Chapter 1

Introduction

Optimization, in simplest terms, is the act of making the best of anything. Broadly speaking, optimization is a set of mathematical principles and techniques for determining the minima/maxima of a function over a set of constraints that express restrictions on the problem. Classical optimization and its various branches now have solid theoretical foundations and are supported by a large collection of advanced algorithms and software. Mathematical optimization, in this day and age, has been transformed into an innovatory tool for powerful modeling and decision-making occurrences in all quantitative disciplines from Computer Science and Engineering to Operations Research and Economics.

Mathematically, an optimization problem comprises of three key ingredients—

- *Decision variable* is a collection of variables that designates a value that can change within the context of the given optimization problem.

- *Constraints* are the logical conditions, allowable values, or scopes for variables in an optimization problem that must be satisfied by the solution to a given problem.

- An *objective function* that expresses the main criteria of the problem is either to be minimized or maximized satisfying the restrictions.

Although this is the era of digital computers, due to the limited representation capability of machines, we often need to round-off the values of the initial data for the sake of calculations. This rounding off causes errors. Sometimes, these errors can be significant and may lead to a wrong or inaccurate conclusion. For instance, in 1991, an inefficient error management algorithm of the guidance system of an American patriot missile battery failed to track and ambush the attacking missile. Also, in 1996, due to overflow, the Ariane 5 rocket launched by the European space agency exploded. It is now known that use of standard interval analysis could have possibly detected these errors in both failures (for details see [80]).

Although there are numerous theories and optimization tools for obtaining optimal solutions to classical optimization problems, due to the error or uncertainty in the given data, it is inadequate to model real-life problems by conventional optimization problems. To handle the uncertainty of the given data, these problems are often modeled by IOPs. Therefore, IOPs provide an alternative means to handle the uncertainty in the optimization problems. Further motivation to give special attention to interval optimization along with stochastic and fuzzy optimization can be found in [111].

1.1 Interval Analysis

When performing mathematical computations, the collected and recorded data frequently contain measurement errors or are otherwise uncertain due to rounding errors and approximations. Making interval bounds is one way to deal with these errors. Contemporary research is focused on developing practical interval algorithms that generate sharp (or nearly sharp) bounds, i.e., interval bounds that can be as narrow as possible, for the solution of numerical computing problems. As a result, it is preferable to gain a better understanding of real interval spaces and interval analysis.

Interval analysis is a new and growing branch of applied mathematics. It gives an idea to computing that treats an interval as a new kind of number. The results produced by the methods of interval analysis with properly-rounded interval arithmetic contain both ordinary machine arithmetic results as well as infinite precision arithmetic results. Thus, we have, at the outset, a completely general mechanism for bounding the accumulation of round off errors in any machine computation. If round off is the only error present, then the widths of the intervals will tend to zero as the length of the machine word increases.

Many real-world problems have imprecise or uncertain knowledge of the underlying parameters that influence the behaviour of the mathematical problems. Generally, one cannot measure the parameters affected by imprecision or uncertainties with exact values. In such cases, a real number cannot be used to determine the parameters. We usually overcome this deficiency by using interval or stochastic values, which is a natural way of incorporating the uncertainties of parameters. The goal of using intervals in such mathematical problems is to provide upper and lower bounds on the parameters.

1.2 Interval-valued Function

Even though intervals are crucial elements in dealing with uncertainty, we are unable to describe the world's various uncertain problems without a proper function whose involving parameters are intervals. A function whose involving parameters are intervals is known as *interval-valued function (IVF)*; an IVF is a mathematical function of one or more variables whose domain is a nonempty subset of a finite dimensional Banach space and whose range is a subset of $\mathbb{R}_I$. For each argument point $x \in \mathcal{X}$, where $\mathcal{X}$ is a finite-dimensional Banach space, an IVF $\mathbf{F} : \mathcal{X} \to \mathbb{R}_I$, is presented by (see [32]):

$$\mathbf{F}(x) = \left[\underline{f}(x),\ \overline{f}(x)\right],$$

where $\underline{f}$ and $\overline{f}$ are real-valued functions on $\mathcal{X}$. The functions $\underline{f}$ and $\overline{f}$ are called the lower and the upper boundary functions of $\mathbf{F}$, respectively.

An IVF can be expressed in another (see [18, 52]) way as:
Let $\widehat{\mathbf{C}} \in \mathbb{R}_I^k$, i.e., $\widehat{\mathbf{C}} = (\mathbf{C}_1,\ \mathbf{C}_2, \ldots, \mathbf{C}_k)^\top$, where $\mathbf{C}_j = [\underline{c_j},\ \overline{c_j}] \in \mathbb{R}_I$ for $j = 1, 2, \cdots, k$. Parametrically, the vector $\widehat{\mathbf{C}}$ is given by the following set

$$\Big\{c(t) \ \Big|\ c(t) = (c_1(t_1), c_2(t_2), \cdots, c_k(t_k))^\top, c_j(t_j) = \underline{c_j} + t_j(\overline{c_j} - \underline{c_j}),$$
$$t = (t_1,\ t_2, \cdots, t_k)^\top,\ 0 \leq t_j \leq 1,\ j = 1, 2, \cdots, k\Big\}.$$

Then an IVF $\mathbf{F}_{\widehat{\mathbf{C}}} : \mathcal{X} \to \mathbb{R}_I$, can be represented as a collection of a bunch of real-valued functions $f_{c(t)}$'s, i.e., for all $x \in \mathcal{X}$,

$$\mathbf{F}_{\widehat{\mathbf{C}}}(x) = \left\{f_{c(t)}(x) | f_{c(t)} : \mathcal{X} \to \mathbb{R},\ c(t) \in \widehat{\mathbf{C}},\ t \in [0,\ 1]^k\right\}.$$

1.3 Interval Optimization Problem

The data collected by decision-makers in real-world problems is always assumed to be real numbers with a specific value. In this scenario, the objective function of optimization problems is a real-valued function. However, there are some optimization problems in which the objective function is uncertain due to imprecise data. For example, suppose that a factory can produce two products, say P_1 and P_2, with input quantities x_1 and x_2, subject to budget constraint $\mathcal{S} \subseteq \mathbb{R}^2$. For selling the products P_1 and P_2 in the market, we assume that the factory can earn c_1 and c_2 dollars for per unit sales of x_1 and x_2 respectively. In this case, the aim is to maximize the objective function $c_1x_1 + c_2x_2$ subject to the budget constraint set $\mathcal{S} \subseteq \mathbb{R}^2$. We know that the prices of products may vary from time to time in the financial market, so it is more reasonable to take the prices to be uncertain quantities in nature. There are three different methodologies available that can model uncertain quantities by using: random variables, fuzzy numbers, and intervals. If the coefficients c_1 and c_2 are random, the problem is transformed into a stochastic optimization problem. Birge and Louveaux in 1997 [20], Kall in 1976 [68], and Vajda in 1972 [103] have explained the main stream of stochastic optimization problems and also introduced some useful methods to solve such optimization problems. If the coefficients c_1 and c_2 are fuzzy in nature, then the problem becomes a fuzzy optimization problem. The stochastic and fuzzy optimization problems are not easy to be tackled. The usual way is to transform them into conventional optimization problems, frequently, these problems are very complicated. On the other hand, if we assume the coefficients c_1 and c_2 to be in the compact intervals of real numbers, then although the prices may fluctuate from time to time, we can always make sure that the prices will fall within the corresponding intervals, in this case, problem becomes an *interval optimization* *problem* (IOP). Under this assumption, the optimization problem will be easier to be solved than a stochastic or fuzzy optimization problem.

As we know that in most of the cases, the coefficients of the objective function in the stochastic optimization problems are considered as random variables with known distributions. However, the specifications of the distributions are very subjective. For example, many researchers invoke the Gaussian (normal) distributions with different parameters in stochastic optimization problems. However very often these specifications do not completely match the real problems. Therefore, the interval-valued optimization problems may provide a better alternative choice to tackle the uncertainty involved in the optimization problems. That is to say, that the coefficients of the objective function in the interval-valued optimization problems are considered to be compact intervals. Although the specifications of compact intervals may still be judged as subjective viewpoint, we can argue that the bounds of the uncertain data (i.e. determining the compact intervals, which give the upper and lower bounds of the possible observed data) are easier to be handled than specified by the Gaussian distributions in the stochastic optimization problems.

Alike conventional optimization problems, IOPs are classified into two categories – unconstraint and constraint.

Let $\mathbf{F} : \mathcal{X} \to \mathbb{R}_I$ be an IVF. An *unconstraint IOP* is defined below.

$$\text{(UIOP):} \qquad \min_{x \in \mathcal{X}} \ \mathbf{F}(x). \tag{1.1}$$

Further, a *constraint IOP* is defined below.

$$\text{(CIOP):} \qquad \min_{x \in \mathcal{S}} \ \mathbf{F}(x), \tag{1.2}$$

where $\mathcal{S} = \{x \in \mathcal{X} \mid \mathbf{G}_j(x) \preceq \mathbf{0} \text{ for all } j \in \mathcal{J},\ \mathbf{H}_l(x) = \mathbf{0} \text{ for all } k \in \mathcal{K}\}$, $\mathbf{G}_j$'s are IVFs on $\mathcal{X}$ for each $j \in \mathcal{J} = \{1, 2, \ldots, p\}$, and $\mathbf{H}_l$'s are IVFs on $\mathcal{X}$ for each $k \in \mathcal{K} = \{1, 2, \ldots, q\}$.

Unlike $\mathbb{R}$, the set of compact intervals $\mathbb{R}_I$ is not thoroughly ordered, which will be explored in the next chapter, the solutions of IOPs can only be ordered partially. The basic difference between IOPs and conventional optimization problems is that in the case of conventional optimization problems, only one global optimum exists; however, in the case of IOPs, conflicting real objectives can cause a situation where no solution is superior to the others. Thus, usually, there are many solutions to an IOP. The feasible solutions which can be improved without causing simultaneous deterioration in at least one criterion can not certainly be the optimal solution of the considered IOP. This concept consequently leads to the foundation of efficient solutions and nondominated solutions.

Definition 1.1. (*Efficient solution* [46]). We say $\bar{x} \in \mathcal{X}$ as an efficient solution (ES) of UIOP (1.1) if there is no $x \neq \bar{x}$ in $\mathcal{X}$ satisfying $\mathbf{F}(x) \prec \mathbf{F}(\bar{x})$.

Definition 1.2. (*Nondominated solution*). If $\bar{x} \in \mathcal{X}$ is an ES of the UIOP (1.1), then $\mathbf{F}(\bar{x})$ is said to be a nondominated solution of the UIOP (1.1).

1.4 Preliminaries

The following basic definitions and basic properties of intervals are used throughout this thesis.

1.4.1 Interval Arithmetic

Throughout the thesis, the bold letters $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots$, are used for denoting the elements of $\mathbb{R}_I$ and an element $\mathbf{J}$ of $\mathbb{R}_I$ is represented by the corresponding small letter: $\mathbf{J} = [\underline{j}, \overline{j}]$.

Arithmetic of intervals is a foundational tool in interval analysis. In the first place, we study interval arithmetic proposed by Moore [81]. Subsequently, we study the concepts of gH-difference and ordering for intervals [63].

For intervals $\mathbf{J} = [\underline{j}, \overline{j}]$ and $\mathbf{K} = [\underline{k}, \overline{k}]$. The *addition* of $\mathbf{J}$ and $\mathbf{K}$, denoted $\mathbf{J} \oplus \mathbf{K}$, is given as

$$\mathbf{J} \oplus \mathbf{K} = \left[\ \underline{j} + \underline{k},\ \overline{j} + \overline{k}\ \right].$$

The *subtraction* of $\mathbf{K}$ from $\mathbf{J}$, denoted $\mathbf{J} \ominus \mathbf{K}$, is given as

$$\mathbf{J} \ominus \mathbf{K} = \left[\ \underline{j} - \overline{k},\ \overline{j} - \underline{k}\ \right].$$

The *multiplication* of $\mathbf{J}$ and $\mathbf{K}$, denoted by $\mathbf{J} \odot \mathbf{K}$, is defined by

$$\mathbf{J} \odot \mathbf{K} = \left[\min\left\{\underline{j}\ \underline{k},\ \underline{j}\overline{k},\ \overline{j}\underline{k},\ \overline{j}\overline{k}\right\},\ \max\left\{\underline{j}\ \underline{k},\ \underline{j}\overline{k},\ \overline{j}\underline{k},\ \overline{j}\overline{k}\right\}\right].$$

The *multiplication* by a real number τ to $\mathbf{J}$, denoted $\tau \odot \mathbf{J}$ or $\mathbf{J} \odot \tau$, is given by

$$\tau \odot \mathbf{J} = \mathbf{J} \odot \tau = \begin{cases} [\tau\underline{j},\ \tau\overline{j}], & \text{if } \tau \geq 0 \\ [\tau\overline{j},\ \tau\underline{j}], & \text{if } \tau < 0. \end{cases}$$

Note that the definition of $\tau \odot \mathbf{J}$ follows from the fact $\tau = [\tau, \tau]$ and the definition of multiplication $\mathbf{J} \odot \mathbf{K}$. We use the following definition for the difference between a pair of intervals being the most general definition of difference (see [52] for the reason why it is most general).

Definition 1.4.1 (gH-difference of intervals [97]). Let $\mathbf{J}$, $\mathbf{K} \in \mathbb{R}_I$. The gH-difference of $\mathbf{J}$ and $\mathbf{K}$, denoted by $\mathbf{J} \ominus_{gH} \mathbf{K}$, is given as

$$\mathbf{J} \ominus_{gH} \mathbf{K} = \left[\min\left\{\underline{j} - \underline{k}, \overline{j} - \overline{k}\right\},\ \max\left\{\underline{j} - \underline{k}, \overline{j} - \overline{k}\right\}\right].$$

It is obvious that unlike the real numbers, intervals are not linearly ordered. Therefore, in order to develop the analysis of IVFs and interval optimization, we use the following order relation in this thesis.

Definition 1.4.2 (Dominance of intervals [18]). Let $\mathbf{J}$, $\mathbf{K} \in \mathbb{R}_I$. Then,

(i) $\mathbf{K}$ is called dominated by $\mathbf{J}$ if $\underline{j} \leq \underline{k}$ and $\overline{j} \leq \overline{k}$, and we write $\mathbf{J} \preceq \mathbf{K}$;

(ii) $\mathbf{K}$ is called strictly dominated by $\mathbf{J}$ if $\mathbf{J} \preceq \mathbf{K}$ and $\mathbf{J} \neq \mathbf{K}$, and then we write $\mathbf{J} \prec \mathbf{K}$. Equivalently, $\mathbf{J} \prec \mathbf{K}$ if and only if any of the these cases hold:

- Case 1. $\underline{j} < \underline{k}$ and $\overline{j} \leq \overline{k}$,
- Case 2. $\underline{j} \leq \underline{k}$ and $\overline{j} < \overline{k}$,
- Case 3. $\underline{j} < \underline{k}$ and $\overline{j} < \overline{k}$;

(iii) if neither $\mathbf{J} \preceq \mathbf{K}$ nor $\mathbf{K} \preceq \mathbf{J}$, then none of $\mathbf{J}$ and $\mathbf{K}$ dominates the other, or $\mathbf{J}$ and $\mathbf{K}$ are not comparable. Identically, $\mathbf{J}$ and $\mathbf{K}$ are not comparable if either '$\underline{j} < \underline{k}$ and $\overline{j} > \overline{k}$' or '$\underline{j} > \underline{k}$ and $\overline{j} < \overline{k}$';

(iv) $\mathbf{K}$ is said to be not dominated by $\mathbf{J}$ if either $\mathbf{K} \preceq \mathbf{J}$ or $\mathbf{J}$ and $\mathbf{K}$ are not comparable, and then we write $\mathbf{J} \nprec \mathbf{K}$.

$(\mathbb{R}_I, \|.\|_{\mathbb{R}_I})$ is a normed quasilinear space (see [75]) with operations $\{\oplus, \ominus_{gH}, \odot\}$, where $\|.\|_{\mathbb{R}_I}$ is defined as following.

Definition 1.4.3 (Norm on $\mathbb{R}_I$ [79]). For an $\mathbf{J} = [\underline{j}, \bar{j}]$ in $\mathbb{R}_I$, the function $\|.\|_{\mathbb{R}_I} : \mathbb{R}_I \to \mathbb{R}^+$, given as

$$\|\mathbf{J}\|_{\mathbb{R}_I} = \max\{|\underline{j}|, |\bar{j}|\},$$

is said to a norm on $\mathbb{R}_I$. In rest of thesis, we use the notation '$\|\cdot\|$' to represent Euclidean norm on $\mathbb{R}^n$.

1.4.2 Basic Properties of Interval Analysis

The following basic properties of interval analysis hold with the help of dominance relation of intervals, norm of a interval, and gH-difference of two intervals, and these are used throughout the thesis.

Lemma 1.3. *Let* $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}, \boldsymbol{D} \in \mathbb{R}_I$. *Then,*

(i) $\|\boldsymbol{A} \oplus \boldsymbol{B}\|_{\mathbb{R}_I} \leq \|\boldsymbol{A}\|_{\mathbb{R}_I} + \|\boldsymbol{B}\|_{\mathbb{R}_I}$,

(ii) if $\boldsymbol{A} \preceq \boldsymbol{C}$ *and* $\boldsymbol{B} \preceq \boldsymbol{D}$, *then* $\boldsymbol{A} \oplus \boldsymbol{B} \preceq \boldsymbol{C} \oplus \boldsymbol{D}$.

Proof. See Appendix A.1. □

Lemma 1.4. *For* $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D} \in \mathbb{R}_I$ *and* $\epsilon > 0$, *we have*

(i) $\|\boldsymbol{A} \ominus_{gH} \boldsymbol{B}\|_{\mathbb{R}_I} < \epsilon \iff \boldsymbol{B} \ominus_{gH} [\epsilon, \epsilon] \prec \boldsymbol{A} \prec \boldsymbol{B} \oplus [\epsilon, \epsilon]$,

(ii) $\boldsymbol{A} \ominus_{gH} [\epsilon, \epsilon] \nprec \boldsymbol{B} \implies \boldsymbol{A} \npreceq \boldsymbol{B}$.

Proof. See Appendix A.2. □

Lemma 1.5. (See [80]). *For* $\boldsymbol{A} \in \mathbb{R}_I$ *and* $\alpha, \beta \in \mathbb{R}$,

$$\boldsymbol{A} \odot (\alpha + \beta) \subseteq \boldsymbol{A} \odot \alpha \oplus \boldsymbol{A} \odot \beta.$$

Lemma 1.6. (See [47]). *For* $\boldsymbol{A},\ \boldsymbol{B} \in \mathbb{R}_I$, $\boldsymbol{A} \preceq \boldsymbol{B} \iff \boldsymbol{A} \ominus_{gH} \boldsymbol{B} \preceq \boldsymbol{0}$.

Lemma 1.7. *For* $\alpha \in [0,1]$ *and* $\boldsymbol{A} \in \mathbb{R}_I, (1-\alpha) \odot \boldsymbol{A} = \boldsymbol{A} \ominus_{gH} \alpha \odot \boldsymbol{A}$.

Proof. Let $\mathbf{A} = [\underline{a}, \overline{a}]$. Then, $(1-\alpha) \odot \mathbf{A} = [(1-\alpha)\underline{a}, (1-\alpha)\overline{a}]$ as $1-\alpha \geq 0$. Also,

$$\begin{aligned} \mathbf{A} \ominus_{gH} \alpha \odot \mathbf{A} &= [\min\{\underline{a} - \alpha\underline{a}, \overline{a} - \alpha\overline{a}\}, \max\{\underline{a} - \alpha\underline{a}, \overline{a} - \alpha\overline{a}\}] \\ &= [\min\{(1-\alpha)\underline{a}, (1-\alpha)\overline{a}\}, \max\{(1-\alpha)\underline{a}, (1-\alpha)\overline{a}\}] \\ &= [(1-\alpha)\underline{a}, (1-\alpha)\overline{a}] \text{ because } 1-\alpha \geq 0. \end{aligned}$$

□

Lemma 1.8. *For* $\boldsymbol{A},\ \boldsymbol{B} \in \mathbb{R}_I$, *we have*

(i) $\boldsymbol{0} \preceq \boldsymbol{A} \ominus_{gH} \boldsymbol{B} \iff \boldsymbol{B} \preceq \boldsymbol{A}$,

(ii) $-1 \odot (\boldsymbol{A} \ominus_{gH} \boldsymbol{B}) = \boldsymbol{B} \ominus_{gH} \boldsymbol{A}$.

Proof. See Appendix A.3. □

1.4.3 Sequence of Intervals

Definition 1.9. (*Infimum of a subset of* $\overline{\mathbb{R}_I}$). Let $\mathbf{X} \subseteq \overline{\mathbb{R}_I}$. We say, an interval $\bar{\mathbf{P}} \in \mathbb{R}_I$ a lower bound of $\mathbf{X}$ if $\bar{\mathbf{P}} \preceq \mathbf{K}$ for all $\mathbf{K}$ in $\mathbf{X}$. A lower bound $\bar{\mathbf{P}}$ of $\mathbf{X}$ is called an infimum of $\mathbf{X}$ if for all lower bounds $\mathbf{R}$ of $\mathbf{X}$ in $\mathbb{R}_I$, $\mathbf{R} \preceq \bar{\mathbf{P}}$. We denote infimum of $\mathbf{X}$ by $\inf \mathbf{X}$.

Example 1.1. *Let* $\boldsymbol{X} = \left\{\left[\frac{1}{n}, 1\right] : n \in \mathbb{N}\right\}$. *The set of lower bounds of* $\boldsymbol{X}$ *is*

$$\{[\alpha, \beta] : -\infty < \alpha \leq 0 \ \textit{and} \ -\infty < \beta \leq 1\}.$$

Therefore, the infimum of $\boldsymbol{X}$ *is* $[0,1]$ *because* $[\alpha,\beta] \preceq [0,1]$ *for all* $-\infty < \alpha \leq 0$ *and* $-\infty < \beta \leq 1$.

Definition 1.10. (*Supremum of a subset of* $\overline{\mathbb{R}_I}$). Let $\mathbf{X} \subseteq \overline{\mathbb{R}_I}$. We say, an interval $\bar{\mathbf{P}} \in \mathbb{R}_I$ an upper bound of $\mathbf{X}$ if $\mathbf{K} \preceq \bar{\mathbf{A}}$ for all $\mathbf{K}$ in $\mathbf{X}$. An upper bound $\bar{\mathbf{P}}$ of $\mathbf{X}$ is called a supremum of $\mathbf{X}$ if for all upper bounds $\mathbf{R}$ of $\mathbf{X}$ in $\mathbb{R}_I$, $\bar{\mathbf{P}} \preceq \mathbf{R}$. We denote supremum of $\mathbf{X}$ by $\sup \mathbf{X}$.

Example 1.2. *Let* $\boldsymbol{X} = \left\{\left[\frac{1}{n^2}+1, 3\right] : n \in \mathbb{N}\right\}$. *The set of upper bounds of* $\boldsymbol{X}$ *is*

$$\{[\alpha,\beta] : 2 \leq \alpha < +\infty \text{ and } 3 \leq \beta < +\infty\}.$$

Therefore, the supremum of $\boldsymbol{X}$ *is* $[2,3]$ *because* $[2,3] \preceq [\alpha,\beta]$ *for all* $2 \leq \alpha < +\infty$ *and* $3 \leq \beta < +\infty$.

Remark 1.11. Let $\mathbf{X} = \left\{[p_\alpha, q_\alpha] \in \overline{\mathbb{R}_I} : \alpha \in \Gamma \text{ and } \Gamma \text{ being an index set }\right\}$. Therefore, by Definitions 1.9 and 1.10, we get $\inf \mathbf{X} = \left[\inf_{\alpha\in\Gamma} p_\alpha,\ \inf_{\alpha\in\Gamma} q_\alpha\right]$ and $\sup \mathbf{X} = \left[\sup_{\alpha\in\Gamma} p_\alpha,\ \sup_{\alpha\in\Gamma} q_\alpha\right]$. It is easy to see that if $\inf \mathbf{X}$ and $\sup \mathbf{X}$ exist for an $\mathbf{X}$, then they are unique.

Definition 1.12. (*Infimum of an IVF*). Let $\mathcal{S}(\neq \emptyset) \subseteq \mathcal{X}$ and $\mathbf{F} : \mathcal{S} \to \overline{\mathbb{R}_I}$ be an extended IVF. Then infimum of $\mathbf{F}$, denoted as $\inf_{\mathcal{S}} \mathbf{F}$, is defined by

$$\inf_{\mathcal{S}} \mathbf{F} = \inf\{\mathbf{F}(x) : x \in \mathcal{S}\}.$$

Definition 1.13. (*Sequence in* $\mathbb{R}_I$). An IVF $\mathbf{I}$ defined on the set of naturals is called a sequence in $\mathbb{R}_I$. We denote a sequence $\mathbf{I}$ by $\{\mathbf{I}(n)\}$.

Example 1.3. *(i)* $\boldsymbol{F} : \mathbb{N} \to \mathbb{R}_I$ *given as* $\boldsymbol{F}(n) = [n, n+1]$ *is a sequence.*

(ii) $\boldsymbol{F} : \mathbb{N} \to \mathbb{R}_I$ *given as* $\boldsymbol{F}(n) = \left[\frac{n}{4}, \frac{n}{2}\right]$ *is also a sequence.*

Definition 1.14. (*Convergent sequence in* $\mathbb{R}_I$). A sequence $\{\mathbf{I}(n)\}$ is called convergent to $\mathbf{U} \in \mathbb{R}_I$ if for each $\epsilon > 0$, there exists an integer $m > 0$ satisfying

$$\|\mathbf{I}(n) \ominus_{gH} \mathbf{U}\|_{\mathbb{R}_I} < \epsilon \text{ for all } n \geq m.$$

The interval $\mathbf{U}$ is known as the limit of the sequence $\{\mathbf{I}(n)\}$ and it is denoted by $\lim\limits_{n\to+\infty} \mathbf{I}(n) = \mathbf{U}$ or $\mathbf{I}(n) \to \mathbf{U}$.

We need the following definition of floor function in the next example.

Definition 1.15. (See [82]). Given any real number x, the floor function denoted by $\lfloor x \rfloor$ is defined as

$$\lfloor x \rfloor = \max\{m \in \mathbb{Z} | m \leq x\}.$$

Example 1.4. *Take the sequence* $\boldsymbol{I}(n) = \left[\frac{1}{n}, 1\right]$, $n \in \mathbb{N}$, *in* $\mathbb{R}_I$.
Let $\epsilon > 0$ *be given. Note that*

$$\|\boldsymbol{I}(n) \ominus_{gH} [0,1]\|_{\mathbb{R}_I} = \left\| \left[\frac{1}{n}, 1\right] \ominus_{gH} [0,1] \right\|_{\mathbb{R}_I} = \left\| \left[0, \frac{1}{n}\right] \right\|_{\mathbb{R}_I} = \frac{1}{n} < \epsilon \textit{ whenever } n > \frac{1}{\epsilon}.$$

So, by taking $m = \lfloor \frac{1}{\epsilon} \rfloor + 1$, *where* $\lfloor \cdot \rfloor$ *is the floor function, we get*

$$\|\boldsymbol{I}(n) \ominus_{gH} [0,1]\|_{\mathbb{R}_I} < \epsilon \textit{ for all } n \geq m.$$

Thus, $\lim\limits_{n\to+\infty} \boldsymbol{I}(n) = \boldsymbol{U} = [0,1]$.

Note 1. Consider a sequence $\{\mathbf{I}(n)\}$ in $\mathbb{R}_I$ with $\mathbf{I}(n) = \left[\underline{i}(n), \overline{i}(n)\right]$, where $\{\underline{i}(n)\}$ and $\left\{\overline{i}(n)\right\}$ are two convergent sequences in $\mathbb{R}$. Then, $\{\mathbf{I}(n)\}$ is convergent and

$$\lim_{n\to+\infty} \mathbf{I}(n) = \left[\lim_{n\to+\infty} \underline{i}(n), \lim_{n\to+\infty} \overline{i}(n)\right].$$

The reason is as follows.

Suppose $\underline{i}(n)$ and $\bar{i}(n)$ are convergent sequences with limits u_1 and u_2, respectively. Then, for each $\epsilon > 0$, we have two positive integers m_1 and m_2 satisfying

$$\begin{aligned} & |\underline{i}(n) - u_1| < \epsilon \text{ for all } n \geq m_1, \text{ and } \left|\bar{i}(n) - u_2\right| < \epsilon \text{ for all } n \geq m_2 \\ \iff & \max\left\{|\underline{i}(n) - u_1|, \left|\bar{i}(n) - u_2\right|\right\} < \epsilon \text{ for all } n \geq m, \text{ where } m = \max\{m_1, m_2\} \\ \iff & \left\|\left[\underline{i}(n), \bar{i}(n)\right] \ominus_{gH} [u_1, u_2]\right\|_{\mathbb{R}_I} < \epsilon \text{ for all } n \geq m \\ \text{i.e.,} \quad & \|\mathbf{I}(n) \ominus_{gH} [u_1, u_2]\|_{\mathbb{R}_I} < \epsilon \text{ for all } n \geq m. \end{aligned}$$

Hence,

$$\lim_{n\to+\infty} \mathbf{I}(n) = [u_1, u_2] = \left[\lim_{n\to+\infty} \underline{i}(n), \lim_{n\to+\infty} \bar{i}(n)\right].$$

Definition 1.16. (*Bounded sequence in* $\mathbb{R}_I$). A sequence $\{\mathbf{I}(n)\}$ is called bounded above if there is an interval $\mathbf{K}_1 \in \mathbb{R}_I$ satisfying

$$\mathbf{I}(n) \preceq \mathbf{K}_1 \text{ for all } n \in \mathbb{N}.$$

$\{\mathbf{I}(n)\}$ is bounded below if there is an interval $\mathbf{K}_2 \in \mathbb{R}_I$ satisfying

$$\mathbf{K}_2 \preceq \mathbf{I}(n) \text{ for all } n \in \mathbb{N}.$$

$\{\mathbf{I}(n)\}$ is bounded if it is both bounded above and below.

Definition 1.17. $\{\mathbf{I}(n)\}$ is called monotonic increasing if $\mathbf{I}(n) \preceq \mathbf{I}(n+1)$ for all $n \in \mathbb{N}$.

Lemma 1.18. *A bounded above monotonic increasing sequence of intervals is convergent and converges to its supremum.*

Proof. Let $\{\mathbf{I}(n)\}$ be a bounded above monotonic increasing sequence with supremum $\mathbf{M}$.

By Definition 1.10, we get

(i) $\mathbf{I}(n) \preceq \mathbf{M}$ for all $n \in \mathbb{N}$ and

(ii) for a given $\epsilon > 0$, there is an integer $m > 0$ satisfying $\mathbf{M} \ominus_{gH} [\epsilon, \epsilon] \prec \mathbf{I}(m)$.

Since $\{\mathbf{I}(n)\}$ is a monotonic increasing,

$$\mathbf{M} \ominus_{gH} [\epsilon, \epsilon] \prec \mathbf{I}(m) \preceq \mathbf{I}(m+1) \preceq \mathbf{I}(m+2) \preceq \cdots \preceq \mathbf{M}.$$

That is, $\mathbf{M} \ominus_{gH} [\epsilon, \epsilon] \prec \mathbf{I}(m) \prec \mathbf{M} \oplus [\epsilon, \epsilon]$ for all $n \geq m$. Therefore, $\{\mathbf{I}(n)\}$ is convergent and $\lim_{n \to +\infty} \mathbf{I}(n) = \mathbf{M}$. □

1.4.4 Few Elements of Convex Analysis

Note 2. *(See [82]).* For any $\bar{x} \in X$ and $S \subseteq X$, we have

$$\partial_f \operatorname{dist}(\bar{x}, S) = \widehat{N}(\bar{x}, S) \cap \mathbb{B},$$

where $\partial_f \operatorname{dist}(\bar{x}, S)$ represents the Fréchet subgradient of the distance function $\operatorname{dist}(\bar{x}, S)$ and $\widehat{N}(\bar{x}, S)$ denotes the Fréchet normal cone to S at $\bar{x}$, defined by

$$\widehat{N}(\bar{x}, S) = \left\{ y \in \mathbb{R}^n : \limsup_{x \xrightarrow{S} \bar{x}} \frac{y^\top (x - \bar{x})}{\|x - \bar{x}\|} \leq 0 \right\},$$

where $x \xrightarrow{S} \bar{x}$ means that $x \to \bar{x}$ with $x \in S$.

1.4.5 Some Basic Definitions and Properties of Interval-valued Functions

This section presents some basic definitions for interval-valued functions which are used throughout the thesis.

Definition 1.4.4 (Convex IVF [110]). Let $W \subseteq \mathbb{R}^n$ be a convex set. An IVF $\mathbf{I} : W \to \mathbb{R}_I$ is called convex on W if for any t_1 and t_2 in W,

$$\mathbf{I}(\gamma_1 t_1 + \gamma_2 t_2) \preceq \gamma_1 \odot \mathbf{I}(t_1) \oplus \gamma_2 \odot \mathbf{I}(t_2) \text{ for all } \gamma_1, \gamma_2 \in [0, 1] \text{ with } \gamma_1 + \gamma_2 = 1.$$

Lemma 1.19 (See [110]). *An IVF $\boldsymbol{I}(t) = [\underline{i}(t), \overline{i}(t)]$ is convex on W if and only if $\underline{i}$ and $\overline{i}$ are convex on W.*

Definition 1.20. (*Proper IVF* [72]). An IVF $\mathbf{I} : W \to \overline{\mathbb{R}_I}$ is called a proper function if there exists an $\bar{x} \in X$ satisfying $\mathbf{I}(\bar{x}) \prec [+\infty, +\infty]$ and $[-\infty, -\infty] \prec \mathbf{I}(x)$ for all $x \in W$.

Definition 1.21. (*Linear IVF* [49]). An IVF $\mathbf{I} : W \to \mathbb{R}_I$ is called linear if it satisfies

(i) $\mathbf{I}(\gamma\ t) = \gamma \odot \mathbf{I}(t)$ for all $t \in W$ and $\gamma \in \mathbb{R}$, and

(ii) for all $t_1,\ t_2 \in W$,

either $\mathbf{I}(t_1) \oplus \mathbf{I}(t_2) = \mathbf{I}(t_1 + t_2)$ or 'none of $\mathbf{I}(t_1) \oplus \mathbf{I}(t_2)$ and $\mathbf{I}(t_1 + t_2)$ dominates the other'.

Definition 1.22. (*gH-limit of an IVF* [110]). Let $\mathbf{I} : W \to \mathbb{R}_I$ be a proper IVF. The function $\mathbf{I}$ is called tending to a limit $\mathbf{U} \in \mathbb{R}_I$ as t tends to $\bar{t}$, denoted by $\lim_{t \to \bar{t}} \mathbf{I}(t)$, if for each $\epsilon > 0$, we have a $\delta > 0$ satisfying

$$\|\mathbf{I}(t) \ominus_{gH} \mathbf{U}\|_{\mathbb{R}_I} < \epsilon \text{ whenever } 0 < \|t - \bar{t}\| < \delta.$$

Definition 1.23. (*gH-continuity* [110]). Let $\mathbf{I} : W \to \mathbb{R}_I$ be an IVF. The function $\mathbf{I}$ is called gH-continuous at $\bar{t} \in W$ if for each $\epsilon > 0,$ we have a $\delta > 0$ satisfying

$$\|\mathbf{I}(t) \ominus_{gH} \mathbf{I}(\bar{t})\|_{\mathbb{R}_I} < \epsilon \text{ whenever } \|t - \bar{t}\| < \delta.$$

Definition 1.24. (*gH-derivative* [26]). Let $W \subseteq \mathbb{R}$. The gH-derivative of a proper extended IVF $\mathbf{I} : W \to \overline{\mathbb{R}_I}$ at $\bar{x} \in W$ is defined by

$$\mathbf{I}'(\bar{x}) = \lim_{\tau \to 0} \frac{1}{\tau} \odot (\mathbf{I}(\bar{x} + \tau) \ominus_{gH} \mathbf{I}(\bar{x})), \text{ provided the limit exists.}$$

Definition 1.25. (*gH-partial derivative* [90]).For an interior point $t_0 = (t_1^0, t_2^0, \ldots, t_n^0)$ of W and $h = (h_1, h_2, \ldots, h_n) \in \mathbb{R}^n$ with $t_0 + h \in W$. Define a function

$$\mathbf{\Phi}_i(x_i) = \mathbf{I}(t_1^0, t_2^0, \ldots, t_{i-1}^0, t_i, t_{i+1}^0, \ldots, t_n^0).$$

If the following limit

$$\lim_{h_i \to 0} \frac{1}{h_i} \odot (\mathbf{\Phi}_i(t_i^0 + h_i) \ominus_{gH} \mathbf{\Phi}_i(t_i^0))$$

exists, then $\mathbf{I}$ has the ith partial derivative at t_0, and is represented by $D_i\mathbf{I}(t_0), i = 1, 2, \ldots, n$.

Note 3. Observe that at t_0, for any $i = 1, 2, \ldots, n$, the partial derivatives of $\mathbf{I}$ are given by

$$D_i\mathbf{I}(t_0) = \left[\min\left\{\frac{\partial \underline{i}}{\partial t_i}(t_0), \frac{\partial \bar{i}}{\partial t_i}(t_0)\right\}, \max\left\{\frac{\partial \underline{i}}{\partial t_i}(t_0), \frac{\partial \bar{i}}{\partial t_i}(t_0)\right\}\right].$$

Definition 1.26. (*gH-gradient* [90]). For function $\mathbf{I}$, the gH-gradient of $\mathbf{I}$ at a point $t_0 \in W$ is given by the vector

$$(D_1\mathbf{I}(t_0), D_2\mathbf{I}(t_0), \ldots, D_n\mathbf{I}(t_0)).$$

This gH-gradient is denoted by $\nabla\mathbf{I}(t_0)$.

Definition 1.27. (*gH-differentiability* [90]). An IVF $\mathbf{I} : W \to \mathbb{R}_I$ is called gH-differentiable at $t_0 \in W$ if $\nabla\mathbf{I}(t_0)$ exists and

$$\lim_{\|h\|\to 0} \frac{\mathbf{I}(\bar{t}+h) \ominus_{gH} \mathbf{I}(\bar{t}) \ominus_{gH} h^\top \odot \nabla\mathbf{I}(t_0)}{\|h\|} = \mathbf{0}.$$

Definition 1.28. The effective domain of an extended IVF $\mathbf{I} : X \to \overline{\mathbb{R}_I}$ denoted by $\text{dom}(\mathbf{I})$ is defined by

$$\text{dom}(\mathbf{I}) = \Big\{x \in X : \mathbf{I}(x) \prec [+\infty, +\infty]\Big\}.$$

Definition 1.29. (*gH-subgradients of convex IVFs* [50]). Let $\mathbf{I} : W \subseteq \mathbb{R}^n \to \overline{\mathbb{R}_I}$ be a proper convex IVF and $\bar{x} \in \text{dom}(\mathbf{I})$. Then, gH-subdifferential of $\mathbf{I}$ at $\bar{x}$, denoted by $\boldsymbol{\partial}\mathbf{I}(\bar{x})$ is defined by

$$\boldsymbol{\partial}\mathbf{I}(\bar{x}) = \Big\{\widehat{\mathbf{G}} \in \mathbb{R}_I^n : (x-\bar{x})^\top \odot \widehat{\mathbf{G}} \preceq \mathbf{I}(x) \ominus_{gH} \mathbf{I}(\bar{x}) \text{ for all } x \in W\Big\}. \tag{1.3}$$

The elements of (1.3) are known as gH-subgradients of $\mathbf{I}$ at $\bar{x}$.

Definition 1.30. (*Weak sharp minima (WSM) for an IVF* [71]). Let $\mathbf{I} : \mathbb{R}^n \to \overline{\mathbb{R}_I}$ be a gH-lsc and convex IVF. Let $\bar{S}$ and S be two nonempty closed convex sets satisfying $\bar{S} \subseteq S \subseteq \mathbb{R}^n$. Further, let $\text{dom}(\mathbf{I}) \cap S \neq \emptyset$. Then, $\bar{S}$ is a set of WSM of $\mathbf{I}$

over the set S with modulus $\gamma > 0$ if

$$\mathbf{I}(\bar{x}) \oplus \gamma\, \text{dist}(x, \bar{S}) \preceq \mathbf{I}(x) \text{ for all } \bar{x} \in \bar{S} \text{ and } x \in S,$$

where

$$\text{dist}(x, \bar{S}) = \inf_{\bar{x} \in \bar{S}} \|x - \bar{x}\|$$

is the distance fiction.

Lemma 1.31. *Let $\boldsymbol{F} : [a, b] \to \mathbb{R}_I$ be a gH-differentiable IVF such that $\bar{x} \in [a, b]$ is a weak efficient solution of the IOP*

$$\min_{x \in [a,b]} \boldsymbol{F}(x). \tag{1.4}$$

Then, we have the following three cases:

(i) if $a < \bar{x} < b$, then $\boldsymbol{0} \in \nabla \boldsymbol{F}(\bar{x})$;

(ii) if $\bar{x} = a$, then $\boldsymbol{0} \preceq \nabla \boldsymbol{F}(\bar{x})$; and

(iii) if $\bar{x} = b$, then $\nabla \boldsymbol{F}(\bar{x}) \preceq \boldsymbol{0}$.

Proof. See Appendix B.1. □

Theorem 1.32. (Mean value theorem for IVFs [77]). *If the IVF $\boldsymbol{F}$ is gH-continuous in $\triangle = [\alpha, \beta]$ and gH-differentiable in (α, β), then $\boldsymbol{F}(\beta) \ominus_{gH} \boldsymbol{F}(\alpha) \subseteq \boldsymbol{F}'(\triangle) \odot (\beta - \alpha)$, where $\boldsymbol{F}'(\triangle) = \bigcup_{\xi \in \triangle} \boldsymbol{F}'(\xi)$.*

The following lemma is an immediate consequence of Theorem 1.32.

Lemma 1.33. (See [77]) *If the IVF $\boldsymbol{F}$ is gH-continuous in $\triangle = [\alpha, \beta]$ and gH-differentiable in (α, β) with $\boldsymbol{0} \preceq \boldsymbol{F}'(x)$ for all $x \in (\alpha, \beta)$, then $\boldsymbol{F}$ is monotonic increasing.*

Definition 1.34. (*gH-Gâteaux derivative* [49]). Let W be a nonempty open subset of $\mathbb{R}^n$ and $\mathbf{I}$ be an IVF on W. If at $\bar{x} \in W$, the limit

$$\mathbf{I}_{\mathscr{G}}(\bar{x})(h) := \lim_{\lambda \to 0+} \frac{1}{\lambda} \odot (\mathbf{I}(\bar{x} + \lambda h) \ominus_{gH} \mathbf{I}(\bar{x}))$$

exists for all $h \in \mathbb{R}^n$ and $\mathbf{I}_{\mathscr{G}}(\bar{x})$ is a gH-continuous linear IVF from $\mathbb{R}^n$ to $\mathbb{R}_I$, then $\mathbf{I}_{\mathscr{G}}(\bar{x})$ is called gH-Gâteaux derivative of $\mathbf{I}$ at $\bar{x}$. If $\mathbf{I}$ has a gH-Gâteaux derivative at $\bar{x}$, then $\mathbf{I}_{\mathscr{G}}(\bar{x})$ is called gH-Gâteaux differentiable at $\bar{x}$.

Theorem 1.35. (Characterization of efficient points [49]). *Let W be a nonempty subset of $\mathbb{R}^n$, $\boldsymbol{I} : W \to \mathbb{R}_I$ be an IVF, and $\bar{x} \in W$ be an efficient point of the IOP (2.1). If the function $\boldsymbol{I}$ has a gH-directional derivative at $\bar{x}$ in the direction $x - \bar{x}$ for any $x \in W$, then*

$$\boldsymbol{I}_{\mathscr{D}}(\bar{x})(x - \bar{x}) \nprec \boldsymbol{0} \text{ for all } x \in W. \tag{1.5}$$

We call $\mathbf{T}$ as an interval vector-valued function (IVVF) if $\mathbf{T} : X \to \mathbb{R}_I^n$ is such that for each $x \in X$, $\mathbf{T}(x)$ is an n-tuple interval vector $(\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_n)^\top$, where $\mathbf{C}_i$ is an element of $\mathbb{R}_I$ for $i = 1, 2, \ldots, n$.

Observe that any vector-valued function $T : X \to \mathbb{R}^n$ can be regarded as an interval vector-valued function.

Definition 1.36. (See [52]). For $t = (t_1, t_2, \ldots, t_n) \in \mathbb{R}^n$ and $\widehat{\mathbf{J}} = (\mathbf{J}_1, \mathbf{J}_2, \ldots, \mathbf{J}_n) \in \mathbb{R}_I^n$ with $\mathbf{J}_i = [\underline{j}_i, \overline{j}_i]$ for all $i = 1, 2, \ldots, n$, the product of t and $\widehat{\mathbf{J}}$, represented as $t^\top \odot \widehat{\mathbf{J}}$, is provided by

$$t^\top \odot \widehat{\mathbf{J}} = t_1 \odot [\underline{j}_1, \overline{j}_1] \oplus t_2 \odot [\underline{j}_2, \overline{j}_2] \oplus \cdots \oplus t_n \odot [\underline{j}_n, \overline{j}_n].$$

Lemma 1.37. (See [81]). *For* $\boldsymbol{A} \in \mathbb{R}_I^n$ *and* $\tau, \gamma \in \mathbb{R}^n$,

$$(\tau + \gamma) \odot \boldsymbol{A} \subseteq \tau \odot \boldsymbol{A} \oplus \gamma \odot \boldsymbol{A}.$$

Definition 1.38. (*gH-continuity* [110]). An IVVF $\mathbf{T} : X \to \mathbb{R}_I^n$ is called gH-continuous at $\bar{t} \in X$ if for each $\epsilon > 0$, there is a $\delta > 0$ with

$$\|\mathbf{T}(t) \ominus_{gH} \mathbf{T}(\bar{t})\|_{\mathbb{R}_I^n} < \epsilon \text{ whenever } \|t - \bar{t}\|_{\mathbb{R}^n} < \delta \text{ and } t \in X.$$

Definition 1.39. (*gH-hemicontinuity*). An IVVF $\mathbf{T} : X \to \mathbb{R}_I^n$ is called gH-hemicontinuous, if for any $t, p \in X$, the mapping $\tau \mapsto \mathbf{T}(t + \tau(p - t))$ defined on $[0, 1]$ is gH-continuous.

Definition 1.40. An IVVF $\mathbf{T} : X \to \mathbb{R}_I^n$ is called pseudomonotone if for all $t, p \in X, t \neq p$,

$$\mathbf{0} \preceq (p - t)^\top \odot \mathbf{T}(t) \implies \mathbf{0} \preceq (p - t)^\top \odot \mathbf{T}(p).$$

Definition 1.41. (*Convergent sequence in* $\mathbb{R}_I^n$ [50]). $\{\mathbf{T}(n)\}$ converges to $\mathbf{L} \in \mathbb{R}_I^n$ if for each $\epsilon > 0$, there exists an $m \in \mathbb{N}$ with

$$\|\mathbf{T}(n) \ominus_{gH} \mathbf{L}\|_{\mathbb{R}_I^n} < \epsilon \text{ for all } n \geq m.$$

1.5 Literature Survey

1.5.1 Literature on Interval Analysis

One of the purposes of interval analysis is to provide upper and lower bounds on the effects of errors and uncertainties on a computed quantity. Several researchers independently had the idea of bounding rounding errors by computing with intervals, for instance, see [36], [95], [105], and [106]. However, interval analysis can be said to have begun with the appearance of R.E. Moore's book 'Interval Analysis' in 1966 [79]. Moore's work transformed this simple idea into a practical tool for error analysis. The interval arithmetic given in [79] is applicable to the intervals with finite end-points. Hanson [57] and Kahan [67] each described incomplete extension of interval arithmetic in which endpoints of intervals are allowed to be infinite. Further, Hansen [58] described a practical interval Newton algorithm in which division by an interval containing zero is allowed.

The conventional interval arithmetic is insufficient to find the additive inverse of a non-degenerate interval. That is, for a given non-degenerate interval $\mathbf{A}$, there may not exist an interval $\mathbf{B}$ with $\mathbf{A} \oplus \mathbf{B} = \mathbf{0}$. Infact, the conventional definition of difference between two compact intervals has the following two deficiencies:

(i) difference between $\mathbf{A}$ and $\mathbf{B}$ is not $\{0\}$, and

(ii) If $\mathbf{C} = \mathbf{A} \ominus \mathbf{B}$, then $\mathbf{A}$ may not be equal to $\mathbf{B} \oplus \mathbf{C}$. For instance, if we take $\mathbf{A} = [1, 3]$ and $\mathbf{B} = [4, 6]$, then $\mathbf{C} = \mathbf{A} \ominus \mathbf{B} = [-5, -1]$, thus $\mathbf{B} \oplus \mathbf{C} = [-1, 5] \neq \mathbf{A}$.

To overcome these deficiencies, Hukuhara [61] proposed a concept of Hukuhara difference ($\ominus_H$) between intervals. Even though this Hukuhara-difference overcame the deficiencies of the usual difference, this difference has a major drawback that $\mathbf{A} \ominus_{gH} \mathbf{B}$ exist only if the width of the interval $\mathbf{A}$ is less than or equal to the width of interval $\mathbf{B}$. For instance, if we take $\mathbf{A} = [1, 2]$ and $\mathbf{B} = [4, 6]$, then the Hukuhara-difference between $\mathbf{A}$ and $\mathbf{B}$ does not exist [27]. To overcome the drawback of Hukuhara-difference, in 2009, Stefanini and Bede [98] generalized the concept of Hukuhara difference, which is known as generalized Hukuhara difference ($\ominus_{gH}$). This generalized difference enables one to evaluate the difference between any pair of intervals.

1.5.2 Literature on Calculus of Interval-valued Functions

To understand the characteristics of an IVF, calculus plays a crucial role. Not only that calculus also help to develop the required techniques to obtain solutions to various optimization problems. Initially, to develop the calculus of IVFs, Hukuhara in 1967 [61] proposed the notion of differentiability of IVFs by using H-difference. However, the Hukuhara differentiability (H-differentiability) is found to be restrictive (see [27]). To remove the deficiencies in H-differentiability, Bede and Gal in 2005 [15] defined strongly generalized derivative (G-derivative) for IVFs and derived a Newton-Leibnitz-type formula. In order to formulate the mean-value theorem for IVFs, Markov in 1979 [77] introduced a new concept of difference of intervals and defined differentiability of IVFs by using this difference. In 2009 [97], Stefanini and Bede defined the generalized Hukuhara differentiability (gH-differentiability) of IVFs by using the concept of generalized Hukuhara difference. For deriving the calculus of IVFs, the notions of gH-derivative, partial derivative, gradient, and differentiability for IVFs are presented in [52, 97, 98].

Showing the restrictiveness of H-differentiability, Chalco et al. [26] introduced the concept of π-derivative for IVFs that generalizes Hukuhara derivative and G-derivative. Chalco-Cano et al. [27] extended the calculus of IVFs based on the modified concept of the gH-difference, known as generalized-Hukuhara differentiability (gH-differentiability), and proved that G-derivative is equivalent to gH-derivative. Thereafter, Lupulescu [74] defined delta generalized Hukuhara differentiability on time scales by using gH-difference. In [96], Stefanini and Bede defined level-wise gH-differentiability. Further, in [51], Ghosh analyzed the notion of gH-differentiability of multi-variable IVFs to propose the Newton method for IOPs. To derive a Karush-Kuhn-Tucker (KKT) condition for IOPs, Guo et al. in 2019 [55] defined gH-symmetric derivative for IVFs. Ramsurat et al. [49] have extended the ideas of directional derivative, Gâteaux derivative, and Fréchet derivative for IVFs to derive the optimality conditions for IOPs. The notion of second-order differentiability of IVFs has been introduced by Van [102] to study the existence of a unique solution of interval differential equations. Recently, Gourav and Ghosh [72] have proposed Ekeland's variational principle for IVFs.

1.5.3 Literature on Interval Optimization Problem

Over the last two decades interval optimization problems (IOPs) have become an important research topic. Tanaka et al. in 1984 [100] discussed the linear programming problem with interval coefficients in the objective function. Tong in 1994 [99] investigated the problems in which the coefficients of the objective function and the constraints were all interval numbers. Chanas and Kuchta in 1996 [29, 30] suggested an approach based on an order relation of interval number to convert the linear optimization problem with uncertainty into a deterministic optimization problem. Liu and Da in 1999 [33] proposed an interval number optimization method based on a fuzzy constraint to deal with linear problems. Sengupta et al. in 2001 [93] studied the linear interval number programming problems in which the coefficients of the objective function and inequality constraints were all interval numbers. They used the concept of the "acceptability index" to give a solution for the uncertain linear programming. Zhang et al. in 1999 [86] assumed interval numbers as random variables with uniform distributions and constructed a possibility degree to solve the multi-criteria decision problem. The above methods point out a fine way for uncertain optimization. Ma in 2002 [76] discussed the solution methods for nonlinear interval number programming (NINP). In this article, a deterministic optimization method is used to obtain the interval of the nonlinear objective function, and a three-objective optimization problem is formulated. By using the concept of Hukuhara differentiability Wu [110], provided KKT conditions for interval optimization problems. In 2013, the KKT conditions, based on gH-differentiability, of optimization problems with IVFs have been illustrated by Chalco-Cano and others [26]. After that, Bhurji and Panda [19] have defined interval-valued function in the parametric form and studied its properties, and provided a technique to study the existence of the efficient solution of an optimization problem with IVFs. Ghosh proposed a Newton method [52] and an updated Newton method [51] to solve IOPs. Several other researchers have also suggested optimality conditions and solution ideas for IOPs, for instance, see [1, 51, 53, 107] and the references therein.

1.6 Objective of the Thesis

The objectives of the thesis are:

- To extend the conventional variational analysis to the case of interval variational analysis. For instance, we extend the conventional Ekeland's variational principle and variational inequalities for IVFs.
- to study the notion of Fréchet subdifferentiability for nonsmooth IVFs, and
- to explore and characterize the efficient solutions of IOPs by using the proposed theory.

1.7 Organization of the Thesis

This thesis consists of six chapters including an introductory chapter and a chapter comprised of conclusion and future scopes. In this chapter, which is the introductory chapter, a concise but adequate literature of the concerned topics is presented. It also defines the objective of the thesis.

In **Chapter 2**, the concept of gH-semicontinuity is introduced for IVFs. Their interrelation with gH-continuity is shown. A characterization of the set of argument minimum of an IVF is provided with the help of gH-Gâteaux differentiability. Further, Ekeland's variational principle for IVFs is given. The proposed Ekeland's variational principle is applied to find variational principle for gH-Gâteaux differentiable IVFs.

In **Chapter 3**, the concept of gH-Fréchet subdifferentiability is introduced. Various calculus results for gH-Fréchet subgradients are provided. gH-Fréchet subdifferentiable set for a gH- Fréchet differentiable IVF is provided. A smooth variational description of gH-Fréchet subgradients is added. A necessary condition for unconstrained WSM is given.

In **Chapter 4**, we briefly discuss the concept of variational inequalities for IVFs. Interval variational inequalities are introduced and a relation between their solution sets is established. We also define interval-valued pseudoconvex and pseudomonotone functions. Stampacchia IVI is used to derive a necessary and sufficient condition for a point to be an efficient solution of an IOP.

Chapter 5 is devoted to study Stampacchia and Minty variational inequalities for IVFs. In the sequel, it is observed that conventional Stampacchia and Minty variational inequalities are special cases of the proposed inequalities. The relation between solution sets of these two variational inequalities is analyzed. Existence and uniqueness results are provided for the solutions of the proposed variational inequalities. Moreover, a necessary optimality condition is added for a constrained interval optimization problem (IOP) using the generalized Hukuhara differentiability. A new first-order necessary condition is given for convex IVFs. By using this new first-order necessary condition for convex IVFs, as an application of the proposed study, a necessary and sufficient optimality condition for a constrained IOP is provided in terms of Stampacchia IVI. Further, a sufficient optimality condition for a constrained IOP is provided in terms of Minty IVI.

Finally, **Chapter 6** completes this work by summarizing the concluding remarks and forecasting some potential avenues for future research.

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