Chapter 2

Ekeland's Variational Principle for Interval-valued Functions

2.1 Introduction

Conventional Ekeland's variational principle (EVP)[37] is likely to be the most useful tool for nonlinear analysis. It has applications in nonlinear analysis [21, 45, 87], convex analysis [39], generalized differential calculus [31, 43, 88], optimization theory [12, 38], sensitivity analysis [12, 39], fixed point theory [11], global analysis [40]. Keeping in mind the theoretical development and application aspects , we propose EVP in this chapter for IVFs.

2.1.1 Motivation

In the literature of IVFs techniques to capture minima are available only for continuous and differentiable IVFs. However, while portraying real-life problems, we may obtain an objective function that is neither differentiable nor continuous[1]. In this work, we present the concepts of gH-semicontinuity and provide results that guarantee the existence of a minima and an approximate minima for an IVF without taking the assumption of continuity or differentiability.

In optimization, it is not always possible to locate an exact optimum [41]. In these cases, one tries to locate approximate optima. EVP [37] is useful to obtain approximate solutions [41]. EVP [37] is also one of the most potent nonlinear analysis tools. It has applications in a variety of fields including optimization theory, fixed point theory, and global analysis as demonstrated by [22, 85]. Because of the numerous applications of EVP in various areas, particularly nonsmooth optimization and control theory, we attempt to investigate this principle for gH-lower semicontinuous IVFs in this chapter.

2.1.2 Contribution

In this chapter, we

- define gH-semicontinuity for IVFs and characterize gH-continuity in terms of gH-lower and upper semicontinuity,

[1]Few fascinating real-life problems whose objective function is non-continuous and non-differentiable can be found in [31].

- provide a characterization of gH-lower semicontinuity and use this to demonstrate that an extended gH-lower semicontinuous, level-bounded, and proper IVF achieves its minimum,
- provide a characterization of the set argument minimum of an IVF, and
- derive EVP for IVFs.

2.2 gH-continuity and gH-semicontinuity of Interval-valued Functions

Throughout this section, an extended IVF is an IVF with domain $\mathcal{X}$ and codomain $\overline{\mathbb{R}_I}$.

Definition 2.1. (*gH-limit of an IVF*). Let $\mathbf{F} : \mathcal{S} \to \mathbb{R}_I$ be an IVF on a nonempty subset $\mathcal{S}$ of $\mathcal{X}$. The function $\mathbf{F}$ is called to be tending to a limit $\mathbf{L} \in \mathbb{R}_I$ as x tends to $\bar{x}$, denoted by $\lim\limits_{x \to \bar{x}} \mathbf{F}(x)$, if for each $\epsilon > 0$, there is a $\delta > 0$ satisfying

$$\|\mathbf{F}(x) \ominus_{gH} \mathbf{L}\|_{\mathbb{R}_I} < \epsilon \text{ whenever } 0 < \|x - \bar{x}\|_{\mathcal{X}} < \delta.$$

Definition 2.2. (*gH-continuity*). Let $\mathbf{F} : \mathcal{S} \to \mathbb{R}_I$ be an IVF on a nonempty subset $\mathcal{S}$ of $\mathcal{X}$. The function $\mathbf{F}$ is called gH-continuous at $\bar{x} \in \mathcal{S}$ if for each $\epsilon > 0$, there is a $\delta > 0$ satisfying

$$\|\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x})\|_{\mathbb{R}_I} < \epsilon \text{ whenever } \|x - \bar{x}\|_{\mathcal{X}} < \delta.$$

Definition 2.3. (*Lower limit and gH-lower semicontinuity of an extended IVF*). The lower limit of an extended IVF $\mathbf{F}$ at $\bar{x} \in \mathcal{X}$, denoted as $\liminf\limits_{x \to \bar{x}} \mathbf{F}(x)$, is defined by

$$\begin{aligned}\liminf_{x\to\bar{x}} \mathbf{F}(x) &= \lim_{\delta\downarrow 0}\left(\inf\{\mathbf{F}(x) : x \in B_\delta(\bar{x})\}\right)\\ &= \sup_{\delta>0}\left(\inf\{\mathbf{F}(x) : x \in B_\delta(\bar{x})\}\right).\end{aligned}$$

$\mathbf{F}$ is called gH-lower semicontinuous (gH-lsc) at a point $\bar{x}$ if

$$\mathbf{F}(\bar{x}) \preceq \liminf_{x\to\bar{x}} \mathbf{F}(x). \tag{2.1}$$

Further $\mathbf{F}$ is called gH-lsc on $\mathcal{X}$ if (2.1) holds for every $\bar{x} \in \mathcal{X}$.

Example 2.1. *Consider the following IVF* $\boldsymbol{F} : \mathbb{R}^2 \to \mathbb{R}_I$*:*

$$\boldsymbol{F}(x_1, x_2) = \begin{cases}[1,2] \odot \sin\left(\frac{1}{x_1}\right) \oplus \cos^2 x_2 & \text{if } \ x_1x_2 \neq 0\\ [-2,-1] & \text{if } \ x_1x_2 = 0.\end{cases}$$

The lower limit of $\boldsymbol{F}$ *at* $(0,0)$ *is given by*

$$\liminf_{(x_1,x_2)\to(0,0)} \boldsymbol{F}(x_1,x_2) = \lim_{\delta\downarrow 0}\left(\inf\{\boldsymbol{F}(x_1,x_2) : (x_1,x_2) \in B_\delta(0,0)\}\right).$$

Note that as $x_1 \to 0$, $\sin\left(\frac{1}{x_1}\right)$ *oscillates between* -1 *and* 1. *Therefore, for any* $\delta > 0$*,*

$$\inf_{(x_1,x_2)\in B_\delta(0,0)} \boldsymbol{F}(x_1,x_2) = [1,2] \odot (-1) = [-2,-1].$$

Also, note that when $(x_1,x_2) = (0,0)$, $\boldsymbol{F}(x_1,x_2) = [-2,-1]$.

Thus,

$$\liminf_{(x_1,x_2)\to(0,0)} \boldsymbol{F}(x_1,x_2) = [-2,-1].$$

Since $\boldsymbol{F}(0,0) = [-2,-1] \preceq [-2,-1] = \liminf_{(x_1,x_2)\to(0,0)} \boldsymbol{F}(x_1,x_2)$*, the function* $\boldsymbol{F}$ *is* gH*-lsc at* $(0,0)$*.*

Note 4. Let $\mathbf{F}$ be an extended IVF with $\mathbf{F}(x) = \left[\underline{f}(x), \overline{f}(x)\right]$, where $\underline{f},\ \overline{f} : \mathcal{X} \to \mathbb{R} \cup \{-\infty, +\infty\}$ are two extended real-valued functions. Then, $\mathbf{F}$ is gH-lsc at $\bar{x} \in \mathcal{X}$ if and only if $\underline{f}$ and $\overline{f}$ both are lsc at $\bar{x}$. The reason is as follows.

$$\begin{aligned}
\underline{f} \text{ and } \overline{f} \text{ are lsc at } \bar{x} \quad &\iff \quad \underline{f}(\bar{x}) \leq \liminf_{x \to \bar{x}} \underline{f}(x) \text{ and } \overline{f}(\bar{x}) \leq \liminf_{x \to \bar{x}} \overline{f}(x) \\
&\iff \quad \left[\underline{f}(\bar{x}), \overline{f}(\bar{x})\right] \preceq \left[\liminf_{x \to \bar{x}} \underline{f}(x), \liminf_{x \to \bar{x}} \overline{f}(x)\right] \\
&\iff \quad \left[\underline{f}(\bar{x}), \overline{f}(\bar{x})\right] \preceq \liminf_{x \to \bar{x}} \left[\underline{f}(x), \overline{f}(x)\right], \text{ by Remark 1.11} \\
&\text{i.e.,} \quad \mathbf{F}(\bar{x}) \preceq \liminf_{x \to \bar{x}} \mathbf{F}(x).
\end{aligned}$$

Note 4 reduces our efforts to check gH-lower semicontinuity of extended IVFs that are given in the form $\mathbf{F}(x) = \left[\underline{f}(x), \overline{f}(x)\right]$. For example, consider $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}_I$ as

$$\mathbf{F}(x_1, x_2) = \begin{cases} \left[\frac{|x_1 x_2|}{2x_1^2 + x_2^2}, \frac{e^{|6x_1 x_2|}}{x_1^2 + x_2^2}\right] & \text{if } \ x_1 x_2 \neq 0 \\ \mathbf{0} & \text{if } \ x_1 x_2 = 0 \end{cases}$$

and take $\bar{x} = (0, 0)$. It is easy to see that both

$$\underline{f}(x_1, x_2) = \begin{cases} \frac{|x_1 x_2|}{2x_1^2 + x_2^2} & \text{if } \ x_1 x_2 \neq 0 \\ 0 & \text{if } \ x_1 x_2 = 0 \end{cases}$$

and

$$\overline{f}(x_1, x_2) = \begin{cases} \frac{e^{|6x_1 x_2|}}{x_1^2 + x_2^2} & \text{if } \ x_1 x_2 \neq 0 \\ 0 & \text{if } \ x_1 x_2 = 0 \end{cases}$$

are lsc at $\bar{x}$. Thus, by Note 4, $\mathbf{F}$ is gH-lsc at $\bar{x}$.

Theorem 2.4. *Let $\mathbf{F}$ be an extended IVF. Then, $\mathbf{F}$ is gH-lsc at $\bar{x} \in \mathcal{X}$ if and only if for each $\epsilon > 0$, there exists a $\delta > 0$ such that $\mathbf{F}(\bar{x}) \ominus_{gH} [\epsilon, \epsilon] \prec \mathbf{F}(x)$ for all $x \in B_\delta(\bar{x})$.*

Proof. Let $\mathbf{F}$ be gH-lsc at $\bar{x}$.

To the contrary, suppose there exists an $\epsilon_0 > 0$ such that for all $\delta > 0$, $\mathbf{F}(\bar{x}) \ominus_{gH} [\epsilon_0, \epsilon_0] \not\prec \mathbf{F}(x)$ for atleast one x in $B_\delta(\bar{x})$.

Then,

$$\begin{aligned} & \mathbf{F}(\bar{x}) \ominus_{gH} [\epsilon_0, \epsilon_0] \not\prec \inf\{\mathbf{F}(x) : x \in B_\delta(\bar{x})\} \text{ for all } \delta > 0 \\ \Longrightarrow \quad & \mathbf{F}(\bar{x}) \ominus_{gH} [\epsilon_0, \epsilon_0] \not\prec \lim_{\delta \downarrow 0}(\inf\{\mathbf{F}(x) : x \in B_\delta(\bar{x})\}) \\ \Longrightarrow \quad & \mathbf{F}(\bar{x}) \ominus_{gH} [\epsilon_0, \epsilon_0] \not\prec \liminf_{x \to \bar{x}} \mathbf{F}(x) \\ \Longrightarrow \quad & \mathbf{F}(\bar{x}) \not\preceq \liminf_{x \to \bar{x}} \mathbf{F}(x), \text{ by (ii) of Lemma 1.4}, \end{aligned}$$

which contradicts that $\mathbf{F}$ is gH-lsc at $\bar{x}$. Thus, for each $\epsilon > 0$, there exists a $\delta > 0$ such that $\mathbf{F}(\bar{x}) \ominus_{gH} [\epsilon, \epsilon] \prec \mathbf{F}(x)$ for all $x \in B_\delta(\bar{x})$.

Conversely, suppose for a given $\epsilon > 0$, there exists a $\delta > 0$ such that $\mathbf{F}(\bar{x}) \ominus_{gH} [\epsilon, \epsilon] \prec \mathbf{F}(x)$ for all $x \in B_\delta(\bar{x})$. Then,

$$\begin{aligned} & \mathbf{F}(\bar{x}) \ominus_{gH} [\epsilon, \epsilon] \preceq \inf\{\mathbf{F}(x) : x \in B_\delta(\bar{x})\} \\ \Longrightarrow \quad & \mathbf{F}(\bar{x}) \ominus_{gH} [\epsilon, \epsilon] \preceq \lim_{\delta \downarrow 0}(\inf\{\mathbf{F}(x) : x \in B_\delta(\bar{x})\}) \\ \Longrightarrow \quad & \mathbf{F}(\bar{x}) \ominus_{gH} [\epsilon, \epsilon] \preceq \liminf_{x \to \bar{x}} \mathbf{F}(x). \end{aligned}$$

As, $\mathbf{F}(\bar{x}) \ominus_{gH} [\epsilon, \epsilon] \preceq \liminf_{x \to \bar{x}} \mathbf{F}(x)$ for every $\epsilon > 0$, we have $\mathbf{F}(\bar{x}) \preceq \liminf_{x \to \bar{x}} \mathbf{F}(x)$. Thus, $\mathbf{F}$ is gH-lsc at $\bar{x}$. □

Definition 2.5. (*Upper limit and gH-upper semicontinuity of an extended IVF*). The upper limit of an extended IVF $\mathbf{F}$ at $\bar{x} \in \mathcal{X}$, denoted $\limsup_{x \to \bar{x}} \mathbf{F}(x)$, is defined as

$$\begin{aligned} \limsup_{x \to \bar{x}} \mathbf{F}(x) &= \lim_{\delta \downarrow 0} (\sup\{\mathbf{F}(x) : x \in B_\delta(\bar{x})\}) \\ &= \inf_{\delta > 0} (\sup\{\mathbf{F}(x) : x \in B_\delta(\bar{x})\}). \end{aligned}$$

$\mathbf{F}$ is called gH-upper semicontinuous (gH-usc) at $\bar{x}$ if

$$\limsup_{x \to \bar{x}} \mathbf{F}(x) \preceq \mathbf{F}(\bar{x}). \tag{2.2}$$

Further, $\mathbf{F}$ is called gH-usc on $\mathcal{X}$ if (2.2) holds for every $\bar{x} \in \mathcal{X}$.

Note 5. Let $\mathbf{F}$ be an extended IVF with $\mathbf{F}(x) = \left[\underline{f}(x), \overline{f}(x)\right]$, where $\underline{f}$, $\overline{f} : \mathcal{X} \to \mathbb{R} \cup \{-\infty, +\infty\}$ be two extended real-valued functions. Then, because of a similar reason as in Note 4, $\mathbf{F}$ is gH-usc at $\bar{x} \in \mathcal{X}$ if and only if $\underline{f}$ and $\overline{f}$ are usc at $\bar{x}$.

Remark 2.6. By Definitions 2.3 and 2.5, it is easy to observe that

$$\limsup_{x \to \bar{x}} \mathbf{F}(x) = -1 \odot \liminf_{x \to \bar{x}} (-1 \odot \mathbf{F}(x)).$$

Theorem 2.7. *Let* $\boldsymbol{F}$ *be an extended IVF. Then,* $\boldsymbol{F}$ *is gH-usc at* $\bar{x} \in \mathcal{X}$ *if and only if for each* $\epsilon > 0$*, there exists a* $\delta > 0$ *such that* $\boldsymbol{F}(x) \prec \boldsymbol{F}(\bar{x}) \oplus [\epsilon, \epsilon]$ *for all* $x \in B_\delta(\bar{x})$.

Proof. Similar to the proof of Theorem 2.4. ☐

Theorem 2.8. *An extended IVF* $\boldsymbol{F}$ *is gH-continuous if and only if* $\boldsymbol{F}$ *is both gH-lower and upper semicontinuous.*

Proof. Let $\mathbf{F}$ be gH-continuous at $\bar{x} \in \mathcal{X}$. Then, for each $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\begin{aligned} & \|\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x})\|_{\mathbb{R}_I} < \epsilon \text{ for all } x \in B_\delta(\bar{x}) \\ \iff & \mathbf{F}(\bar{x}) \ominus_{gH} [\epsilon, \epsilon] \prec \mathbf{F}(x) \prec \mathbf{F}(\bar{x}) \oplus [\epsilon, \epsilon] \text{ for all } x \in B_\delta(\bar{x}), \\ & \text{ by (i) of Lemma 1.4} \\ \iff & \mathbf{F}(\bar{x}) \text{ is } gH\text{-lsc and } gH\text{-usc at } \bar{x}, \text{ by Theorems 2.4 and 2.7.} \end{aligned}$$

☐

Lemma 2.9. *Let* $\mathbf{F}_1$ *and* $\mathbf{F}_2$ *be two proper extended IVFs, and* $\mathcal{S}$ *be a nonempty subset of* $\mathcal{X}$*. Then,*

(i) $\inf_{x\in\mathcal{S}} \mathbf{F}_1(x) \oplus \inf_{x\in\mathcal{S}} \mathbf{F}_2(x) \preceq \inf_{x\in\mathcal{S}}\{\mathbf{F}_1(x) \oplus \mathbf{F}_2(x)\}$ *and*

(ii) $\sup_{x\in\mathcal{S}}\{\mathbf{F}_1(x) \oplus \mathbf{F}_2(x)\} \preceq \sup_{x\in\mathcal{S}} \mathbf{F}_1(x) \oplus \sup_{x\in\mathcal{S}} \mathbf{F}_2(x)$.

Proof. Let $\boldsymbol{\alpha}_1 = \inf_{x\in\mathcal{S}} \mathbf{F}_1(x)$ and $\boldsymbol{\alpha}_2 = \inf_{x\in\mathcal{S}} \mathbf{F}_2(x)$. Then,

$$\begin{aligned}
& \boldsymbol{\alpha}_1 \preceq \mathbf{F}_1(x) \text{ for all } x \in \mathcal{S} \text{ and } \boldsymbol{\alpha}_2 \preceq \mathbf{F}_2(x) \text{ for all } x \in \mathcal{S} \\
\Longrightarrow\quad & \boldsymbol{\alpha}_1 \oplus \boldsymbol{\alpha}_2 \preceq \mathbf{F}_1(x) \oplus \mathbf{F}_2(x) \text{ for all } x \in \mathcal{S}, \text{ by (ii) of Lemma 1.3} \\
\Longrightarrow\quad & \boldsymbol{\alpha}_1 \oplus \boldsymbol{\alpha}_2 \preceq \inf_{x\in\mathcal{S}}(\mathbf{F}_1(x) \oplus \mathbf{F}_2(x)) \\
\text{i.e.,}\quad & \inf_{x\in\mathcal{S}} \mathbf{F}_1(x) \oplus \inf_{x\in\mathcal{S}} \mathbf{F}_2(x) \preceq \inf_{x\in\mathcal{S}}\{\mathbf{F}_1(x) \oplus \mathbf{F}_2(x)\}.
\end{aligned}$$

Part (ii) can be similarly proved. □

Theorem 2.10. *Let* $\mathbf{F}_1$ *and* $\mathbf{F}_2$ *be two proper extended IVFs, and* $\mathcal{S}$ *be a nonempty subset of* $\mathcal{X}$*. Then,*

(i) $\liminf_{x\to\bar{x}} \mathbf{F}_1(x) \oplus \liminf_{x\to\bar{x}} \mathbf{F}_2(x) \preceq \liminf_{x\to\bar{x}}(\mathbf{F}_1 \oplus \mathbf{F}_2)(x)$ *and*

(ii) $\limsup_{x\to\bar{x}}(\mathbf{F}_1 \oplus \mathbf{F}_2)(x) \preceq \limsup_{x\to\bar{x}} \mathbf{F}_1(x) \oplus \limsup_{x\to\bar{x}} \mathbf{F}_2(x)$.

Proof.

$$\begin{aligned}
\liminf_{x\to\bar{x}} \mathbf{F}_1(x) \oplus \liminf_{x\to\bar{x}} \mathbf{F}_2(x) &= \lim_{\delta\downarrow 0} \inf_{x\in B_\delta(\bar{x})} \mathbf{F}_1(x) \oplus \lim_{\delta\downarrow 0} \inf_{x\in B_\delta(\bar{x})} \mathbf{F}_2(x), \\
&\quad \text{by Definition 2.3} \\
&\preceq \lim_{\delta\downarrow 0}\left(\inf_{x\in B_\delta(\bar{x})} \mathbf{F}_1(x) \oplus \inf_{x\in B_\delta(\bar{x})} \mathbf{F}_2(x)\right) \\
&\preceq \lim_{\delta\downarrow 0} \inf_{x\in B_\delta(\bar{x})}(\mathbf{F}_1 \oplus \mathbf{F}_2)(x), \text{ by (i) of Lemma 2.9}
\end{aligned}$$

$$= \liminf_{x \to \bar{x}} (\mathbf{F}_1 \oplus \mathbf{F}_2)(x).$$

This completes the proof of (i).

Part (ii) can be similarly proved. □

Theorem 2.11. *Let* $\mathbf{F}_1$ *and* $\mathbf{F}_2$ *be two proper and gH-lsc extended IVFs. Then,* $\mathbf{F}_1 \oplus \mathbf{F}_2$ *is gH-lsc.*

Proof. Take $\bar{x} \in \mathcal{X}$. Since $\mathbf{F}_1$ and $\mathbf{F}_2$ are gH-lsc at $\bar{x}$, we have

$$\begin{aligned}
& \mathbf{F}_1(\bar{x}) \preceq \liminf_{x \to \bar{x}} \mathbf{F}_1(x) \text{ and } \mathbf{F}_2(\bar{x}) \preceq \liminf_{x \to \bar{x}} \mathbf{F}_2(x) \\
\Longrightarrow\ & \mathbf{F}_1(\bar{x}) \oplus \mathbf{F}_2(\bar{x}) \preceq \liminf_{x \to \bar{x}} \mathbf{F}_1(x) \oplus \liminf_{x \to \bar{x}} \mathbf{F}_2(x), \text{ by (ii) of Lemma 1.3} \\
\Longrightarrow\ & (\mathbf{F}_1 \oplus \mathbf{F}_2)(\bar{x}) \preceq \liminf_{x \to \bar{x}} (\mathbf{F}_1 \oplus \mathbf{F}_2)(x), \text{ by (i) of Theorem 2.10} \\
\Longrightarrow\ & \mathbf{F}_1 \oplus \mathbf{F}_2 \text{ is } gH\text{-lsc at } \bar{x}.
\end{aligned}$$

Since $\bar{x}$ is arbitrarily chosen, so $\mathbf{F}_1 \oplus \mathbf{F}_2$ is gH-lsc on $\mathcal{X}$. □

Lemma 2.12. (Characterization of lower limits of IVFs). *Let* $\boldsymbol{F}$ *be an extended IVF. Then,*

$$\liminf_{x \to \bar{x}} \boldsymbol{F}(x) = \inf \left\{ \boldsymbol{\alpha} \in \overline{\mathbb{R}_I} : \textit{there exists a sequence } \{x_k\} \to \bar{x} \textit{ with } \boldsymbol{F}(\{x_k\}) \to \boldsymbol{\alpha} \right\}.$$

Proof. Let $\bar{\boldsymbol{\alpha}} = \liminf_{x \to \bar{x}} \mathbf{F}(x)$. Assume that sequence $\{x_k\} \to \bar{x}$ with $\mathbf{F}(\{x_k\}) \to \boldsymbol{\alpha}$. In the below, we show that $\bar{\boldsymbol{\alpha}} \preceq \boldsymbol{\alpha}$.

Since $\{x_k\} \to \bar{x}$, for any $\delta > 0$, there exists $k_\delta \in \mathbb{N}$ such that $\{x_k\} \in B_\delta(\bar{x})$ for every $k \geq k_\delta$.

Therefore,

$$\inf\{\mathbf{F}(x) : x \in B_\delta(\bar{x})\} \preceq \mathbf{F}(\{x_k\}) \text{ for any } \delta > 0$$

$$\implies \inf\{\mathbf{F}(x) : x \in B_\delta(\bar{x})\} \preceq \lim_{k\to+\infty} \mathbf{F}(\{x_k\}) \text{ for any } \delta > 0$$

$$\implies \inf\{\mathbf{F}(x) : x \in B_\delta(\bar{x})\} \preceq \boldsymbol{\alpha} \text{ for any } \delta > 0$$

$$\implies \lim_{\delta\downarrow 0} \inf\{\mathbf{F}(x) : x \in B_\delta(\bar{x})\} \preceq \boldsymbol{\alpha}$$

$$\implies \liminf_{x\to\bar{x}} \mathbf{F}(x) = \bar{\boldsymbol{\alpha}} \preceq \boldsymbol{\alpha}.$$

Next, we show that there exists a sequence $\{x_k\} \to \bar{x}$ with $\mathbf{F}(\{x_k\}) \to \bar{\boldsymbol{\alpha}}$.
Consider a nonnegative sequence $\{\delta_k\}$ with $\delta_k \downarrow 0$, and construct a sequence $\{\bar{\boldsymbol{\alpha}}_k\} = \inf\{\mathbf{F}(x) : x \in B_{\delta_k}(\bar{x})\}$.
As $\delta_k \downarrow 0$, by Definition 2.3 of lower limit, $\{\bar{\boldsymbol{\alpha}}_k\} \to \bar{\boldsymbol{\alpha}}$. Also, by definition of infimum, for a given $\epsilon > 0$ and $k \in \mathbb{N}$, there exists $\{x_k\} \in B_{\delta_k}(\bar{x})$ such that $\mathbf{F}(\{x_k\}) \preceq \{\bar{\boldsymbol{\alpha}}_k\}$. That is, $\{\bar{\boldsymbol{\alpha}}_k\} \preceq \mathbf{F}(\{x_k\}) \preceq \{\boldsymbol{\alpha}_k\}$, where $\{\boldsymbol{\alpha}_k\} \to \bar{\boldsymbol{\alpha}}$.
Note that $\{x_k\} \in B_{\delta_k}(\bar{x})$ and $\delta_k \downarrow 0$. Therefore, as $k \to +\infty$, $\{x_k\} \to \bar{x}$. Also, note that $\mathbf{F}(\{x_k\})$ is a monotonic increasing bounded sequence and therefore, by Lemma 1.18, $\mathbf{F}(\{x_k\})$ converges to $\bar{\boldsymbol{\alpha}}$, and the proof is complete. □

Lemma 2.13. (Characterization of upper limits of IVFs). *Let* $\boldsymbol{F}$ *be an extended IVF. Then,*

$$\limsup_{x\to\bar{x}} \boldsymbol{F}(x) = \sup\left\{\boldsymbol{\alpha} \in \overline{\mathbb{R}_I} : \textit{there exists a sequence } \{x_k\} \to \bar{x} \textit{ with } \boldsymbol{F}(\{x_k\}) \to \boldsymbol{\alpha}\right\}.$$

Proof. Similar to the proof of Lemma 2.12. □

Definition 2.14. (*Level set of an IVF*). Let $\mathbf{F}$ be an extended IVF. For an $\boldsymbol{\alpha} \in \overline{\mathbb{R}_I}$, the level set of $\mathbf{F}$, denoted as $\text{lev}_{\boldsymbol{\alpha}\not\prec}\mathbf{F}$, is defined by

$$\text{lev}_{\boldsymbol{\alpha}\not\prec}\mathbf{F} = \{x \in \mathcal{X} : \boldsymbol{\alpha} \not\prec \mathbf{F}(x)\}.$$

Example 2.2. *Consider* $\boldsymbol{F} : \mathbb{R}^2 \to \overline{\mathbb{R}_I}$ *as* $\boldsymbol{F}(x) = [1,2] \odot x_1^2 \oplus [3,4] \odot e^{x_2^2}$ *and* $\boldsymbol{\alpha} = [-1, 10]$*. Then,*

$$\begin{aligned}
lev_{\boldsymbol{\alpha}\not\prec}\boldsymbol{F} &= \left\{(x_1,x_2) \in \mathbb{R}^2 : [-1,10] \not\prec [1,2] \odot x_1^2 \oplus [3,4] \odot e^{x_2^2}\right\} \\
&= \left\{(x_1,x_2) \in \mathbb{R}^2 : [-1,10] \not\prec \left[x_1^2 + 3e^{x_2^2}, 2x_1^2 + 4e^{x_2^2}\right]\right\} \\
&= \Big\{(x_1,x_2) \in \mathbb{R}^2 : \left[x_1^2 + 3e^{x_2^2}, 2x_1^2 + 4e^{x_2^2}\right] \preceq [-1,10] \textit{ or} \\
&\qquad\qquad [-1,10] \textit{ and } \left[x_1^2 + 3e^{x_2^2}, 2x_1^2 + 4e^{x_2^2}\right] \textit{ are not comparable}\Big\} \\
&= \Big\{(x_1,x_2) \in \mathbb{R}^2 : [-1,10] \textit{ and } \left[x_1^2 + 3e^{x_2^2}, 2x_1^2 + 4e^{x_2^2}\right] \\
&\qquad\qquad \textit{are not comparable}\Big\} \\
&= \Big\{(x_1,x_2) \in \mathbb{R}^2 : \text{`}x_1^2 + 3e^{x_2^2} < -1 \textit{ and } 2x_1^2 + 4e^{x_2^2} > 10\text{'} \textit{ or} \\
&\qquad\qquad \text{`}x_1^2 + 3e^{x_2^2} > -1 \textit{ and } 2x_1^2 + 4e^{x_2^2} < 10\text{'}\Big\} \\
&= \left\{(x_1,x_2) \in \mathbb{R}^2 : x_1^2 + 3e^{x_2^2} > -1 \textit{ and } 2x_1^2 + 4e^{x_2^2} < 10\right\} \\
&= \left\{(x_1,x_2) \in \mathbb{R}^2 : 2x_1^2 + 4e^{x_2^2} < 10\right\}.
\end{aligned}$$

Hence,

$$lev_{\boldsymbol{\alpha}\not\prec}\boldsymbol{F} = \left\{(x_1,x_2) \in \mathbb{R}^2 : x_1^2 + 2e^{x_2^2} < 5\right\}.$$

Definition 2.15. (*Level-bounded IVF*). An extended IVF $\mathbf{F}$ is said to be level-bounded if for any $\boldsymbol{\alpha} \in \mathbb{R}_I$, $\text{lev}_{\boldsymbol{\alpha}\not\prec}\mathbf{F}$ is bounded.

Lemma 2.16. *Let* $\boldsymbol{F}$ *be an extended IVF and* $\bar{x} \in \mathcal{X}$*. Then,*

$$\inf_{\{\{x_k\}\}} (\liminf \boldsymbol{F}(\{x_k\})) \not\prec \liminf_{x \to \bar{x}} \boldsymbol{F}(x), \tag{2.3}$$

where the infimum on the left-hand side is taken over all sequences $\{x_k\} \to \bar{x}$*.*

Proof. Let $\mathbf{M} = \liminf_{x \to \bar{x}} \mathbf{F}(x)$ and $\mathbf{L} = \inf_{\{\{x_k\}\}} \liminf \mathbf{F}(\{x_k\})$.
If $\mathbf{M} = -\infty$, there is nothing to prove.

Next, let $\mathbf{M} = +\infty$ and $\{x_k\}$ be an arbitrary sequence converging to $\bar{x}$. We show that $\mathbf{F}(\{x_k\}) \to +\infty$. Since $\mathbf{M} = +\infty$, for any given $\alpha > 0$, there exists a $\delta > 0$ such that $[\alpha, \alpha] \prec \inf_{x \in B_\delta(\bar{x})} \mathbf{F}(x)$. Since $\{x_k\} \to \bar{x}$, there exists an integer $m > 0$ such that $\{x_k\} \in B_\delta(\bar{x})$ for all $n \geq m$. Thus, $[\alpha, \alpha] \prec \mathbf{F}(\{x_k\})$ for all $n \geq m$, and hence $\mathbf{F}(\{x_k\}) \to +\infty$.

Finally, let $[-\infty, -\infty] \prec \mathbf{M} \prec [+\infty, +\infty]$, i.e., $\mathbf{M} \in \mathbb{R}_I$. Suppose that there exists $\epsilon_0 > 0$ such that for all $\delta > 0$, $\inf_{x \in B_\delta(\bar{x})} \mathbf{F}(x) \preceq \mathbf{M} \ominus_{gH} [\epsilon_0, \epsilon_0]$. Then,

$$\begin{aligned} & \lim_{\delta \downarrow 0} \inf_{x \in B_\delta(\bar{x})} \mathbf{F}(x) \preceq \mathbf{M} \ominus_{gH} [\epsilon_0, \epsilon_0] \\ \Longrightarrow \quad & \liminf_{x \to \bar{x}} \mathbf{F}(x) \preceq \mathbf{M} \ominus_{gH} [\epsilon_0, \epsilon_0] \\ \text{i.e.,} \quad & \mathbf{M} \preceq \mathbf{M} \ominus_{gH} [\epsilon_0, \epsilon_0], \end{aligned}$$

which is not true. Thus, for a given $\epsilon > 0$, there exists a $\delta > 0$ such that $\inf_{x \in B_\delta(\bar{x})} \mathbf{F}(x) \npreceq \mathbf{M} \ominus_{gH} [\epsilon, \epsilon]$. This implies $\mathbf{F}(x) \npreceq \mathbf{M} \ominus_{gH} [\epsilon, \epsilon]$ for all $x \in B_\delta(\bar{x})$. Let $\{x_k\}$ be a sequence converging to $\bar{x}$. Since $\{x_k\} \in B_\delta(\bar{x})$ for large enough k, we have $\liminf \mathbf{F}(\{x_k\}) \npreceq \mathbf{M} \ominus_{gH} [\epsilon, \epsilon]$ for any $\epsilon > 0$. Thus, $\liminf \mathbf{F}(\{x_k\}) \nprec \mathbf{M}$ for any sequence converging to $\bar{x}$, and hence $\mathbf{L} \nprec \mathbf{M}$. Therefore, (2.3) holds. ☐

Theorem 2.17. *Let* $\boldsymbol{F}$ *be an extended IVF. Then,* $\boldsymbol{F}$ *is* gH*-lsc on* $\mathcal{X}$ *if and only if the level set* $lev_{\boldsymbol{\alpha} \nprec} \boldsymbol{F}$ *is closed for every* $\boldsymbol{\alpha} \in \mathbb{R}_I$*.*

Proof. Let $\mathbf{F}$ be gH-lsc on $\mathcal{X}$. For a fixed $\boldsymbol{\alpha} \in \mathbb{R}_I$, suppose that $\{x_k\} \subseteq \text{lev}_{\boldsymbol{\alpha} \nprec} \mathbf{F}$ such that $\{x_k\} \to \bar{x}$. Then,

$$\begin{aligned} & \boldsymbol{\alpha} \nprec \mathbf{F}(\{x_k\}) \\ \Longrightarrow \quad & \boldsymbol{\alpha} \nprec \liminf \mathbf{F}(\{x_k\}) \\ \Longrightarrow \quad & \boldsymbol{\alpha} \nprec \liminf_{x \to \bar{x}} \mathbf{F}(x), \text{ by Lemma 2.16} \\ \Longrightarrow \quad & \boldsymbol{\alpha} \nprec \mathbf{F}(\bar{x}) \text{ since } \mathbf{F} \text{ is } gH\text{-lsc at } \bar{x}. \end{aligned}$$

Thus, $\bar{x} \in \text{lev}_{\boldsymbol{\alpha}\not\prec}\mathbf{F}$, and hence $\text{lev}_{\boldsymbol{\alpha}\not\prec}\mathbf{F}$ is closed.

Since $\boldsymbol{\alpha} \in \mathbb{R}_I$ is arbitrarly chosen, so $\text{lev}_{\boldsymbol{\alpha}\not\prec}\mathbf{F}$ is closed for every $\boldsymbol{\alpha} \in \mathbb{R}_I$.

Conversely, suppose the level set $\text{lev}_{\boldsymbol{\alpha}\not\prec}\mathbf{F}$ is closed for every $\boldsymbol{\alpha} \in \mathbb{R}_I$. Fix an $\bar{x} \in \mathcal{X}$. To prove that $\mathbf{F}$ is gH-lsc at $\bar{x}$, we need to show that

$$\mathbf{F}(\bar{x}) \preceq \liminf_{x \to \bar{x}} \mathbf{F}(x).$$

Let $\bar{\boldsymbol{\alpha}} = \liminf_{x \to \bar{x}} \mathbf{F}(x)$. The case of $\bar{\boldsymbol{\alpha}} = +\infty$ is trivial; so assume $\bar{\boldsymbol{\alpha}} \prec [+\infty, +\infty]$. By Lemma 2.12, there exists a sequence $\{x_k\} \to \bar{x}$ with $\mathbf{F}(\{x_k\}) \to \bar{\boldsymbol{\alpha}}$. For any $\boldsymbol{\alpha}$ such that $\bar{\boldsymbol{\alpha}} \prec \boldsymbol{\alpha}$, it will eventually be true that $\boldsymbol{\alpha} \not\prec \mathbf{F}(\{x_k\})$, or in other words, that $\{x_k\} \in \text{lev}_{\boldsymbol{\alpha}\not\prec}\mathbf{F}$. Since $\text{lev}_{\boldsymbol{\alpha}\not\prec}\mathbf{F}$ is closed, $\bar{x} \in \text{lev}_{\boldsymbol{\alpha}\not\prec}\mathbf{F}$.

Thus, $\boldsymbol{\alpha} \not\prec \mathbf{F}(\bar{x})$ for every $\boldsymbol{\alpha}$ such that $\bar{\boldsymbol{\alpha}} \prec \boldsymbol{\alpha}$, then $\bar{\boldsymbol{\alpha}} \not\prec \mathbf{F}(\bar{x})$. Therefore, either $\mathbf{F}(\bar{x}) \preceq \bar{\boldsymbol{\alpha}}$ or $\bar{\boldsymbol{\alpha}}$ and $\mathbf{F}(\bar{x})$ are not comparable. But since $\bar{\boldsymbol{\alpha}} = \liminf_{x \to \bar{x}} \mathbf{F}(x)$, so $\bar{\boldsymbol{\alpha}}$ is comparable with $\mathbf{F}(\bar{x})$, and hence $\mathbf{F}(\bar{x}) \preceq \bar{\boldsymbol{\alpha}}$.

Since $\bar{x} \in \mathcal{X}$ is arbitrarily chosen, $\mathbf{F}$ is gH-lsc on $\mathcal{X}$. This completes the proof. □

Definition 2.18. (*Indicator function*). Consider a subset $\mathcal{S}$ of $\mathcal{X}$. The indicator function of $\mathcal{S}, \delta_{\mathcal{S}}(s) : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\delta_{\mathcal{S}}(s) = \begin{cases} \mathbf{0} & \text{if } s \in \mathcal{S} \\ +\infty & \text{if } s \notin \mathcal{S}. \end{cases}$$

Remark 2.19. (i) It is easy to see that $\delta_{\mathcal{S}}$ is proper if and only if $\mathcal{S}$ is nonempty.

(ii) By Theorem 2.17, $\delta_{\mathcal{S}}$ is gH-lsc if and only if $\mathcal{S}$ is closed.

Definition 2.20. (*Argument minimum of an IVF*). Let $\mathbf{F}$ be an extended IVF. Then, the argument minimum of $\mathbf{F}$, denoted as $\underset{x\in\mathcal{X}}{\operatorname{argmin}}\,\mathbf{F}(x)$, is defined by

$$\underset{x\in\mathcal{X}}{\operatorname{argmin}}\,\mathbf{F}(x) = \begin{cases} \left\{x \in \mathcal{X} : \mathbf{F}(x) = \inf\limits_{y\in\mathcal{X}} \mathbf{F}(y)\right\} & \text{if } \inf\limits_{y\in\mathcal{X}} \mathbf{F}(y) \neq +\infty \\ \emptyset & \text{if } \inf\limits_{y\in\mathcal{X}} \mathbf{F}(y) = +\infty. \end{cases}$$

Example 2.3. *Consider* $\boldsymbol{F} : \mathbb{R}^2 \to \overline{\mathbb{R}_I}$ *as*

$$\boldsymbol{F}(x_1, x_2) = \begin{cases} \left[-\frac{1}{|x_1|}, e^{-\frac{1}{|x_1|}+x_2^2}\right] & \textit{if } \ x_1 \neq 0 \\ [-\infty, 0] & \textit{if } \ x_1 = 0. \end{cases}$$

Then, $\inf\limits_{(x_1,x_2)\in\mathbb{R}^2} \mathbf{F}(x_1, x_2) = [-\infty, 0]$.

$$\begin{aligned} \underset{x\in\mathbb{R}^2}{\operatorname{argmin}}\,\mathbf{F}(x) &= \left\{(x_1, x_2) \in \mathbb{R}^2 : \boldsymbol{F}(x_1, x_2) = \inf_{x\in\mathbb{R}^2} \mathbf{F}(x_1, x_2) = [-\infty, 0]\right\} \\ &= \{(0, x_2) : x_2 \in \mathbb{R}\}. \end{aligned}$$

Therefore, $\underset{x\in\mathbb{R}^2}{\operatorname{argmin}}\,\mathbf{F}(x) = \{(0, x_2) : x_2 \in \mathbb{R}\}$.

Theorem 2.21. (Minimum attained by an extended IVF). *Let* $\boldsymbol{F}$ *be gH-lsc, level-bounded and proper extended IVF. Then, the set* $\operatorname{argmin}_{\mathcal{X}} \boldsymbol{F}$ *is nonempty and compact.*

Proof. Let $\bar{\boldsymbol{\alpha}} = \inf \mathbf{F}$. So, $\bar{\boldsymbol{\alpha}} \prec [+\infty, +\infty]$ because $\mathbf{F}$ is proper.

Note that $\text{lev}_{\boldsymbol{\alpha}\nprec}\mathbf{F} \neq \emptyset$ for any $\boldsymbol{\alpha}$ that satisfies $\bar{\boldsymbol{\alpha}} \prec \boldsymbol{\alpha} \prec [+\infty, +\infty]$. Also, as $\mathbf{F}$ is level-bounded, $\text{lev}_{\boldsymbol{\alpha}\nprec}\mathbf{F}$ is bounded and by Theorem 2.17, it is also closed. Thus, $\text{lev}_{\boldsymbol{\alpha}\nprec}\mathbf{F}$ is nonempty compact for $\bar{\boldsymbol{\alpha}} \prec \boldsymbol{\alpha} \prec [+\infty, +\infty]$ and are nested as $\text{lev}_{\boldsymbol{\alpha}\nprec}\mathbf{F} \subseteq$

$\text{lev}_{\boldsymbol{\beta}\nprec}\mathbf{F}$ when $\boldsymbol{\alpha} \prec \boldsymbol{\beta}$. Therefore,

$$\bigcap_{\bar{\boldsymbol{\alpha}} \prec \boldsymbol{\alpha} \prec +\infty} \text{lev}_{\boldsymbol{\alpha}\nprec}\mathbf{F} = \text{lev}_{\bar{\boldsymbol{\alpha}}\nprec}\mathbf{F} = \underset{\mathcal{X}}{\text{argmin}}\,\mathbf{F}$$

is nonempty and compact. □

Next, we present a theorem which gives a characterization of the argument minimum set of an IVF in terms of gH-Gâteaux differentiability.

Theorem 2.22. (Characterization of the set argument minimum of an IVF). *Let* $\boldsymbol{F}$ *be an extended IVF and* $\bar{x} \in \text{argmin}_{x\in\mathcal{X}} \boldsymbol{F}(x)$. *If the function* $\boldsymbol{F}$ *has a* gH*-Gâteaux derivative at* $\bar{x}$ *in every direction* $h \in \mathcal{X}$, *then*

$$\boldsymbol{F}_{\mathscr{G}}(\bar{x})(h) = \boldsymbol{0} \textit{ for all } h \in \mathcal{X}.$$

Proof. Observe that any $\bar{x} \in \text{argmin}_{x\in\mathcal{X}} \mathbf{F}(x)$, is also an efficient point. Then, the proof follows from proof of the Theorem 4.2 in [49]. □

2.3 Ekeland's Variational Principle and its Applications

In this section, we present the main results—Ekeland's variational principle for IVFs along with its application for gH-Gâteaux differentiable IVFs.

Theorem 2.23. (EVP for real-valued functions [37]). *Let* $f : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ *be a lsc extended real-valued function and* $\epsilon > 0$. *Assume that*

$$\inf_{\mathcal{X}} f \textit{ is finite and } f(\bar{x}) < \inf_{\mathcal{X}} f + \epsilon.$$

Then, for any $\delta > 0$, there exists an $x_0 \in \mathcal{X}$ such that

(i) $\|x_0 - \bar{x}\|_{\mathcal{X}} < \frac{\epsilon}{\delta}$,

(ii) $f(x_0) \leq f(\bar{x})$, *and*

(iii) $\underset{x \in \mathcal{X}}{\operatorname{argmin}}\{f(x) + \delta\|x - x_0\|_{\mathcal{X}}\} = \{x_0\}$.

Lemma 2.24. *Let $\bar{x} \in \mathcal{X}$ and $\boldsymbol{A} \in \mathbb{R}_I$. Then, $\{x \in \mathcal{X} : \boldsymbol{A} \nprec \|x - \bar{x}\|_{\mathcal{X}}\}$ is a bounded set.*

Proof. Let $\mathbf{A} = [\underline{a}, \overline{a}]$. Then,

$$\begin{aligned}
& \{x \in \mathcal{X} : \mathbf{A} \nprec \|x - \bar{x}\|_{\mathcal{X}}\} \\
= \;& \{x \in \mathcal{X} : [\underline{a}, \overline{a}] \nprec \|x - \bar{x}\|_{\mathcal{X}}\} \\
= \;& \{x \in \mathcal{X} : \|x - \bar{x}\|_{\mathcal{X}} \preceq [\underline{a}, \overline{a}] \text{ or '}[\underline{a}, \overline{a}] \text{ and } \|x - \bar{x}\|_{\mathcal{X}} \text{ are not comparable'}\} \\
= \;& \{x \in \mathcal{X} : \text{'}\|x - \bar{x}\|_{\mathcal{X}} \leq \underline{a} \text{ and } \|x - \bar{x}\|_{\mathcal{X}} \leq \overline{a}\text{'} \\
& \qquad\qquad \text{or '}[\underline{a}, \overline{a}] \text{ and } \|x - \bar{x}\|_{\mathcal{X}} \text{ are not comparable'}\} \\
= \;& \{x \in \mathcal{X} : \text{'}\|x - \bar{x}\|_{\mathcal{X}} \leq \underline{a}\text{' or '}\|x - \bar{x}\|_{\mathcal{X}} < \underline{a} \text{ and } \|x - \bar{x}\|_{\mathcal{X}} > \overline{a}\text{'} \\
& \qquad\qquad \text{or '}\|x - \bar{x}\|_{\mathcal{X}} > \underline{a} \text{ and } \|x - \bar{x}\|_{\mathcal{X}} < \overline{a}\text{'}\} \\
= \;& \{x \in \mathcal{X} : \|x - \bar{x}\|_{\mathcal{X}} \leq \underline{a} \text{ or } \underline{a} < \|x - \bar{x}\|_{\mathcal{X}} < \overline{a}\},
\end{aligned}$$

which is a bounded set.

Hence, for any $\bar{x} \in \mathcal{X}$ and $\mathbf{A} \in \mathbb{R}_I$, $\{x \in \mathcal{X} : \mathbf{A} \nprec \|x - \bar{x}\|_{\mathcal{X}}\}$ is bounded. □

Theorem 2.25. (EVP for IVFs). *Let $\boldsymbol{F} : \mathcal{X} \to \mathbb{R}_I \cup \{+\infty\}$ be a gH-lsc extended IVF and $\epsilon > 0$. Assume that*

$$\inf_{\mathcal{X}} \boldsymbol{F} \textit{ is finite and } \boldsymbol{F}(\bar{x}) \prec \inf_{\mathcal{X}} \boldsymbol{F} \oplus [\epsilon, \epsilon].$$

Then, for any $\delta > 0$*, there exists an* $x_0 \in \mathcal{X}$ *such that*

(i) $\|x_0 - \bar{x}\|_{\mathcal{X}} < \frac{\epsilon}{\delta}$,

(ii) $\boldsymbol{F}(x_0) \preceq \boldsymbol{F}(\bar{x})$, *and*

(iii) $\underset{x \in \mathcal{X}}{\operatorname{argmin}}\{\boldsymbol{F}(x) \oplus \delta\|x - x_0\|_{\mathcal{X}}\} = \{x_0\}$.

Proof. Let $\bar{\boldsymbol{\alpha}} = \inf_{\mathcal{X}} \mathbf{F}$ and $\overline{\mathbf{F}}(x) = \mathbf{F}(x) \oplus \delta\|x - \bar{x}\|_{\mathcal{X}}$.

Since $\overline{\mathbf{F}}$ is the sum of two gH-lsc and proper IVFs, $\overline{\mathbf{F}}$ is gH-lsc by Theorem 2.11. Also,

$$\begin{aligned}
\text{lev}_{\boldsymbol{\alpha} \not\prec} \overline{\mathbf{F}} &= \left\{x \in \mathcal{X} : \boldsymbol{\alpha} \not\prec \overline{\mathbf{F}}(x)\right\} \\
&= \{x \in \mathcal{X} : \boldsymbol{\alpha} \not\prec \mathbf{F}(x) \oplus \delta\|x - \bar{x}\|_{\mathcal{X}}\} \\
&\subseteq \{x \in \mathcal{X} : \boldsymbol{\alpha} \not\prec \bar{\boldsymbol{\alpha}} \oplus \delta\|x - \bar{x}\|_{\mathcal{X}}\} \\
&= \left\{x \in \mathcal{X} : \frac{\boldsymbol{\alpha} \ominus_{gH} \bar{\boldsymbol{\alpha}}}{\delta} \not\prec \|x - \bar{x}\|_{\mathcal{X}}\right\} \\
&= \{x \in \mathcal{X} : \mathbf{A} \not\prec \|x - \bar{x}\|_{\mathcal{X}}\}, \text{ where } \mathbf{A} = \frac{\boldsymbol{\alpha} \ominus_{gH} \bar{\boldsymbol{\alpha}}}{\delta}.
\end{aligned}$$

Therefore, by Lemma 2.24, $\overline{\mathbf{F}}$ is level-bounded. Clearly, $\overline{\mathbf{F}}$ is proper. Hence, by Theorem 2.21, $C = \operatorname{argmin}_{\mathcal{X}} \overline{\mathbf{F}}$ is nonempty and compact.

Let us consider the function $\tilde{\mathbf{F}} = \mathbf{F} \oplus \delta_C$ on $\mathcal{X}$. Note that $\tilde{\mathbf{F}}$ is proper and level-bounded. Since C is nonempty and compact, so by Remark 2.19, δ_C is gH-lsc. Thus, by Theorem 2.11, $\tilde{\mathbf{F}}$ is gH-lsc, and hence by Theorem 2.21, $\operatorname{argmin}_{\mathcal{X}} \tilde{\mathbf{F}}$ is nonempty.

Let $x_0 \in \operatorname{argmin}_{\mathcal{X}} \tilde{\mathbf{F}}$. Then, over the set C, $\mathbf{F}$ is minimum at x_0.

Since $x_0 \in C$, $\overline{\mathbf{F}}(x_0) \prec \overline{\mathbf{F}}(x)$ for $x \notin C$. This implies that for any $x \notin C$,

$$\mathbf{F}(x_0) \oplus \delta\|x_0 - \bar{x}\|_{\mathcal{X}} \prec \mathbf{F}(x) \oplus \delta\|x - \bar{x}\|_{\mathcal{X}}$$

$$\implies \quad \mathbf{F}(x_0) \prec \mathbf{F}(x) \oplus \delta\|x-\bar{x}\|_{\mathcal{X}} \ominus_{gH} \delta\|x_0-\bar{x}\|_{\mathcal{X}}.$$

Hence, $\mathbf{F}(x_0) \prec \mathbf{F}(x)\oplus\delta\|x-x_0\|_{\mathcal{X}}$ for all $x \notin C$ with $x \neq x_0$, and thus $\operatorname{argmin}_{x\in\mathcal{X}}\{\mathbf{F}(x)\oplus \delta\|x-x_0\|_{\mathcal{X}}\} = \{x_0\}$.

Also, as $x_0 \in C$, we have $\overline{\mathbf{F}}(x_0) \preceq \overline{\mathbf{F}}(\bar{x})$, which implies

$$\begin{aligned}
& \overline{\mathbf{F}}(x_0) \preceq \mathbf{F}(\bar{x}) \text{ because } \overline{\mathbf{F}}(\bar{x}) = \mathbf{F}(\bar{x}) \\
\implies \quad & \mathbf{F}(x_0) \oplus \delta\|x_0-\bar{x}\|_{\mathcal{X}} \preceq \mathbf{F}(\bar{x}) \\
\implies \quad & \mathbf{F}(x_0) \preceq \mathbf{F}(\bar{x}) \ominus_{gH} \delta\|x_0-\bar{x}\|_{\mathcal{X}} \\
\implies \quad & \mathbf{F}(x_0) \prec \bar{\boldsymbol{\alpha}} \oplus [\epsilon,\epsilon] \ominus_{gH} \delta\|x_0-\bar{x}\|_{\mathcal{X}} \text{ because } \mathbf{F}(\bar{x}) \prec \inf_{\mathcal{X}} \mathbf{F} \oplus [\epsilon,\epsilon] \\
\implies \quad & \delta\|x_0-\bar{x}\|_{\mathcal{X}} \prec \bar{\boldsymbol{\alpha}} \oplus [\epsilon,\epsilon] \ominus_{gH} \mathbf{F}(x_0) \\
\implies \quad & \delta\|x_0-\bar{x}\|_{\mathcal{X}} \prec [\epsilon,\epsilon] \text{ because } \bar{\boldsymbol{\alpha}} \ominus_{gH} \mathbf{F}(x_0) \preceq \mathbf{0} \\
\implies \quad & \|x_0-\bar{x}\|_{\mathcal{X}} < \frac{\epsilon}{\delta}.
\end{aligned}$$

This completes the proof. □

Note 6. It is to note that if the IVF $\mathbf{F}$ considered in Theorem 2.25 is degenerate IVF, i.e., $\mathbf{F} = \underline{f} = \overline{f}$, then Theorem 2.25 reduces to the conventional Ekeland's variational principle (Theorem 2.23). Hence, Ekeland's variational principle for IVFs (Theorem 2.25) is a true generalization of conventional Ekeland's variational principle (Theorem 2.23).

Example 2.4. *In this example, we verify Theorem 2.25 for the IVF* $\boldsymbol{F} : \mathbb{R}^2 \to \mathbb{R}_I$ *given by*

$$\boldsymbol{F}(x_1,x_2) = \left[|x_1x_2|, e^{|x_1x_2|}\right].$$

It is easy to see that $\underline{f}$ *and* $\overline{f}$ *are lsc, and hence by Note 4,* $\boldsymbol{F}$ *is gH-lsc. Observe that* $\boldsymbol{F}(0,0) = [0,1] \prec \boldsymbol{F}(x_1,x_2)$ *for all* $x_1,x_2 \in \mathbb{R}^2$*. Therefore, by Definition 1.12,* $[0,1]$ *is the infimum of* $\boldsymbol{F}$*. Let* $\bar{x} = (1,1)$ *and* $\epsilon = 2$*. Note that* $\boldsymbol{F}(\bar{x}) = [1,e] \prec [0,1] + [\epsilon,\epsilon] =$

$[2,3]$. *Thus, all the hypotheses of Theorem 2.25 are satisfied. We verify Theorem 2.25, by taking* $\delta = 4$. *For* $x_0 = \left(\frac{3}{4}, \frac{3}{4}\right)$ *observe the following.*

1. $\|x_0 - \bar{x}\| < \frac{\epsilon}{\delta} = \frac{2}{4}$,

2. $\boldsymbol{F}(x_0) = [\frac{9}{16}, e^{\frac{9}{16}}] \preceq [1, e] = \boldsymbol{F}(\bar{x})$.

3. $\underset{x \in \mathbb{R}^2}{\operatorname{argmin}} [|x_1 x_2|, e^{|x_1 x_2|}] \oplus 4\|x - x_0\| = \{x_0\}$.

Similarly, Theorem 2.25 can be verified for other values of δ.

Next, we give an application of EVP for IVFs. In order to do that we need the concept of norm of a bounded linear IVF. By a bounded linear IVF (see [49]), we mean a linear IVF $\mathbf{G} : \mathcal{X} \to \mathbb{R}_I$ for which there exists a nonnegative real number C such that

$$\|\mathbf{G}(x)\|_{\mathbb{R}_I} \leq C\|x\|_{\mathcal{X}} \text{ for all } x \in \mathcal{X}.$$

In the next lemma, we introduce norm for a bounded linear IVF.

Lemma 2.26. (Norm of a bounded linear IVF). *Let* $\boldsymbol{G} : \mathcal{X} \to \mathbb{R}_I$ *be a bounded linear IVF. Then,*

$$\|\boldsymbol{G}\| = \sup_{\substack{x \in \mathcal{X} \\ \|x\|_{\mathcal{X}} = 1}} \|\boldsymbol{G}(x)\|_{\mathbb{R}_I}$$

is a norm on the set of all bounded linear IVFs on $\mathcal{X}$.

Proof. Observe that $\|\mathbf{G}\| \geq 0$ for any bounded linear IVF $\mathbf{G}$ and $\|\mathbf{G}\| = 0$ if and only if $\mathbf{G} = \mathbf{0}$. Let $\gamma \in \mathbb{R}$. We see that

$$\begin{aligned} & \|\gamma \odot \mathbf{G}\| \\ = \; & \sup_{\substack{x \in \mathcal{X} \\ \|x\|_{\mathcal{X}} = 1}} \|(\gamma \odot \mathbf{G})(x)\|_{\mathbb{R}_I} = \sup_{\substack{x \in \mathcal{X} \\ \|x\|_{\mathcal{X}} = 1}} |\gamma| \|\mathbf{G}(x)\|_{\mathbb{R}_I} \end{aligned}$$

$$= |\gamma| \sup_{\substack{x \in \mathcal{X} \\ \|x\|_{\mathcal{X}}=1}} \|\mathbf{G}(x)\|_{\mathbb{R}_I} = |\gamma| \|\mathbf{G}\|.$$

Further,

$$\begin{aligned} \|\mathbf{G_1} \oplus \mathbf{G_2}\| &= \sup_{\substack{x \in \mathcal{X} \\ \|x\|_{\mathcal{X}}=1}} \|(\mathbf{G_1} \oplus \mathbf{G_2})(x)\|_{\mathbb{R}_I} \\ &= \sup_{\substack{x \in \mathcal{X} \\ \|x\|_{\mathcal{X}}=1}} \|\mathbf{G_1}(x) \oplus \mathbf{G_2}(x)\|_{\mathbb{R}_I} \\ &\leq \sup_{\substack{x \in \mathcal{X} \\ \|x\|_{\mathcal{X}}=1}} (\|\mathbf{G_1}(x)\|_{\mathbb{R}_I} + \|\mathbf{G_2}(x)\|_{\mathbb{R}_I}), \text{ by (i) of Lemma 1.3} \\ &= \sup_{\substack{x \in \mathcal{X} \\ \|x\|_{\mathcal{X}}=1}} \|\mathbf{G_1}(x)\|_{\mathbb{R}_I} + \sup_{\substack{x \in \mathcal{X} \\ \|x\|_{\mathcal{X}}=1}} \|\mathbf{G_2}(x)\|_{\mathbb{R}_I} \\ &= \|\mathbf{G_1}\| + \|\mathbf{G_2}\|. \end{aligned}$$

Hence, the result follows. □

Theorem 2.27. *Let $\boldsymbol{G} : \mathcal{X} \to \mathbb{R}_I$ be a linear IVF. If $\boldsymbol{G}$ is gH-continuous on $\mathcal{X}$, then $\boldsymbol{G}$ is a bounded linear IVF.*

Proof. By the hypothesis, $\mathbf{G}$ is gH-continuous at the zero vector of $\mathcal{X}$. Therefore, by Lemma 4.2 in [49], $\mathbf{G}$ is a bounded linear IVF. □

As an application of Theorem 2.25, we give a variational principle for gH-Gâteaux differentiable IVFs.

Theorem 2.28. (Variational principle for gH-Gâteaux differentiable IVFs). *Let $\boldsymbol{F} : \mathcal{X} \to \mathbb{R}_I \cup \{+\infty\}$ be a gH-lsc and gH-Gâteaux differentiable extended IVF, and $\epsilon > 0$. Suppose that*

$$\inf_{\mathcal{X}} \boldsymbol{F} \textit{ is finite and } \boldsymbol{F}(\bar{x}) \prec \inf_{\mathcal{X}} \boldsymbol{F} \oplus [\epsilon, \epsilon].$$

Then, for any $\delta > 0$, there exists an $x_0 \in \mathcal{X}$ such that *(i)* $\|x_0 - \bar{x}\|_{\mathcal{X}} < \frac{\epsilon}{\delta}$,

(ii) $\boldsymbol{F}(x_0) \preceq \boldsymbol{F}(\bar{x})$, *and*

(iii) $\|\boldsymbol{F}_{\mathscr{G}}(x_0)\| \leq \delta$.

Proof. By Theorem 2.25, there exists an $x_0 \in \mathcal{X}$ that satisfies (i) and (ii), and $x_0 \in \operatorname{argmin}_{x \in \mathcal{X}}\{\mathbf{F}(x) \oplus \delta\|x - x_0\|_{\mathcal{X}}\}$. Therefore, $\mathbf{F}(x_0) \preceq \mathbf{F}(x) \oplus \delta\|x - x_0\|_{\mathcal{X}}$ and hence

$$\mathbf{F}(x_0) \ominus_{gH} \delta\|x - x_0\|_{\mathcal{X}} \preceq \mathbf{F}(x). \tag{2.4}$$

Take any $h \in \mathcal{X}$ and set $x = x_0 + th$ in the equation (2.4) with $t > 0$. Then, we get

$$\mathbf{F}(x_0) \ominus_{gH} \delta\|th\|_{\mathcal{X}} \preceq \mathbf{F}(x_0 + th).$$

Thus,

$$-\delta\|h\|_{\mathcal{X}} \preceq \frac{1}{t} \odot \left(\mathbf{F}(x_0 + th) \ominus_{gH} \mathbf{F}(x_0)\right).$$

Letting $t \to 0+$, we get

$$-\delta\|h\|_{\mathcal{X}} \preceq \mathbf{F}_{\mathscr{G}}(x_0)(h).$$

Taking the infimum on both sides over all $h \in \mathcal{X}$ with $\|h\|_{\mathcal{X}} = 1$, we get

$$-\delta \leq -\|\mathbf{F}_{\mathscr{G}}(x_0)\|, \text{ or, } \|\mathbf{F}_{\mathscr{G}}(x_0)\| \leq \delta.$$

This completes the proof. □

The importance of the Theorem 2.28 is that in the absence of points belonging to the set $\operatorname{argmin}_{x \in \mathcal{X}} \mathbf{F}(x)$, we can capture a point x_0 that almost minimizes $\mathbf{F}$. In other words, the equations $\mathbf{F}(x_0) = \inf_{\mathcal{X}} \mathbf{F}$ and $\mathbf{F}_{\mathscr{G}}(x_0) = \mathbf{0}$ can be satisfied to any prescribed accuracy $\delta > 0$.

2.4 Concluding remarks

In this chapter, the concept of gH-semicontinuity has been introduced for IVFs. Their interrelation with gH-continuity has been shown. By using a characterization of gH-lower semicontinuity for IVFs, it has been reported that an extended gH-lsc, level-bounded and proper IVF always attains its minimum. A characterization of the set of argument minimum of an IVF has been provided with the help of gH-Gâteaux differentiability. We have further presented Ekeland's variational principle for IVFs. The proposed Ekeland's variational principle has been applied to find variational principle for gH-Gâteaux differentiable IVFs.

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