{x}) \text{ and } \mathbf{H}'(\bar{x}) = \widehat{\mathbf{K}}. \tag{3.18}$$

Since $\widehat{\mathbf{H}} \in \partial_f(\mathbf{F}_1 \ominus_{gH} \mathbf{F}_2)(\bar{x})$, we get

$$\mathbf{0} \preceq \liminf_{x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \odot ((\mathbf{F}_1 \ominus_{gH} \mathbf{F}_2)(x) \ominus_{gH} (\mathbf{F}_1 \ominus_{gH} \mathbf{F}_2)(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{H}}).$$

Therefore, for any $\epsilon > 0$, we get a $\delta > 0$ such that whenever $0 < \|x - \bar{x}\| < \delta$, we have

$$\begin{aligned}
& \mathbf{0} \preceq \mathbf{F}_1(x) \ominus_{gH} \mathbf{F}_2(x) \ominus_{gH} (\mathbf{F}_1(\bar{x}) \ominus_{gH} \mathbf{F}_2(\bar{x})) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{H}} \oplus \epsilon\|x - \bar{x}\| \\
\implies & (x - \bar{x})^\top \odot \widehat{\mathbf{H}} \preceq \mathbf{F}_1(x) \ominus_{gH} \mathbf{F}_2(x) \ominus_{gH} (\mathbf{F}_1(\bar{x}) \ominus_{gH} \mathbf{F}_2(\bar{x})) \oplus \epsilon\|x - \bar{x}\|, \\
& \text{ by (i) of Lemma 1.8} \\
\implies & (x - \bar{x})^\top \odot \widehat{\mathbf{H}} \preceq \mathbf{F}_1(x) \ominus_{gH} \mathbf{H}(x) \ominus_{gH} (\mathbf{F}_1(\bar{x}) \ominus_{gH} \mathbf{H}(\bar{x})) \oplus \epsilon\|x - \bar{x}\|, \\
& \text{because } \mathbf{H}(x) \preceq \mathbf{F}_2(x) \text{ for any } x \in X \text{ and } \mathbf{H}(\bar{x}) = \mathbf{F}_2(\bar{x}).
\end{aligned}$$

Thus,

$$\widehat{\mathbf{H}} \in \partial_f(\mathbf{F}_1 \ominus_{gH} \mathbf{H})(\bar{x}).$$

Also, by Lemma 3.10, we get $\partial_f(\mathbf{F}_1 \ominus_{gH} \mathbf{H})(\bar{x}) \subseteq \partial_f \mathbf{F}_1(\bar{x}) \ominus_{gH} \partial_f \mathbf{H}(\bar{x})$.
Hence, by using (3.18), we get

$$\widehat{\mathbf{H}} \in \partial_f \mathbf{F}_1(\bar{x}) \ominus_{gH} \widehat{\mathbf{K}},$$

which proves the difference rule (3.17). □

To conclude this section, we derive a rule for calculating gH-Fréchet subgradients of the minimum function,

$$\begin{aligned}
(\wedge \mathbf{F}_i)(x) \;:=\; & \inf\{\mathbf{F}_i | i = 1, 2, \ldots, n\}, \text{ where } \mathbf{F}_i : \mathbb{R}^n \to \overline{\mathbb{R}_I} \text{ is a proper IVF for} \\
& \text{each } i \text{ and } n \geq 2.
\end{aligned}$$

Denote

$$I(x) := \{j \in \{1, 2, \ldots, n\} | \mathbf{F}_j(x) = (\wedge \mathbf{F}_i)(x)\}.$$

Theorem 3.12. *The following inclusion holds:*

$$\partial_f (\wedge \boldsymbol{F}_i)(\bar{x}) \subseteq \bigcap_{j \in I(\bar{x})} \partial_f \boldsymbol{F}_j(\bar{x}).$$

Proof. Take $\widehat{\mathbf{G}} \in \partial_f(\wedge \mathbf{F}_i)(\bar{x})$. Then, by Definition 3.1, we have

$$\mathbf{0} \preceq \liminf_{x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \odot ((\wedge \mathbf{F}_i)(x) \ominus_{gH} (\wedge \mathbf{F}_i)(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{G}}).$$

That is, for any $\epsilon > 0$, we get a $\delta > 0$ such that whenever $0 < \|x - \bar{x}\| < \delta$, we have

$$\mathbf{0} \preceq (\wedge \mathbf{F}_i)(x) \ominus_{gH} (\wedge \mathbf{F}_i)(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{G}} \oplus \epsilon \|x - \bar{x}\|.$$

Therefore, for x such that $0 < \|x - \bar{x}\| < \delta$ and for any $j \in I(\bar{x})$, we have

$$\begin{aligned}
(x - \bar{x})^\top \odot \widehat{\mathbf{G}} &\preceq (\wedge \mathbf{F}_i)(x) \ominus_{gH} (\wedge \mathbf{F}_i)(\bar{x}) \oplus \epsilon \|x - \bar{x}\|, \text{ by (i) of Lemma 1.8} \\
&= (\wedge \mathbf{F}_i)(x) \ominus_{gH} (\mathbf{F}_j)(\bar{x}) \oplus \epsilon \|x - \bar{x}\| \\
&\preceq \mathbf{F}_j(x) \ominus_{gH} \mathbf{F}_j(\bar{x}) \oplus \epsilon \|x - \bar{x}\|,
\end{aligned}$$

which proves that $\widehat{\mathbf{G}} \in \partial_f \mathbf{F}_j(\bar{x})$, and hence the proof is complete. □

3.3 Necessary optimality conditions for IOPs with nondifferentiable IVFs

With the help of the studied concepts in Section 3.2, we now proceed to identify optimality conditions for the following unconstrained IOP:

$$\min_{x \in \mathbb{R}^n} \mathbf{F}(x), \tag{3.19}$$

where $\mathbf{F} : \mathbb{R}^n \to \overline{\mathbb{R}_I}$ is a proper extended IVF. By an optimum solution of (3.19), we refer to the following concept. A point $\bar{x} \in \mathbb{R}^n$ is called a *weak efficient solution* of (3.19) if $\mathbf{F}(\bar{x}) \preceq \mathbf{F}(x)$ for all $x \in \mathbb{R}^n$ (see [71]).

Theorem 3.13. *If $\bar{x}$ is a weak efficient solution of (3.19), then $\widehat{\mathbf{0}} \in \partial_f \boldsymbol{F}(\bar{x})$.*

Proof. Since $\bar{x}$ is a weak efficient solution of (3.19),

$$\begin{aligned}
&\mathbf{F}(\bar{x}) \preceq \mathbf{F}(x) \text{ for all } x \in \mathbb{R}^n \\
\implies &\mathbf{0} \preceq \mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \text{ for all } x \in \mathbb{R}^n, \text{ by (i) of Lemma 1.8} \\
\implies &\mathbf{0} \preceq \frac{1}{\|x - \bar{x}\|} \odot \left(\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{0}}\right) \text{ for all } x \in \mathbb{R}^n \\
\implies &\mathbf{0} \preceq \liminf_{x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \odot \left(\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{0}}\right) \\
\implies &\widehat{\mathbf{0}} \in \partial_f \mathbf{F}(\bar{x}).
\end{aligned}$$

□

We next consider the following example to verify Theorem 3.13.

Example 3.2. *Consider the following IOP:*

$$\min_{x\in\mathbb{R}} \boldsymbol{F}(x) = \begin{cases} [-(x+1), x^2-1], & x \leq -1 \\ [0, 1-x^2|], & -1 \leq x \leq 1 \\ [x-1, x^2-1], & x \geq 1. \end{cases} \tag{3.20}$$

The graph of the IVF $\boldsymbol{F}$ is illustrated in Figure 3.2 by the grey shaded region. From

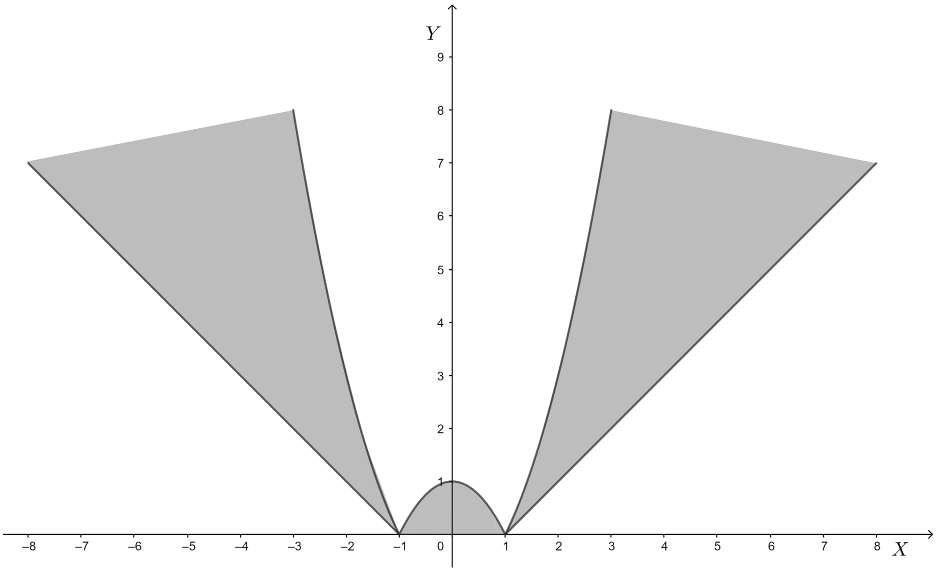

FIGURE 3.2: Objective function of the IOP (3.20) of Example 3.2

Figure 3.2, it is clear that $\boldsymbol{F}$ is not convex. Also, observe that $\boldsymbol{F}(-1) = \boldsymbol{F}(1) \preceq \boldsymbol{F}(x)$ for all $x \in \mathbb{R}$. Hence, -1 and 1 are two weak efficient solutions of (3.20).

At $\bar{x} = -1$,

$$\begin{aligned} (\boldsymbol{F}'(x))_+ : &= \lim_{d\to 0^+} \frac{1}{d} \odot (\boldsymbol{F}(\bar{x}+d) \ominus_{gH} \boldsymbol{F}(\bar{x})) \\ &= \lim_{d\to 0^+} \frac{1}{d} \odot (\boldsymbol{F}(d-1) \ominus_{gH} \boldsymbol{F}(-1)) \\ &= \lim_{d\to 0^+} \frac{1}{d} \odot [0, 1-(d-1)^2] \\ &= \lim_{d\to 0^+} \frac{1}{d} \odot [0, -d^2+2d] \\ &= [0, 2]. \end{aligned}$$

and

$$\begin{aligned}(\boldsymbol{F}'(\bar{x}))_- : &= \lim_{d\to 0^-} \frac{1}{d} \odot (\boldsymbol{F}(\bar{x}+d) \ominus_{gH} \boldsymbol{F}(\bar{x})) \\ &= \lim_{d\to 0^-} \frac{1}{d} \odot (\boldsymbol{F}(d-1) \ominus_{gH} \boldsymbol{F}(-1)) \\ &= \lim_{d\to 0^-} \frac{1}{d} \odot [-d, d^2-2d] \\ &= [-2,-1] \neq (\boldsymbol{F}'(x))_+.\end{aligned}$$

Therefore, at the point $x = -1$, $\boldsymbol{F}$ is not gH-differentiable. Similarly, it can be proved that $\boldsymbol{F}$ is not gH-differentiable at the point $x = 1$. Note that

$$\begin{aligned}&\liminf_{x\to -1} \frac{1}{|x+1|} \odot (\boldsymbol{F}(x) \ominus_{gH} \boldsymbol{F}(-1) \ominus_{gH} (x+1)^\top \odot \boldsymbol{0}) \\ = \ &\liminf_{x\to -1} \frac{1}{|x+1|} \odot \boldsymbol{F}(x) \\ = \ &\liminf_{x\to -1} \frac{1}{|x+1|} \odot [\underline{f}(x), \overline{f}(x)], \ \textit{where} \\ &\underline{f}(x) = \begin{cases} -(x+1), & x \leq -1 \\ 0, & -1 \leq x \leq 1 \\ x-1, & x \geq 1 \end{cases} \ \textit{and} \ \overline{f}(x) = \begin{cases} x^2-1, & x \leq -1 \\ 1-x^2, & -1 \leq x \leq 1 \\ x^2-1, & x \geq 1. \end{cases}\end{aligned}$$

It can be easily seen that

$$\liminf_{x\to -1} \frac{1}{|x+1|} \underline{f}(x) = 0 \ \textit{and} \ \liminf_{x\to -1} \frac{1}{|x+1|} \overline{f}(x) = 2.$$

Therefore,

$$\liminf_{x\to -1} \frac{1}{|x+1|} \odot \boldsymbol{F}(x) = [0,2].$$

Clearly,

$$\boldsymbol{0} \preceq [0,2] \ = \ \liminf_{x\to -1} \frac{1}{|x+1|} \odot \boldsymbol{F}(x)$$

$$= \liminf_{x \to -1} \frac{1}{|x+1|} \odot (\boldsymbol{F}(x) \ominus_{gH} \boldsymbol{F}(-1) \ominus_{gH} (x+1)^\top \odot \boldsymbol{0}).$$

Thus, by Definition 3.1, $\boldsymbol{0} \in \partial_f \boldsymbol{F}(-1)$, which verifies Theorem 3.13 for the weak efficient solution $x = -1$. Similarly, Theorem 3.13 can be verified for the weak efficient solution $x = 1$.

Remark 3.14. One may think that the optimality condition given in Theorem 3.13 is useful only to solve unconstrained IOPs. However, this is not the case. The reason is as follows. Similarly as in the conventional optimization theory, the need to study extended IVFs arises when we seek to convert a constrained IOP into an unconstrained IOP. For instance, consider the following IOP.

$$\min_{x \in X} \mathbf{F}(x), \tag{3.21}$$

where $\mathbf{F} : X \to \mathbb{R}_I$ is an IVF. Then (3.21) can be restated as

$$\min_{x \in \mathbb{R}^n} \mathbf{F}_0(x),$$

where

$$\mathbf{F}_0(x) = \begin{cases} \mathbf{F}(x), & x \in X \\ [+\infty, +\infty], & \text{otherwise.} \end{cases}$$

Most rules with infinity are intuitively clear except possibly $0 \times (+\infty)$ and $\infty - \infty$. Throughout the article, we are dealing with minimization problems, we follow the following convention adopted in [89].

$$0 \times (+\infty) = (+\infty) \times 0 = 0 \text{ and } \infty - \infty = \infty.$$

However, we would like to ascertain that we really need not get worried about $\infty - \infty$

as the IVFs considered in this chapter are proper IVFs. Thus, every constrained IOP can be converted into unconstrained IOP with the help of extended IVFs. Theorem 3.13 can be useful to solve constrained IOPs as well.

Note 10. The converse of Theorem 3.13 is not true. For example, consider $\mathbf{F} : \mathbb{R} \to \mathbb{R}_I$ as $\mathbf{F}(x) = [1, 2] \odot x^3$. Then,

$$\liminf_{x \to 0} \frac{1}{|x - 0|} \odot (\mathbf{F}(x) \ominus_{gH} \mathbf{F}(0) \ominus_{gH} (x - 0) \odot \mathbf{0}) = \liminf_{x \to 0} \frac{1}{|x|} \odot \left([1, 2] \odot x^3\right) = \mathbf{0}.$$

Therefore, $\mathbf{0} \in \partial_f \mathbf{F}(0)$. However, 0 is not a weak efficient solution of $\mathbf{F}$ as $\mathbf{F}(-1) = [-2, -1] \prec \mathbf{F}(0)$.

In the next theorem (Theorem 3.15), we provide a necessary condition for $\bar{x} \in \mathbb{R}^n$ to be a weak efficient solution of an unconstrained IOP whose objective function is given as difference of two IVFs.

Theorem 3.15. (Necessary optimality condition for minimizing difference IVFs). *Let $\bar{x} \in \mathbb{R}^n$ be a weak efficient solution of the difference IVF $\boldsymbol{F} = \boldsymbol{F}_1 \ominus_{gH} \boldsymbol{F}_2$, where both $\boldsymbol{F}_1$, $\boldsymbol{F}_2 : \mathbb{R}^n \to \overline{\mathbb{R}_I}$ are proper extended IVFs, finite at $\bar{x}$. Then, the following inclusion holds*

$$\partial_f \boldsymbol{F}_2(\bar{x}) \subseteq \partial_f \boldsymbol{F}_1(\bar{x}).$$

Proof. Since $\bar{x}$ is a weak efficient solution of $\mathbf{F}$, by Theorem 3.13, $\widehat{\mathbf{0}} \in \partial_f \mathbf{F}(\bar{x})$. Therefore,

$$\mathbf{0} \preceq \liminf_{x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \odot \left((\mathbf{F}_1 \ominus_{gH} \mathbf{F}_2)(x) \ominus_{gH} (\mathbf{F}_1 \ominus_{gH} \mathbf{F}_2)(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{0}}\right),$$

$$\text{i.e.,} \quad \mathbf{0} \preceq \liminf_{x \to \bar{x}} \frac{\mathbf{F}_1(x) \ominus_{gH} \mathbf{F}_2(x) \ominus_{gH} (\mathbf{F}_1(\bar{x}) \ominus_{gH} \mathbf{F}_2(\bar{x}))}{\|x - \bar{x}\|}.$$

Thus, for each $\epsilon > 0$, we get a $\delta_1 > 0$ such that whenever $0 < \|x - \bar{x}\| < \delta_1$, we have

$$\mathbf{0} \preceq \mathbf{F}_1(x) \ominus_{gH} \mathbf{F}_2(x) \ominus_{gH} (\mathbf{F}_1(\bar{x}) \ominus_{gH} \mathbf{F}_2(\bar{x})) \oplus \epsilon\|x - \bar{x}\|.$$

This implies

$$\mathbf{F}_2(x) \ominus_{gH} \mathbf{F}_2(\bar{x}) \preceq \mathbf{F}_1(x) \ominus_{gH} \mathbf{F}_1(\bar{x}) \oplus \epsilon\|x - \bar{x}\|. \tag{3.22}$$

Next to prove that $\partial_f \mathbf{F}_2(\bar{x}) \subseteq \partial_f \mathbf{F}_1(\bar{x})$, consider $\widehat{\mathbf{G}} \in \partial_f \mathbf{F}_2(\bar{x})$. Then,

$$\mathbf{0} \preceq \liminf_{x \to \bar{x}} \frac{\mathbf{F}_2(x) \ominus_{gH} \mathbf{F}_2(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{G}}}{\|x - \bar{x}\|}.$$

Thus, for each $\epsilon > 0$, we get a $\delta_2 > 0$ such that whenever $0 < \|x - \bar{x}\| < \delta_2$, we have

$$\mathbf{0} \preceq \mathbf{F}_2(x) \ominus_{gH} \mathbf{F}_2(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{G}} \oplus \epsilon\|x - \bar{x}\|.$$

Therefore, by using (i) of Lemma 1.8, we get

$$(x - \bar{x})^\top \odot \widehat{\mathbf{G}} \preceq \mathbf{F}_2(x) \ominus_{gH} \mathbf{F}_2(\bar{x}) \oplus \epsilon\|x - \bar{x}\|. \tag{3.23}$$

By taking $\delta = \min\{\delta_1,\ \delta_2\}$ and using (3.22) and (3.23), we get

$$(x - \bar{x})^\top \odot \widehat{\mathbf{G}} \preceq \mathbf{F}_1(x) \ominus_{gH} \mathbf{F}_1(\bar{x}) \oplus \epsilon\|x - \bar{x}\| \text{ whenever } 0 < \|x - \bar{x}\| < \delta.$$

This implies $\widehat{\mathbf{G}} \in \partial_f \mathbf{F}_1(\bar{x})$. Hence, $\partial_f \mathbf{F}_2(\bar{x}) \subseteq \partial_f \mathbf{F}_1(\bar{x})$. □

It is well known that the notion of conventional WSM introduced in Burke and Ferris [23], plays an important role in the sensitivity analysis and convergence analysis of conventional optimization problems. Recently, Krishan et al. [71] extended the notion of WSM for IVFs and showed its applications to solve linear and quadratic

IOPs. Adding to the literature of WSM for IVFs, in Corollary 3.16, we provide a necessary condition for a subset S of $\mathbb{R}^n$ to be a set of WSM of an extended IVF $\mathbf{F}$.

Corollary 3.16. (Necessary condition for unconstrained weak sharp minima). *Let S be the set of WSM for the function $\boldsymbol{F} : \mathbb{R}^n \to \overline{\mathbb{R}_I}$ relative to the whole space $\mathbb{R}^n$ with modulus α. Then, for every $\bar{x} \in S$, we have*

$$\alpha \mathbb{B} \cap \widehat{N}(\bar{x}, S) \subseteq \partial_f \boldsymbol{F}(\bar{x}),$$

where $\mathbb{B} \subseteq \mathbb{R}^n$ stands for the closed unit ball and $\widehat{N}(\bar{x}, S)$ denotes the Fréchet normal cone to S at $\bar{x}$.

Proof. By definition of WSM, we have

$$\mathbf{F}(y) \oplus \alpha \operatorname{dist}(x, S) \preceq \mathbf{F}(x) \text{ for all } x \in \mathbb{R}^n \text{ and } y \in S.$$

Thus, every $y \in S$ is a weak efficient solution to the unconstrained problem of minimizing the difference function $\mathbf{G}(x) := \mathbf{F}(x) \ominus_{gH} \alpha \operatorname{dist}(x, S)$. Therefore, by Theorem 3.15, we get

$$\alpha \partial_f \operatorname{dist}(\bar{x}, S) \subseteq \partial_f \mathbf{F}(\bar{x}). \tag{3.24}$$

By Note 2, we have

$$\partial_f \operatorname{dist}(\bar{x}, S) = \widehat{N}(\bar{x}, S) \cap \mathbb{B}.$$

Thus, by (3.24), we get $\alpha \mathbb{B} \cap \widehat{N}(\bar{x}, S) \subseteq \partial_f \mathbf{F}(\bar{x})$. □

Example 3.3. *In this Example, we verify corollary 3.16 for the IVF*

$$\boldsymbol{F}(x) = \begin{cases} [-(x+2), -(x-3)], & x < -2 \\ [0, 5], & x \in [-2, 2] \\ [x-2, x+3], & x > 2. \end{cases}$$

In Fig. 3.3, we have depicted the graph of $\boldsymbol{F}(x)$ by light grey-shaded region. The

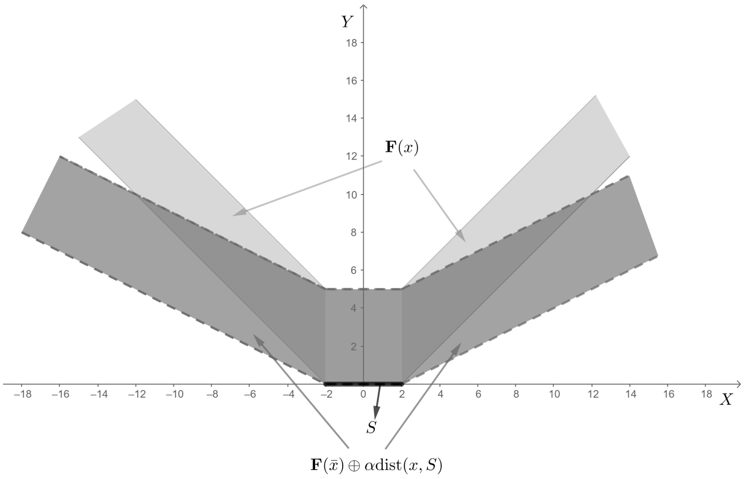

Figure 3.3: Objective function **F** of Example 3.3 and the location of the set of WSM of **F**

dark grey-shaded region with dashed lines show the graph of $\boldsymbol{F}(\bar{x}) \oplus \alpha\,\mathrm{dist}(x, S)$. From the graphs, notice that for any $x \in \mathbb{R}$,

$$\boldsymbol{F}(\bar{x}) \oplus \alpha\,\mathrm{dist}(x, S) \preceq \boldsymbol{F}(x) \text{ for all } \bar{x} \in S \text{ and } \alpha = \frac{1}{2}.$$

Hence, by Definition 1.30 of WSM, $S = [-2, 2]$ *is the set of WSM of given* $\boldsymbol{F}$ *relative to the whole space* $\mathbb{R}$ *with modulus* $\alpha = \frac{1}{2}$. *Note that*

$$\widehat{N}(\bar{x}, S) = \begin{cases} 0, & \text{if } x \in (-2, 2) \\ [0, \infty), & \text{if } x = 2 \\ (-\infty, 0], & \text{if } x = -2. \end{cases}$$

Therefore, for all $\bar{x} \in (-2, 2), \alpha\mathbb{B} \cap \widehat{N}(\bar{x}, S) = \{0\}$. *Also,* $\partial_f \boldsymbol{F}(\bar{x}) = 0$ *for all* $\bar{x} \in (-2, 2)$. *Therefore,*

$$\alpha\mathbb{B} \cap \widehat{N}(\bar{x}, S) \subseteq \partial_f \boldsymbol{F}(\bar{x}) \text{ for all } \bar{x} \in (-2, 2). \tag{3.25}$$

At $\bar{x} = 2$, $\alpha\mathbb{B} \cap \widehat{N}(\bar{x}, S) = [0, \frac{1}{2}]$. *It can be easily seen that*

$$(x - 2) \odot [0, 1] \preceq \boldsymbol{F}(x) \ominus_{gH} \boldsymbol{F}(2) \text{ for all } x \in \mathbb{R}.$$

Therefore, $[0, 1] \in \partial_f \boldsymbol{F}(2)$. *Thus,*

$$\alpha\mathbb{B} \cap \widehat{N}(2, S) \subseteq \partial_f \boldsymbol{F}(2). \tag{3.26}$$

At $\bar{x} = -2$, $\alpha\mathbb{B} \cap \widehat{N}(\bar{x}, S) = [-\frac{1}{2}, 0]$. *It is obvious that*

$$(x + 2) \odot [-1, 0] \preceq \boldsymbol{F}(x) \ominus_{gH} \boldsymbol{F}(-2) \text{ for all } x \in \mathbb{R}.$$

Therefore, $[-1, 0] \in \partial_f \boldsymbol{F}(-2)$. *Thus,*

$$\alpha\mathbb{B} \cap \widehat{N}(-2, S) \subseteq \boldsymbol{F}(-2). \tag{3.27}$$

Hence, by (3.25), (3.26) and (3.27), we have

$$\alpha\mathbb{B} \cap \widehat{N}(\bar{x}, S) \subseteq \partial_f \boldsymbol{F}(\bar{x}) \text{ for all } \bar{x} \in S.$$

3.4 Concluding Remarks

In this chapter, the concept of gH-Fréchet subdifferentiability has been introduced. Various calculus results for gH-Fréchet subgradients has been provided. It has been shown that for a gH- Fréchet differentiable IVF, gH-Fréchet subdifferentiable set reduces to a singleton, i.e., $\partial_f \mathbf{F}(\bar{x}) = \{\nabla \mathbf{F}(\bar{x})\}$. A smooth variational description of gH-Fréchet subgradients has been given. By using the proposed notion of subdifferentiability, necessary optimality condition for unconstrained IOPs with nondifferentiable IVFs has been given. A necessary condition for unconstrained WSM has been given.

Based on the proposed research, in future, one make work in the following directions.

(i) In the literature, several optimality conditions are provided for unconstrained and constrained smooth IOPs (for instance, see ([47, 110]), and references therein). However, the optimality conditions for nonsmooth IOPs are not much explored. Therefore, one may work on the application of gH-Fréchet subdifferentiability in constrained interval optimization with nondifferentiable IVFs.

(ii) Recently, the authors of [53] presented gH-gradient based algorithms to solve convex IOPs. However, for nonsmooth IOPs there are no algorithms available in the literature of IOPs. Thus, one may try to develop a gH-Fréchet subgradient method to solve IOPs with nonconvex and nondifferentiable IVFs.

(iii) As IVFs are the special case of set-valued functions and IOPs are the special case of set optimization problems, similar results can be extended for set-valued functions and set optimization problems.

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