PREFACE

Optimization is a way of characterizing, locating, and computing a function's maxima and minima for a set of acceptable points or certain predetermined conditions. It was initially combined with differential calculus as part of mathematical analysis. Fermat presented the first idea of differential calculus and the rule for computing maxima and minima in 1638. Fermat defined the optimality condition as $f'(x) = 0$ to obtain the extremum of a differentiable algebraic function f. Fermat's rule still holds true for differentiable functions with multiple variables and differentiable functions defined on topological and Hilbert spaces.

Optimization is not limited to the mathematical realm. Optimization methods can be used in a variety of fields to find solutions that maximize or minimize some study parameters, such as manufacturing a good or service, minimizing production costs, and maximizing profits. These instances frequently have unique structures, such as convex, nonconvex, linear, nonlinear, quadratic, semidefinite, dynamic, integer, stochastic programming, and so on. To operate them, optimization sources vast theoretical foundations and advanced algorithms.

In modeling many real-life problems, uncertainty can be effectively dealt with fuzzy and stochastic optimization. However, it is not always possible to specify the membership function or probability distribution in an inexact environment. So, the use of fuzzy and stochastic optimization may not appropriately represent the system in such cases. Moore [79] introduced interval analysis to determine imprecise or uncertain parameters in such problems. Interval analysis is based on the depiction of an uncertain variable by an interval and offers a natural way of incorporating parameter uncertainties. Modeling of a system under interval uncertainty requires no additional information such as membership function and probability distribution.

Also, it is often possible to establish a variation range in the data. Therefore, interval optimization may provide an alternative choice for the consideration of uncertainty in optimization problems. Many good examples of the use of interval optimization in solving real-life problems can be found in [44, 91, 92, 104, 108, 112].

Moore was the first to introduce interval analysis [79]. Interval analysis, which is based on the depiction of an uncertain variable by an interval, provides a natural way of incorporating parameter uncertainties. Moore [79] in his book explained the importance of interval analysis from both a theoretical and practical standpoint. After the pioneering work by Moore the subject has evolved rapidly since its introduction nearly 60 years ago. For a good survey of the developments and applications of interval arithmetic and interval programming problems, we refer to [2, 60, 80, 107].

In this thesis, the author discusses some properties of interval analysis, smooth and nonsmooth analysis of interval-valued functions (IVFs), interval variational analysis, and optimality conditions for interval optimization problems (IOPs). The basic introduction of optimization, interval analysis with its origin, interval-valued functions, interval optimization problem with its origin are given at the begining of the first chapter. Interval arithmetic and some important properties of intervals are also explained. The definitions of continuity and convexity for IVFs along with their basic results, are explained in the last section of the first chapter.

Further we have proposed Ekeland's variational principle for IVFs. In order to arrive at the variational principle, we study the concept of sequence of intervals and extend the idea of gH-semicontinuity for IVFs. A condition is given that is both necessary and sufficient for an IVF to be gH-continuous in relation to the gH-lower and upper semicontinuity. Furthermore, we show that the level sets of the IVF can be used to characterize gH-lower semicontinuity. We guarantee the existence of a minimum for an extended gH-lower semicontinuous, level-bounded, and proper IVF using this characterization result. The proposed Ekeland's variational principle is applied to find approximate minima of a gH-lower semicontinuous and gH-Gâteaux differentiable IVF.

Next, to deal with nondifferentiable IVFs (not necessarily convex), we present the notion of Fréchet subdifferentiability or gH-Fréchet subdifferentiability. In the sequel, we explore its relationship with gH-differentiability and develop various analytical results for gH-Fréchet subgradients of extended IVFs. By using the proposed notion of subdifferentiability, we derive new necessary optimality conditions for unconstrained interval optimization problems (IOPs) with nondifferentiable IVFs. We also provide a necessary condition for unconstrained weak sharp minima of an extended IVF in terms of the proposed notion of subdifferentiability.

From the Stampacchia and Minty's work on variational inequalities for interval-valued functions (IVFs) it is observed that conventional Stampacchia and Minty variational inequalities are special cases of the proposed inequalities. The relation between the solution sets of these two variational inequalities is analyzed. Existence and uniqueness results are established for the solutions of the proposed variational inequalities. Moreover, a necessary optimality condition is given for a constrained interval optimization problem (IOP) using the generalized Hukuhara differentiability. It is observed that the first-order characterization of convex IVFs given in the literature is not true and a new first-order necessary condition is provided for convex IVFs. By using this new first-order necessary condition for convex IVFs and as an application of the proposed study, a necessary and sufficient optimality condition for a constrained IOP is discussed in terms of Stampacchia IVI. Further, a sufficient optimality condition for a constrained IOP is provided in terms of Minty IVI.