Chapter 6

A robust domain decomposition method for time delayed singularly perturbed parabolic reaction-diffusion problems with Robin boundary conditions

In the present work, we consider the following singularly perturbed time delayed parabolic reaction-diffusion problem

$$\mathcal{L}u(x,t) := Lu(x,t) + b(x,t)u(x,t-\tau) = f(x,t), \quad (x,t) \in \Omega, \tag{6.1}$$

with initial condition

$$u(x,t) = g_b(x,t), \quad (x,t) \in \gamma_b = [0,1] \times [-\tau, 0],$$

and Robin boundary conditions

$$\begin{cases} \Gamma_\ell u(x,t) :\equiv u(x,t) - \sqrt{\varepsilon}u_x(x,t) = g_\ell(t), & \text{on } \gamma_\ell = \{(0,t) : 0 \le t \le T\}, \\ \Gamma_r u(x,t) :\equiv u(x,t) + \sqrt{\varepsilon}u_x(x,t) = g_r(t), & \text{on } \gamma_r = \{(1,t) : 0 \le t \le T\}, \end{cases} \tag{6.2}$$

where

$$Lu(x,t) = u_t(x,t) - \varepsilon u_{xx}(x,t) + a(x,t)u(x,t).$$

We define $\Omega := D \times (0, T], D = (0, 1)$. Suppose $\gamma = \gamma_\ell \cup \gamma_b \cup \gamma_r$ and $\overline{\gamma} = \gamma_\ell \cup \gamma_o \cup \gamma_r$ where $\gamma_o = [0, 1] \times \{0\}$. Also $0 < \varepsilon \leq 1, \tau > 0$ and $a + b \geq \beta > 0,\ b \leq 0$. The terminal time T is assumed to be $T = k\tau$, $k > 0$. The data of the problem (6.1)-(6.2) is supposed to be enough smooth and satisfy the compatibility conditions to ensure that $u \in C^{(4,2)}(\overline{\Omega})$ [76].

There are only a few research articles available in the literature that study singularly perturbed problems with Robin boundary conditions; e.g. [76, 77, 92]. In this work, we develop a domain decomposition based Schwarz waveform relaxation (SWR) method for solving problem (6.1)-(6.2). This is the first time Robin boundaries have been analyzed using the domain decomposition approach for singularly perturbed delayed reaction-diffusion problem. We split the domain into three overlapping subdomains (one is a regular subdomain and two are layer subdomains) with the help of a subdomain parameter. The problem (6.1) on each subdomain is discretized by using backward Euler scheme in time and the standard central difference scheme in space on uniform mesh while a specific finite difference scheme is used to approximate the boundaries (6.2) in order to ensure the desired accuracy. Then, using the barrier function approach we prove that the proposed method is robust convergent having the order of accuracy one in time and almost two in space. More importantly, we showed that only one iteration is needed for small values of the perturbation parameter ε. The convergence analysis is supported by the numerical results.

The work is arranged as follows. The continuous maximum principle and the derivatives bounds of the solution are discussed in Section 6.1. In Section 6.2 we introduce an algorithm which is analyzed in Section 6.3. Numerical results are presented in Section 6.4 for two test problems and some concluding remarks are included in Section 6.5.

6.1 A priori bounds on solution derivatives

We discuss some bounds on the derivatives of the solution of (6.1)-(6.2). For this purpose, first we introduce a continuous maximum principle for the operator $\mathcal{L}$ which is defined as $\mathcal{L}u(x,t) := Lu(x,t) + b(x,t)u(x,t-\tau),\ (x,t) \in \Omega$.

Lemma 6.1. *Suppose* $\Gamma_\ell y(0,t) \geq 0,\ \Gamma_r y(1,t) \geq 0$ *on* $(0,T]$ *and* $y(x,t) \geq 0$ *on* γ_b. *Then* $Ly(x,t) \geq 0$ *on* Ω *implies that* $y(x,t) \geq 0$ *on* $\overline{\Omega}$.

Proof. Suppose y attains its minima at $(\overline{x},\overline{t})$ and $y(\overline{x},\overline{t}) = min_{(x,t)\in\overline{\Omega}} y(x,t) < 0$. Clearly, $(\overline{x},\overline{t})$ is not inside of γ_b. Let $(\overline{x},\overline{t}) \in \Omega$. Then

$$Ly(\overline{x},\overline{t}) = (y_t - \varepsilon y_{xx} + ay)(\overline{x},\overline{t}) < 0,$$

as $y_t(\overline{x},\overline{t}) = 0$ and $y_{xx}(\overline{x},\overline{t}) > 0$, which contradicts the assumption.
Let $(\overline{x},\overline{t}) \in \gamma_\ell$. Then

$$\Gamma_\ell y(\overline{x},\overline{t}) = y(\overline{x},\overline{t}) - \sqrt{\varepsilon} y_x(\overline{x},\overline{t}) < 0,$$

as $y_x(\overline{x},\overline{t}) = 0$, which contradicts the assumption. A similar contradiction can be obtained assuming $(\overline{x},\overline{t}) \in \gamma_r$. Hence, we have the proof. □

Lemma 6.2. *Suppose* z *be a function satisfying* $\Gamma_\ell z(0,t) \geq 0,\ \Gamma_r z(1,t) \geq 0$ *on* $(0,T]$ *and* $z(x,t) \geq 0$ *on* γ_b. *Then* $\mathcal{L}z(x,t) \geq 0$ *on* Ω *implies that* $z(x,t) \geq 0$ *on* $\overline{\Omega}$.

Proof. Assuming $y = z$ for $(x,t) \in \overline{D} \times [-\tau,\tau]$. Therefore, we have $y(0,t) \geq 0,\ y(1,t) \geq 0$ on $(0,\tau]$, and $y(x,t) \geq 0$ on γ_b. Further,

$$Ly(x,t) \geq -b(x,t)y(x,t-\tau) \geq 0, \quad (x,t) \in D \times (0,\tau],$$

as $b \leq 0$ and $y \geq 0$ for $(x,t) \in \overline{D} \times [-\tau, 0]$. Therefore, $z = y \geq 0$ for $(x,t) \in \overline{D} \times [0,\tau]$ by applying Lemma 6.1. Repeating the above arguments and knowing the fact that $z \geq 0$ for $(x,t) \in \overline{D} \times [(j-1)\tau, j\tau]$, we can show that $z \geq 0$, $\forall$ $(x,t) \in \overline{D} \times [j\tau, (j+1)\tau]$, $j \geq 1$. □

The continuous solution u of (6.1)-(6.2) satisfies the following estimates that can be proved using arguments in [76]:

$$\left\| \frac{\partial^{s_1+s_2} u}{\partial x_1^s \partial t_2^s} \right\|_{\overline{\Omega}} \leq C\varepsilon^{-s_1/2} \quad \text{for} \;\; 0 \leq s_1 + 2s_2 \leq 4. \tag{6.3}$$

A further decomposition of the solution u is required to prove the convergence analysis of the numerical method. The actual solution u is decomposed as $u = v + w$. The component v solves

$$\begin{aligned}
&Lv = -bv(x,t-\tau) + f, && \text{in } \Omega,\\
&\Gamma_\ell v = \Gamma_\ell v_0,\;\; \Gamma_r v = \Gamma_r v_0, && \text{in } \gamma_\ell \cup \gamma_r,\\
&v = g_b, && \text{in } \gamma_b,
\end{aligned}$$

where v_0 is the solution of the problem (by setting $\varepsilon = 0$ in (6.1)-(6.2)):

$$\begin{aligned}
&\partial_t v_0 + a v_0 = -b v_0(x,t-\tau) + f, && \text{in } \Omega,\\
&v_0 = g_b, && \text{in } \gamma_b.
\end{aligned}$$

The component w solves

$$\begin{aligned}
&Lw = -bw(x,t-\tau), && \text{in } \Omega,\\
&\Gamma_\ell w = g_\ell - \Gamma_\ell v_0,\;\; \Gamma_r w = g_r - \Gamma_r v_0, && \text{in } \gamma_\ell \cup \gamma_r,\\
&w = 0, && \text{in } \gamma_b.
\end{aligned}$$

Using the arguments in [76], we can prove that the solution components v and w satisfy the following bounds:

$$||\partial_x^{s_1} v||_{\overline{\Omega}} \le C(1 + \varepsilon^{(2-s_1)/2}), \tag{6.4}$$

$$|\partial_x^{s_1} w(x,t)| \le C\varepsilon^{-s_1/2}\left(e^{-x\sqrt{\beta/\varepsilon}} + e^{-(1-x)\sqrt{\beta/\varepsilon}}\right), \tag{6.5}$$

for $(x,t) \in \overline{\Omega}$, $s_1 = 0, \ldots, 4$.

6.2 Domain decomposition method

The computational domain Ω is divided into three overlapping subdomains $\Omega_p = D_p \times (0,T]$ such that $p = \ell, m, r$ where $\Omega_\ell = (0, 2\rho) \times (0,T]$ and $\Omega_r = (1-2\rho, 1) \times (0,T]$ are the layer subdomains and $\Omega_m = (\rho, 1-\rho) \times (0,T]$ is regular subdomain with subdomain parameter

$$\rho = \min\left\{\frac{1}{4},\ \ 2\sqrt{\frac{\varepsilon}{\beta}}\ln N\right\}. \tag{6.6}$$

Each subdomain $\Omega_p = D_p \times (0,T]$, where $D_p = (c,d)$ is discretized using a rectangular mesh which is formed by the uniform meshes in both the directions. We define $\gamma_{b,p} = \overline{D}_p \times \overline{\omega}_o$, where $\overline{\omega}_o = [-\tau, 0]$. The uniform mesh $\overline{D}_p^N$ in spatial direction is defined as $\overline{D}_p^N = \{x_i = c + ih_p, h_p = (d-c)/N\}_{i=0}^N$, while $\overline{\omega}_o^{m_\tau}$, $\overline{\omega}^M$ are the meshes in time direction with uniform length $\Delta t = T/M$, where the subintervals m_τ and $M = nm_\tau$ are obtained by dividing $[-\tau, 0]$ and $[0,T]$. Defining $D_p^N = \overline{D}_p^N \cap D_p$ and $\omega^M = \overline{\omega}^M \cap (0,T]$, then on each subdomain, meshes $\Omega_p^{N,M}$ and $\gamma_{b,p}^{N,m_\tau}$ are defined as $\Omega_p^{N,M} = D_p^N \times \omega^M$ and $\gamma_{b,p}^{N,m_\tau} = \overline{D}_p^N \times \overline{\omega}_o^{m_\tau}$ where N and M are the discretization parameters in space and time direction respectively.

On each subdomain $\Omega_p^{N,M}$, $p = \ell, m, r$, the backward Euler and central difference schemes are employed to discretize the time and spatial variables respectively. Thus, we have

$$L_p^{N,M} U_p(x_i, t_j) + b(x_i, t_j) U_p(x_i, t_{j-m_\tau}) = f(x_i, t_j), \tag{6.7}$$

where

$$L_p^{N,M} U_p(x_i, t_j) = \delta_t U_p(x_i, t_j) - \varepsilon \delta_x^2 U_p(x_i, t_j) + a(x_i, t_j) U_p(x_i, t_j).$$

The boundary conditions are discretized with a special second order scheme as follows:

$$\begin{cases} \Gamma_\ell^{N,M} U_p(0, t_j) \equiv U(0, t_j) - \sqrt{\varepsilon} F_x^+ U(0, t_j) + \frac{h_\ell}{2\sqrt{\varepsilon}} (aU + \delta_t U)(0, t_j) \\ \qquad\qquad = g_\ell(t_j) + \frac{h_\ell}{2\sqrt{\varepsilon}} (-b(0, t_j) U(0, t_{j-m_\tau}) + f(0, t_j)), \\ \Gamma_r^{N,M} U_p(1, t_j) \equiv U(1, t_j) + \sqrt{\varepsilon} F_x^- U(1, t_j) + \frac{h_r}{2\sqrt{\varepsilon}} (aU + \delta_t U)(1, t_j) \\ \qquad\qquad = g_r(t_j) + \frac{h_r}{2\sqrt{\varepsilon}} (-b(1, t_j) U(1, t_{j-m_\tau}) + f(1, t_j)), \\ U_p(x_i, t_j) \qquad = g_b(x_i, t_j). \end{cases} \tag{6.8}$$

Here,

$$[\delta_x^2 Y]_{i,j} = \frac{Y_{i+1,j} - 2Y_{i,j} + Y_{i-1,j}}{h_p^2},$$

$$[\delta_t Y]_{i,j} := \frac{Y_{i,j} - Y_{i,j-1}}{\Delta t}, [F_x^+ Y]_{i,j} := \frac{Y_{i+1,j} - Y_{i,j}}{h_p}, [F_x^- Y]_{i,j} := \frac{Y_{i,j} - Y_{i-1,j}}{h_p}.$$

Setting $\overline{\Omega}^{N,M} := (\overline{\Omega}_\ell^{N,M} \setminus \overline{\Omega}_m) \cup \overline{\Omega}_m^{N,M} \cup (\overline{\Omega}_r^{N,M} \setminus \overline{\Omega}_m)$, the numerical solution of (6.1)-(6.2) is computed using the step by step process defined as follows. Suppose the initial approximation $U^{[0]}$ is

$$U^{[0]}(x_i, t_j) = \begin{cases} 0, & (x_i, t_j) \in (0, 1) \times (0, T], \\ u(x_i, t_j), & (x_i, t_j) \in [0, 1] \times [-\tau, 0], \end{cases}$$

and compute $U_p^{[k]},\ p=\ell,m,r$, for $k\geq 1$, by solving following set of equations

$$\begin{cases} [L_\ell^{N,M}U_\ell^{[k]}]_{i,j}+b_{i,j}U_{\ell;i,j-m_\tau}^{[k]}=f_{i,j}, & (x_i,t_j)\in\Omega_\ell^{N,M},\\ U_\ell^{[k]}(x_i,t_j)=g_b(x_i,t_j), & (x_i,t_j)\in\gamma_{b,\ell}^{N,m_\tau},\\ \Gamma_\ell^{N,M}U_\ell^{[k]}(0,t_j)=g_\ell(t_j)+\frac{h_\ell}{2\sqrt{\varepsilon}}(-b(0,t_j)U(0,t_{j-m_\tau})+f(0,t_j)), & t_j\in\omega^M,\\ U_\ell^{[k]}(2\rho,t_j)=\mathcal{I}_jU^{[k-1]}(2\rho,t_j), & t_j\in\omega^M, \end{cases}$$

$$\begin{cases} [L_r^{N,M}U_r^{[k]}]_{i,j}+b_{i,j}U_{r;i,j-m_\tau}^{[k]}=f_{i,j}, & (x_i,t_j)\in\Omega_\ell^{N,M},\\ U_r^{[k]}(x_i,t_j)=g_b(x_i,t_j), & (x_i,t_j)\in\gamma_{b,r}^{N,m_\tau},\\ U_r^{[k]}(1-2\rho,t_j)=\mathcal{I}_jU^{[k-1]}(1-2\rho,t_j), & t_j\in\omega^M,\\ \Gamma_r^{N,M}U_r^{[k]}(1,t_j)=g_r(t_j)+\frac{h_r}{2\sqrt{\varepsilon}}(-b(1,t_j)U(1,t_{j-m_\tau})+f(1,t_j)), & t_j\in\omega^M, \end{cases}$$

$$\begin{cases} [L_m^{N,M}U_m^{[k]}]_{i,j}+b_{i,j}U_{m;i,j-m_\tau}^{[k]}=f_{i,j}, & (x_i,t_j)\in\Omega_m^{N,M},\\ U_m^{[k]}(x_i,t_j)=g_b(x_i,t_j), & (x_i,t_j)\in\gamma_{b,m}^{N,m_\tau},\\ U_m^{[k]}(\rho,t_j)=\mathcal{I}_jU_\ell^{[k]}(\rho,t_j), & t_j\in\omega^M,\\ U_m^{[k]}(1-\rho,t_j)=\mathcal{I}_jU_r^{[k]}(1-\rho,t_j), & t_j\in\omega^M, \end{cases}$$

Iterate the above process until $||U^{[k+1]}-U^{[k]}||_{\overline{\Omega}^{N,M}}\leq\delta$ is not reached where δ is a user defined threshold and the approximate solution $U^{[k]}$ of (6.1)-(6.2) is defined by

$$U^{[k]}(x_i,t_j)=\begin{cases} U_\ell^{[k]}(x_i,t_j), & (x_i,t_j)\in\overline{\Omega}_\ell^{N,M}\setminus\overline{\Omega}_m,\\ U_m^{[k]}(x_i,t_j), & (x_i,t_j)\in\overline{\Omega}_m^{N,M},\\ U_r^{[k]}(x_i,t_j), & (x_i,t_j)\in\overline{\Omega}_r^{N,M}\setminus\overline{\Omega}_m. \end{cases}\tag{6.9}$$

6.3 Convergence analysis

To prove the convergence of the proposed method first we are introducing the maximum principle for the discrete operators $[L_p^{N,M}U_p]_{i,j}$ and $[\mathcal{L}_p^{N,M}U_p]_{i,j} = [L_p^{N,M}U_p]_{i,j} + b_{i,j}U_{p;i,j-m_\tau}$ for all $(x_i, t_j) \in \Omega_p^{N,M}$. Suppose the operator $\mathcal{G}_p^{N,M},\ p = \ell, m, r$, is defined as $\mathcal{G}_\ell^{N,M}Z(0,t_j) = \Gamma_\ell^{N,M}Z(0,t_j)$, $\mathcal{G}_r^{N,M}Z(1,t_j) = \Gamma_r^{N,M}Z(1,t_j)$ and $\mathcal{G}_p^{N,M}Z(a,t_j) = Z(a,t_j)$ at the points $a = \rho, 2\rho, 1-\rho, 1-2\rho$ for $t_j \in \omega^M$.

Lemma 6.3. *Suppose $Z_p,\ p = \ell, m, r$, be a mesh function satisfying $\mathcal{G}_p^{N,M}Z_p(x_0,t_j) \geq 0$, $\mathcal{G}_p^{N,M}Z_p(x_N,t_j) \geq 0$, $t_j \in \omega^M$ and $Z_p(x_i,t_j) \geq 0$, $(x_i,t_j) \in \gamma_{b,p}^{N,m_\tau}$. Then $L_p^{N,M}Z_p(x_i,t_j) \geq 0$, $(x_i,t_j) \in \Omega_p^{N,M}$ implies that $Z_p(x_i,t_j) \geq 0$, $(x_i,t_j) \in \overline{\Omega}_p^{N,M}$.*

Proof. We prove the result for $p = \ell$. A similar argument can be used for $p = m, r$. Suppose the mesh function Z_ℓ attains its minima at $(\overline{x}_i, \overline{t_j})$ and

$$Z_\ell(\overline{x}_i, \overline{t_j}) = min_{(x_i,t_j)\in\overline{\Omega}_\ell^{N,M}} Z_\ell(x_i,t_j) < 0.$$

Clearly, $(\overline{x}_i, \overline{t_j})$ is not the member of $\gamma_{b,p}^{N,M}$. If $(\overline{x}_i, \overline{t_j}) \in \Omega_\ell^{N,M}$, then

$$L_\ell^{N,M}Z_\ell(\overline{x}_i, \overline{t_j}) = (\delta_t Z_\ell - \varepsilon\delta_x^2 Z_\ell + aZ_\ell)(\overline{x}_i, \overline{t_j}) < 0,$$

as $\delta_t Z_\ell(\overline{x}_i, \overline{t_j}) \leq 0$ and $\delta_x^2 Z_\ell(\overline{x}_i, \overline{t_j}) \geq 0$, which contradicts the assumption.

Now, suppose $(\overline{x}_i, \overline{t_j}) \in \gamma_\ell^{N,M}$ then

$$\mathcal{G}_p^{N,M}Z_\ell(\overline{x}_i, \overline{t_j}) = \left(Z_\ell - \sqrt{\varepsilon}F_x^+ Z_\ell + \frac{h_\ell}{2\sqrt{\varepsilon}}(aZ_\ell + \delta_t Z_\ell)\right)(\overline{x}_i, \overline{t_j}) < 0,$$

as $\delta_t Z_\ell(\overline{x}_i, \overline{t_j}) \leq 0$ and $F_x^+ Z_\ell(\overline{x}_i, \overline{t_j}) \geq 0$, which contradicts the assumption. Hence, the proof is complete. □

Lemma 6.4. *Suppose* $Y_p,\ p=\ell,m,r,$ *be a mesh function satisfying* $\mathcal{G}_p^{N,M}Y_p(x_0,t_j)\geq 0,\ \mathcal{G}_p^{N,M}Y_p(x_N,t_j)\geq 0,\ t_j\in\omega^M$ *and* $Y_p(x_i,t_j)\geq 0,\ (x_i,t_j)\in\gamma_{b,p}^{N,m_\tau}$. *Then* $\mathcal{L}_p^{N,M}Y_p(x_i,t_j)\geq 0,(x_i,t_j)\in\Omega_p^{N,M}$ *implies that* $Y_p(x_i,t_j)\geq 0,\ (x_i,t_j)\in\overline{\Omega}_p^{N,M}$.

Proof. Define $\overline{\Omega}_{p,q_1}^{N,m_\tau}=\overline{D}_p^N\times\overline{\omega}_{q_1}^{m_\tau},\ p=\ell,m,r,\ q_1=0,1\ldots,n,$ where $\overline{\omega}_{q_1}^{m_\tau}$ is defined by splitting $[(q_1-1)\tau,q_1\tau]$ into m_τ intervals of equal length. Further, we introduce $\overline{\Omega}_{p;q_1,q_2}^{N,m_\tau}=\overline{D}_p^N\times\overline{\omega}_{q_1,q_2}^{m_\tau}$, where $\overline{\omega}_{q_1,q_2}^{m_\tau}$ is obtained by splitting the interval $[(q_1-1)\tau,q_2\tau]$ into $(q_2-q_1+1)m_\tau$ intervals of equal length.

Assuming $Z_p(x_i,t_j)=Y_p(x_i,t_j),\ (x_i,t_j)\in\overline{\Omega}_{p;0,1}^{N,m_\tau}$. Therefore, we have

$$Z_p(x_i,t_j)\geq 0,\ (x_i,t_j)\in\gamma_{b,p}^{N,m_\tau}\ \text{and}\ \mathcal{G}_p^{N,M}Z_p(x_0,t_j)\geq 0,\ \mathcal{G}_p^{N,M}Z_p(x_N,t_j)\geq 0,\ t_j\in\omega_1^{m_\tau},$$

and

$$[L_p^{N,M}Z_p]_{i,j}\geq -b_{i,j}Z_p(x_i,t_{j-m_\tau})\geq 0,\quad (x_i,t_j)\in\Omega_{p,1}^{N,m_\tau}.$$

Hence, Lemma 6.3 implies that $Y_p(x_i,t_j)=Z_p(x_i,t_j)\geq 0,\ (x_i,t_j)\in\overline{\Omega}_{p,1}^{N,m_\tau}$. Similarly, for $q_1\geq 2$, we can establish $Y_p(x_i,t_j)\geq 0,\ (x_i,t_j)\in\overline{\Omega}_{p,q_1}^{N,m_\tau}$ using the fact $Y_p(x_i,t_j)\geq 0,\ (x_i,t_j)\in\overline{\Omega}_{p,q_1-1}^{N,m_\tau}$. □

Next, we define the quantities that will appear in further analysis.

$$\begin{aligned}
\mu_\rho &= \max\left\{\max_{t_j\in\omega^M}|(\widetilde{U}_m-\widetilde{U}_\ell)(\rho,t_j)|,\ \max_{t_j\in\omega^M}|(\widetilde{U}_m-\widetilde{U}_r)(1-\rho,t_j)|\right\},\\
\mu_{2\rho} &= \max\left\{\max_{t_j\in\omega^M}|(\widetilde{U}_\ell-\widetilde{U}_m)(2\rho,t_j)|,\ \max_{t_j\in\omega^M}|(\widetilde{U}_r-\widetilde{U}_m)(1-2\rho,t_j)|\right\},\\
\mu^{[k]} &= \max\left\{\max_{t_j\in\omega^M}|(\widetilde{U}_\ell-\mathcal{I}_jU^{[k-1]})(2\rho,t_j)|,\ \max_{t_j\in\omega^M}|(\widetilde{U}_r-\mathcal{I}_jU^{[k-1]})(1-2\rho,t_j)|\right\},\\
\xi^{[k]} &= \max\left\{||\widetilde{U}_\ell-U^{[k]}||_{\overline{\Omega}_\ell^{N,M}\setminus\overline{\Omega}_m},\ ||\widetilde{U}_m-U^{[k]}||_{\overline{\Omega}_m^{N,M}},\ ||\widetilde{U}_r-U^{[k]}||_{\overline{\Omega}_r^{N,M}\setminus\overline{\Omega}_m}\right\}.
\end{aligned}$$

Further, we define the following auxiliary problems that we shall use to prove the convergence of the method.

$$\begin{cases} [L_\ell^{N,M}\widetilde{U}_\ell]_{i,j} + b_{i,j}\widetilde{U}_{\ell;i,j-m_\tau} = f_{i,j}, & (x_i,t_j) \in \Omega_\ell^{N,M}, \\ \widetilde{U}_\ell(x_i,t_j) = u(x_i,t_j), & (x_i,t_j) \in \gamma_{b,\ell}^{N,m_\tau}, \\ \Gamma_\ell^{N,M}\widetilde{U}_\ell(0,t_j) = \Gamma_\ell^{N,M}U^{[k]}(0,t_j),\ \widetilde{U}_\ell(2\rho,t_j) = u(2\rho,t_j), & t_j \in \omega^M, \end{cases}$$

$$\begin{cases} [L_m^{N,M}\widetilde{U}_m]_{i,j} + b_{i,j}\widetilde{U}_{m;i,j-m_\tau} = f_{i,j}, & (x_i,t_j) \in \Omega_m^{N,M}, \\ \widetilde{U}_m(x_i,t_j) = u(x_i,t_j), & (x_i,t_j) \in \gamma_{b,m}^{N,m_\tau}, \\ \widetilde{U}_m(\rho,t_j) = u(\rho,t_j),\ \widetilde{U}_m(1-\rho,t_j) = u(1-\rho,t_j), & t_j \in \omega^M, \end{cases}$$

$$\begin{cases} [L_r^{N,M}\widetilde{U}_r]_{i,j} + b_{i,j}\widetilde{U}_{r;i,j-m_\tau} = f_{i,j}, & (x_i,t_j) \in \Omega_r^{N,M}, \\ \widetilde{U}_r(x_i,t_j) = u(x_i,t_j), & (x_i,t_j) \in \gamma_{b,r}^{N,m_\tau}, \\ \widetilde{U}_r(1-2\rho,t_j) = u(1-2\rho,t_j),\ \Gamma_r^{N,M}\widetilde{U}_r(1,t_j) = \Gamma_r^{N,M}U^{[k]}(1,t_j), & t_j \in \omega^M. \end{cases}$$

Lemma 6.5. *The numerical solutions $\widetilde{U}_p$ defined by the auxiliary problems and the solution u of (6.1)-(6.2) satisfy*

$$||u - \widetilde{U}_p||_{\overline{\Omega}_p^{N,M}} \le C(\Delta t + N^{-2}\ln^2 N). \tag{6.10}$$

Proof. For $(x_i,t_j) \in \Omega_p^{N,M},\ p = \ell, m, r,$

$$\begin{aligned} \mathcal{L}_p^{N,M}(u - \widetilde{U}_p)(x_i,t_j) &= \mathcal{L}_p^{N,M}u(x_i,t_j) - \mathcal{L}u(x_i,t_j) \\ &= (\delta_t u - u_t)(x_i,t_j) - \varepsilon\left(\delta_x^2 u - u_{xx}\right)(x_i,t_j). \end{aligned} \tag{6.11}$$

Applying Taylor expansions and bounds in (6.3) with $h_\ell \le C\sqrt{\varepsilon}N^{-1}\ln N$ we have

$$\left|[\mathcal{L}_\ell^{N,M}(u - \widetilde{U}_\ell)]_{i,j}\right| \le \frac{1}{2}(t_j - t_{j-1})\,\|u_{tt}(x,.)\|_{[t_{j-1},t_j]} + \frac{\varepsilon}{12}h_\ell^2\,\|u_{xxxx}(.,t_j)\|_{[x_{i-1},x_{i+1}]} \tag{6.12}$$

$$\leq C(\Delta t + N^{-2}\ln^2 N).$$

Next, for $t_j \in \omega^M$ we have $(u - \widetilde{U}_\ell)(\rho, t_j) = 0$ and equations (6.2) and (6.8) follows

$$\begin{aligned}
\Gamma_\ell^{N,M}(u - \widetilde{U}_\ell)(0,t_j) &= \left(u - \sqrt{\varepsilon}F_x^+ u + \frac{h_\ell}{2\sqrt{\varepsilon}}(au + \delta_t u) - (u - \sqrt{\varepsilon}u_x) - \frac{h_\ell}{2\sqrt{\varepsilon}}f\right)(0,t_j) \\
&\quad - \frac{h_\ell}{2\sqrt{\varepsilon}}\left(-b(0,t_j)u(0,t_{j-m_\tau})\right) \\
&= \sqrt{\varepsilon}\left(u_x - F_x^+ u\right)(0,t_j) + \frac{h_\ell}{2\sqrt{\varepsilon}}((au - f + u_t)(0,t_j) + b(0,t_j)u(0,t_{j-m_\tau})) \\
&\quad + \frac{h_\ell}{2\sqrt{\varepsilon}}\left(\delta_t u - u_t\right)(0,t_j) \\
&= \sqrt{\varepsilon}\left(u_x - F_x^+ u + \frac{h_\ell}{2}u_{xx}\right)(0,t_j) + \frac{h_\ell}{2\sqrt{\varepsilon}}\left(\delta_t u - u_t\right)(0,t_j).
\end{aligned}$$

Now, using Taylor expansions and bounds in (6.3), we get

$$\begin{aligned}
\left|\Gamma_\ell^{N,M}(u - \widetilde{U}_\ell)(0,t_j)\right| &\leq \frac{\sqrt{\varepsilon}}{6}h_\ell^2 \left\|u_{xxx}(.,t_j)\right\|_{[x_{i-1},\S_{i+1}]} + \frac{h_\ell}{4\sqrt{\varepsilon}}(t_j - t_{j-1})\left\|u_{tt}(x_i,.)\right\|_{[t_{j-1},t_j]} \\
&\leq C(\Delta t + N^{-2}\ln^2 N).
\end{aligned}$$

Further, applying Lemma 6.4 to the mesh function $\varphi^{\pm}(x_i,t_j) = C(\Delta t + N^{-2}\ln^2 N) \pm (u - \widetilde{U}_\ell)(x_i,t_j)$, we get

$$||u - \widetilde{U}_\ell||_{\overline{\Omega}_\ell^{N,M}} \leq C(\Delta t + N^{-2}\ln^2 N).$$

Similarly,

$$||u - \widetilde{U}_r||_{\overline{\Omega}_r^{N,M}} \leq C(\Delta t + N^{-2}\ln^2 N).$$

Now, we consider two cases to find an estimate for $||u - \widetilde{U}_m||_{\overline{\Omega}_m^{N,M}}$.

Case 1: If $\rho = 1/4$, then $h_m = 1/(2N)$ and $\varepsilon^{-1} \leq C\ln^2 N$. Now, using (6.3) and equation (6.11) with Taylor expansions we get

$$\left|[\mathcal{L}_m^{N,M}(u-\widetilde{U}_m)]_{i,j}\right| \leq C(\Delta t + N^{-2}\ln^2 N), \quad (x_i,t_j) \in \Omega_m^{N,M}.$$

Case 2: If $\rho = 2\sqrt{\frac{\varepsilon}{\alpha}}\ln N$, then $h_m \leq CN^{-1}$. So by using Taylor expansion and (6.3), we get

$$|(\delta_t u - u_t)(x_i,t_j)| \leq C\Delta t \;\text{ for }\; (x_i,t_j) \in \Omega_m^{N,M}.$$

And to calculate the bound for the term $\varepsilon\,|(\delta_x^2 u - u_{xx})(x_i,t_j)|$ we use the solution decomposition $u = v + w$ such that

$$\varepsilon\left|\left(\delta_x^2 u - u_{xx}\right)(x_i,t_j)\right| \;\leq\; \varepsilon\left|\left(\delta_x^2 v - v_{xx}\right)(x_i,t_j)\right| + \varepsilon\left|\left(\delta_x^2 w - w_{xx}\right)(x_i,t_j)\right|.$$

Now, by using Taylor expansions and bounds in (6.4), (6.5), we get

$$\begin{aligned}\varepsilon\left|\left(\delta_x^2 u - u_{xx}\right)u(x_i,t_j)\right| &\leq C\varepsilon h_m^2\,\|v_{xxxx}(.,t_j)\|_{[x_{i-1},x_{i+1}]} + C\varepsilon\,\|w_{xx}(.,t_j)\|_{[x_{i-1},x_{i+1}]}\\ &\leq CN^{-2}.\end{aligned}$$

Thus, for $(x_i,t_j) \in \Omega_m^{N,M}$, we have

$$\left|[\mathcal{L}_m^{N,M}(u-\widetilde{U}_m)]_{i,j}\right| \leq C(\Delta t + N^{-2}\ln^2 N).$$

Now, applying Lemma 6.4 to the mesh function $C(\Delta t+N^{-2}\ln^2 N)\pm(u-\widetilde{U}_m)(x_i,t_j)$, we get

$$||u-\widetilde{U}_m||_{\overline{\Omega}_m^{N,M}} \leq C(\Delta t + N^{-2}\ln^2 N).$$

Hence, the desired result can be obtained by combining the above bounds. □

In the underlying theorem we have shown that the method converges uniformly for $\rho = 2\sqrt{\frac{\varepsilon}{\beta}}\ln N$.

Theorem 6.6. *The solution of (6.1)-(6.2) and the first iterate of the proposed scheme satisfy*

$$||u - U^{[1]}||_{\overline{\Omega}^{N,M}} \leq C(\Delta t + N^{-2}\ln^2 N).$$

Proof. Suppose

$$\Psi^{\pm}(x_i,t_j) = \Psi_\ell(x_i,t_j) \pm (\widetilde{U}_\ell - U_\ell^{[1]})(x_i,t_j),$$

is the mesh function, where Ψ_ℓ is the solution of following problem

$$\begin{cases} \delta_t\Psi_\ell(i,j) - \varepsilon\delta_x^2\Psi_\ell(i,j) + \frac{\beta}{2}\Psi_\ell(i,j) + \frac{\beta}{2}\Psi_\ell(i,j-m_\tau) = 0, & (x_i,t_j) \in \Omega_\ell^{N,M}, \\ \Psi_\ell(x_i,t_j) = \mu^{[1]}\frac{\varphi_2\zeta_1^i - \varphi_1\zeta_2^i}{\varphi_2\zeta_1^N - \varphi_1\zeta_2^N}, & (x_i,t_j) \in \gamma_{b,\ell}^{N,m_\tau}, \\ \widetilde{\Gamma}_\ell^{N,M}\Psi_\ell(0,t_j) = 0, & t_j \in \omega^M, \\ \Psi_\ell(\rho,t_j) = \mu^{[1]}, & t_j \in \omega^M, \end{cases} \tag{6.13}$$

where the operator $\widetilde{\Gamma}_\ell^{N,M}$ is defined as

$$\widetilde{\Gamma}_\ell^{N,M}\Psi_\ell(0,t_j) = \Psi_\ell(0,t_j) - \sqrt{\varepsilon}F_x^+\Psi_\ell(0,t_j) + \frac{h_\ell}{2\sqrt{\varepsilon}}(\beta\Psi_\ell + \delta_t\Psi_\ell)(0,t_j),$$

$\varphi_1 = 2\sqrt{\varepsilon}h_\ell - 2\varepsilon(\zeta_1 - 1) + h_\ell^2 a$, $\varphi_2 = 2\sqrt{\varepsilon}h_\ell - 2\varepsilon(\zeta_2 - 1) + h_\ell^2 a$ and $\zeta_1 = \beta_1 + \beta_2,\ \zeta_2 = \beta_1 - \beta_2$ with

$$\beta_1 = 1 + \left(\frac{\rho}{N}\sqrt{\frac{\beta}{\varepsilon}}\right)^2,\ \beta_2 = 2\left(\frac{\rho}{N}\sqrt{\frac{\beta}{\varepsilon}}\right)\sqrt{1 + \left(\frac{\rho}{N}\sqrt{\frac{\beta}{\varepsilon}}\right)^2},$$

and $\widetilde{U}_\ell - U_\ell^{[1]}$ satisfies

$$\mathcal{L}_\ell^{N,M}(\widetilde{U}_\ell - U_\ell^{[1]})(x_i,t_j) = 0,\quad (x_i,t_j) \in \Omega_\ell^{N,M},\ (\widetilde{U}_\ell - U_\ell^{[1]})(x_i,t_j) = 0,\quad (x_i,t_j) \in \gamma_{b,\ell}^{N,m_\tau}$$

$$\text{and } \Gamma_\ell^{N,M}(\widetilde{U}_\ell - U_\ell^{[1]})(0,t_j) = 0, \quad |(\widetilde{U}_\ell - U_\ell^{[1]})(2\rho,t_j)| \le \mu^{[1]}, \quad t_j \in \omega^M.$$

The problem (6.13) has the solution such that

$$\Psi_\ell(x_i,t_j) = \mu^{[1]} \frac{\varphi_2\zeta_1^i - \varphi_1\zeta_2^i}{\varphi_2\zeta_1^N - \varphi_1\zeta_2^N}, \tag{6.14}$$

which satisfies $\Psi_\ell(x_i,t_j) \ge 0, (x_i,t_j) \in \Omega_\ell^{N,M}$ and it is monotonically increasing. Also, we have $\Psi^{\pm}(x_i,t_j) \ge 0$ for $(x_i,t_j) \in \gamma_{b,\ell}^{N,m_\tau}$, $\Gamma_\ell^{N,M}\Psi^{\pm}(0,t_j) \ge 0, \Psi^{\pm}(2\rho,t_j) \ge 0$ for $t_j \in \omega^M$, and $\mathcal{L}_\ell^{N,M}\Psi^{\pm}(x_i,t_j) \ge 0$, for $(x_i,t_j) \in \Omega_\ell^{N,M}$. Thus, by applying Lemma 6.4 to the mesh function $\Psi^{\pm}(x_i,t_j)$, we get

$$|(\widetilde{U}_\ell - U_\ell^{[1]})(x_i,t_j)| \le \Psi_\ell(x_i,t_j) \quad \text{for } (x_i,t_j) \in \overline{\Omega}_\ell^{N,M}.$$

Since $x_i \le \rho$ for $(x_i,t_j) \in \overline{\Omega}_\ell^{N,M} \backslash \overline{\Omega}_m$, the equation (6.14) implies that

$$\Psi_\ell(x_i,t_j) \le \mu^{[1]} \frac{\varphi_2\zeta_1^{N/2} - \varphi_1\zeta_2^{N/2}}{\varphi_2\zeta_1^N - \varphi_1\zeta_2^N} = \mu^{[1]} \frac{\lambda\zeta_1^{N/2} - \zeta_2^{N/2}}{\lambda\zeta_1^N - \zeta_2^N}, \ \lambda = \varphi_2/\varphi_1,$$

$$= \frac{\mu^{[1]}}{\lambda\zeta_1^{N/2} + \zeta_2^{N/2}} \left[\frac{\lambda^2\zeta_1^N - \zeta_2^N}{\lambda\zeta_1^N - \zeta_2^N} \right].$$

Therefore, we have

$$\Psi_\ell(x_i,t_j) \le \frac{\mu^{[1]}}{\lambda\zeta_1^{N/2} + \zeta_2^{N/2}} \left[\frac{\lambda^2 - \zeta_2^N/\zeta_1^N}{\lambda - \zeta_2^N/\zeta_1^N} \right] \le \mu^{[1]} \frac{C\lambda}{\lambda\zeta_1^{N/2} + \zeta_2^{N/2}},$$

as $\lambda > 1$ and $\zeta_2/\zeta_1 < 1$, we have $\left[\frac{\lambda^2 - \zeta_2^N/\zeta_1^N}{\lambda - \zeta_2^N/\zeta_1^N}\right] \le C\lambda$. Hence, we get

$$\Psi_\ell(x_i,t_j) \le \frac{\mu^{[1]}C}{\zeta_1^{N/2}}.$$

Further, using the arguments in [3, Lemma 5.1] and $\mu^{[1]} \le C$, we get $\Psi_\ell(x_i, t_j) \le CN^{-2}$ for $(x_i, t_j) \in \overline{\Omega}_\ell^{N,M} \backslash \overline{\Omega}_m$. Thus

$$||\widetilde{U}_\ell - U_\ell^{[1]}||_{\overline{\Omega}_\ell^{N,M} \backslash \overline{\Omega}_m} \le CN^{-2}. \tag{6.15}$$

Similarly

$$||\widetilde{U}_r - U_r^{[1]}||_{\overline{\Omega}_r^{N,M} \backslash \overline{\Omega}_m} \le CN^{-2}. \tag{6.16}$$

Now, take $\widetilde{U}_m - U_m^{[1]}$, which satisfies the following

$$[\mathcal{L}_m^{N,M}(\widetilde{U}_m - U_m^{[1]})]_{i,j} = 0, \quad (x_i, t_j) \in \Omega_m^{N,M}, \; (\widetilde{U}_m - U_m^{[1]})(x_i, t_j) = 0, \; (x_i, t_j) \in \gamma_{b,m}^{N,m_\tau},$$

$$|(\widetilde{U}_m - U_m^{[1]})(\eta, t_j)| \le \mu_\rho + CN^{-2}, \quad (\eta, t_j) \in \{\rho, 1-\rho\} \times \omega^M.$$

Thus, Lemma 6.4 gives

$$||\widetilde{U}_m - U_m^{[1]}||_{\overline{\Omega}_m^{N,M}} \le \mu_\rho + CN^{-2}. \tag{6.17}$$

Hence

$$\xi^{[1]} \le \mu_\rho + CN^{-2}. \tag{6.18}$$

Also, we note that $\mu_\rho \le C(\Delta t + N^{-2}\ln^2 N)$ using Lemma 6.5 and $(\rho, t_j) \in \overline{\Omega}_\ell^{N,M}$, $(1-\rho, t_j) \in \overline{\Omega}_r^{N,M}$. Thus, we get the result on combining (6.18) and Lemma 6.5. □

For $\sigma = 1/4$, the method is proved to be uniformly convergent in the following theorem.

Theorem 6.7. *The exact solution of problem (6.1)-(6.2) and the approximate solution $U^{[k]}$ defined by (6.9) satisfy*

$$||u - U^{[k]}||_{\overline{\Omega}^{N,M}} \le C\,(5/6)^k + C(\Delta t + N^{-2}\ln^2 N). \tag{6.19}$$

Proof. Assume the mesh function

$$\Phi^{\pm}(x_i,t_j) = \frac{x_i+\sqrt{\varepsilon}}{2\rho+\sqrt{\varepsilon}}\mu^{[1]} \pm (\widetilde{U}_\ell - U_\ell^{[1]})(x_i,t_j),$$

where $\widetilde{U}_\ell - U_\ell^{[1]}$ satisfies

$$\mathcal{L}_\ell^{N,M}(\widetilde{U}_\ell - U_\ell^{[1]})(x_i,t_j) = 0, \ \ (x_i,t_j) \in \Omega_\ell^{N,M}, \ (\widetilde{U}_\ell - U_\ell^{[1]})(x_i,t_j) = 0, \ (x_i,t_j) \in \gamma_{b,\ell}^{N,m_\tau},$$

$$\Gamma_\ell^{N,M}(\widetilde{U}_\ell - U_\ell^{[1]})(0,t_j) = 0, \ |(\widetilde{U}_\ell - U_\ell^{[1]})(2\rho,t_j)| \le \mu^{[1]}, \ \ t_j \in \omega^M.$$

For $t_j \in \omega^M$,

$$\begin{aligned}
\Gamma_\ell^{N,M}\Phi^{\pm}(0,t_j) &= \Gamma_\ell^{N,M}\left[\frac{x_i+\sqrt{\varepsilon}}{2\rho+\sqrt{\varepsilon}}\mu^{[1]}\right] \pm \Gamma_\ell^{N,M}(\widetilde{U}_\ell - U_\ell^{[1]})(0,t_j)\\
&= \frac{\mu^{[1]}}{2\rho+\sqrt{\varepsilon}}\left[x_0+\sqrt{\varepsilon}-\sqrt{\varepsilon}\left(\frac{x_1-x_0}{h_\ell}\right)+\frac{h_\ell(a_{i,j}+b_{i,j})}{2\sqrt{\varepsilon}}(x_0+\sqrt{\varepsilon})\right]\\
&= \frac{\mu^{[1]}}{2(2\rho+\sqrt{\varepsilon})}\left(x_0+\frac{h_\ell(a_{i,j}+b_{i,j})}{2\sqrt{\varepsilon}}(x_0+\sqrt{\varepsilon})\right) \ge 0,
\end{aligned}$$

i.e. $\Phi^{\pm}(x_i,t_j) \ge 0, \ \ (x_i,t_j) \in \gamma_{b,\ell}^{N,m_\tau}$ and $\Gamma_\ell^{N,M}\Phi^{\pm}(0,t_j) \ge 0, \ \Phi^{\pm}(2\rho,t_j) \ge 0, \ t_j \in \omega^M$. For $(x_i,t_j) \in \Omega_\ell^{N,M}$,

$$\mathcal{L}_\ell^{N,M}\Phi^{\pm}(x_i,t_j) = (a_{i,j}+b_{i,j})\left(\frac{x_i+\sqrt{\varepsilon}}{2\rho+\sqrt{\varepsilon}}\right)\mu^{[1]} \ge 0.$$

Then, Lemma 6.4 leads to

$$|(\widetilde{U}_\ell - U_\ell^{[1]})(x_i,t_j)| \le \left(\frac{x_i+\sqrt{\varepsilon}}{2\rho+\sqrt{\varepsilon}}\right)\mu^{[1]} \quad \text{for } (x_i,t_j) \in \overline{\Omega}_\ell^{N,M}.$$

Hence, for $(x_i,t_j) \in \overline{\Omega}_\ell^{N,M}\backslash\overline{\Omega}_m$, we get

$$||\widetilde{U}_\ell - U_\ell^{[1]}||_{\overline{\Omega}_\ell^{N,M}\backslash\overline{\Omega}_m} \le \frac{5}{6}\mu^{[1]}, \quad \text{as } x_i \le \rho. \tag{6.20}$$

Similarly, we can show that

$$||\widetilde{U}_r - U_r^{[1]}||_{\overline{\Omega}_r^{N,M} \setminus \overline{\Omega}_m} \leq \frac{5}{6}\mu^{[1]}. \tag{6.21}$$

Next, for the term $\widetilde{U}_m - U_m^{[1]}$, as $(\rho, t_j) \in \overline{\Omega}_\ell^{N,M}$ and $(1-\rho, t_j) \in \overline{\Omega}_r^{N,M}$, we have the following

$$\begin{aligned}
\mathcal{L}_m^{N,M}(\widetilde{U}_m - U_m^{[1]})(x_i, t_j) &= 0, \quad (x_i, t_j) \in \Omega_m^{N,M}, \\
(\widetilde{U}_m - U_m^{[1]})(x_i, t_j) &= 0, \quad (x_i, t_j) \in \gamma_{b,m}^{N,m_\tau}, \\
|(\widetilde{U}_m - U_m^{[1]})(\eta, t_j)| &\leq \mu_\rho + \frac{5}{6}\mu^{[1]}, \quad (\eta, t_j) \in \{\rho, 1-\rho\} \times \omega^M.
\end{aligned}$$

So, applying Lemma 6.4, we get

$$||\widetilde{U}_m - U_m^{[1]}||_{\overline{\Omega}_m^{N,M}} \leq \mu_\rho + \frac{5}{6}\mu^{[1]}. \tag{6.22}$$

Next, we calculate the bound on $\xi^{[2]}$ for which we need to estimate $\mu^{[2]}$ first. We note that $(2\rho, t_j), (1-2\rho, t_j) \in \overline{\Omega}_m^{N,M}$. Consequently

$$\begin{aligned}
|(\widetilde{U}_\ell - \mathcal{I}_j U^{[1]})(2\rho, t_j)| &\leq \mu_{2\rho} + \mu_\rho + \frac{5}{6}\mu^{[1]}, \\
|(\widetilde{U}_r - \mathcal{I}_j U^{[1]})(1-2\rho, t_j)| &\leq \mu_{2\rho} + \mu_\rho + \frac{5}{6}\mu^{[1]}.
\end{aligned}$$

Therefore, we have $\mu^{[2]} \leq \mu_{2\rho} + \mu_\rho + \frac{5}{6}\mu^{[1]}$. Hence

$$\max\{\xi^{[1]}, \mu^{[2]}\} \leq \lambda + \frac{5}{6}\mu^{[1]}, \quad \lambda = \mu_{2\rho} + \mu_\rho.$$

And repeating the above process, we have

$$\max\{\xi^{[k]}, \mu^{[k+1]}\} \leq \lambda + \frac{5}{6}\mu^{[k]}.$$

Simplifying the above expression we obtain

$$\mu^{[k]} \leq 6\lambda + \left(\frac{5}{6}\right)^{k-1} \mu^{[1]}.$$

Therefore

$$\xi^{[k]} \leq 6\lambda + \left(\frac{5}{6}\right)^{k} \mu^{[1]}. \tag{6.23}$$

Further, Lemma 6.5 gives $\lambda \leq C(\Delta t + N^{-2}\ln^2 N)$, as $(2\rho, t_j)$, $(1-2\rho, t_j) \in \overline{\Omega}_m^{N,M}$ and $(\rho, t_j) \in \overline{\Omega}_\ell^{N,M}$ and $(1-\rho, t_j) \in \overline{\Omega}_r^{N,M}$. Also, we have $\mu^{[1]} \leq C$. Hence, we get the proof on combining (6.23) and Lemma 6.5. □

6.4 Numerical results

Here, two test problems are considered to illustrate the numerical scheme discussed earlier. We compute the uniform error and convergence rate as $E^{N,M} = \max_\varepsilon E_\varepsilon^{N,M}$ and $R^{N,M} = \log_2\left(\frac{E^{N,M}}{E^{2N,4M}}\right)$ taking $\varepsilon \in \{10^{-1}, 10^{-1}, ..., 10^{-8}\}$ and different values of discretization parameters N, M. The computation of errors $E_\varepsilon^{N,M}$ is given below. The stopping criterion of the algorithm for the considered problems is $||U^{[k+1]} - U^{[k]}||_{\overline{\Omega}^{N,M}} \leq N^{-2}$.

Example 6.1. Consider

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} - \varepsilon\frac{\partial^2 u(x,t)}{\partial x^2} + (1 + xe^{-t})u(x,t) - 0.9u(x,t-1) = f(x,t) & (x,t) \in \Omega := D \times (0,2], \\ u(x,t) - \sqrt{\varepsilon}\frac{\partial u}{\partial x}(x,t) = g_\ell(t) & (x,t) \in \{0\} \times (0,2], \\ u(x,t) + \sqrt{\varepsilon}\frac{\partial u}{\partial x}(x,t) = g_r(t) & (x,t) \in \{1\} \times (0,2], \\ u(x,t) = g_b(x,t) & (x,t) \in [0,1] \times [-1,0], \end{cases}$$

where g_b, f, g_ℓ and g_r are determined with the help of exact solution

$$u(x,t) = t\left[\frac{e^{-x/\sqrt{\varepsilon}} + e^{(x-1)/\sqrt{\varepsilon}}}{1 + e^{-1/\sqrt{\varepsilon}}} - \cos^2 \pi x\right].$$

The maximum pointwise errors for Example are evaluated as $E_{\varepsilon}^{N,M} = ||u - U^{N,M}||_{\overline{\Omega}^{N,M}}$, where u is the actual solution and $U^{N,M}$ is the approximate solution obtained by stopping the iterative process. In Table 6.1 we present the maximum pointwise

TABLE 6.1: Maximum pointwise errors $E_{\varepsilon}^{N,M}$, uniform errors $E^{N,M}$ and uniform convergence rate $R^{N,M}$ for Example 6.1.

$\varepsilon = 10^{-p}$	$N = 2^6$, $M = 4^2$	$N = 2^7$, $M = 4^3$	$N = 2^8$, $M = 4^4$	$N = 2^9$, $M = 4^5$	$N = 2^{10}$, $M = 4^6$
$p = 1$	1.683E-04	4.279E-05	1.067E-05	2.675E-06	6.686E-07
2	7.490E-04	1.872E-04	4.679E-05	1.169E-05	2.924E-06
3	7.205E-03	1.811E-03	4.535E-04	1.134E-04	2.835E-05
4	6.669E-02	1.776E-02	4.511E-03	1.132E-03	2.834E-04
5	7.316E-02	2.647E-02	8.832E-03	2.815E-03	8.708E-04
6	7.316E-02	2.647E-02	8.832E-03	2.815E-03	8.708E-04
7	7.316E-02	2.647E-02	8.833E-03	2.815E-03	8.708E-04
8	7.316E-02	2.647E-02	8.833E-03	2.815E-03	8.708E-04
$E^{N,M}$	7.316E-02	2.647E-02	8.833E-03	2.815E-03	8.708E-04
$R^{N,M}$	1.466	1.583	1.649	1.692	

TABLE 6.2: Required iteration counts to attain the convergence for Example 6.1.

$\varepsilon = 10^{-p}$	$N = 2^6$, $M = 4^2$	$N = 2^7$, $M = 4^3$	$N = 2^8$, $M = 4^4$	$N = 2^9$, $M = 4^5$	$N = 2^{10}$, $M = 4^6$
$p = 1$	5	6	6	7	7
2	2	2	2	2	2
3	1	1	1	1	1
4	1	1	1	1	1
5	1	1	1	1	1
6	1	1	1	1	1
7	1	1	1	1	1
8	1	1	1	1	1

errors $E_{\varepsilon}^{N,M}$, uniform errors $E^{N,M}$ and uniform convergence rate $R^{N,M}$ for Example 6.1, which verify the theoretical outcomes that we have proved earlier in Theorems 6.6 and 6.7. The iteration counts mentioned in Table 6.2 display the number of iteration needed to satisfy the stopping criterion for the algorithm. From Table 6.2 we can notice that only one iteration is enough to stop the iterative procedure for

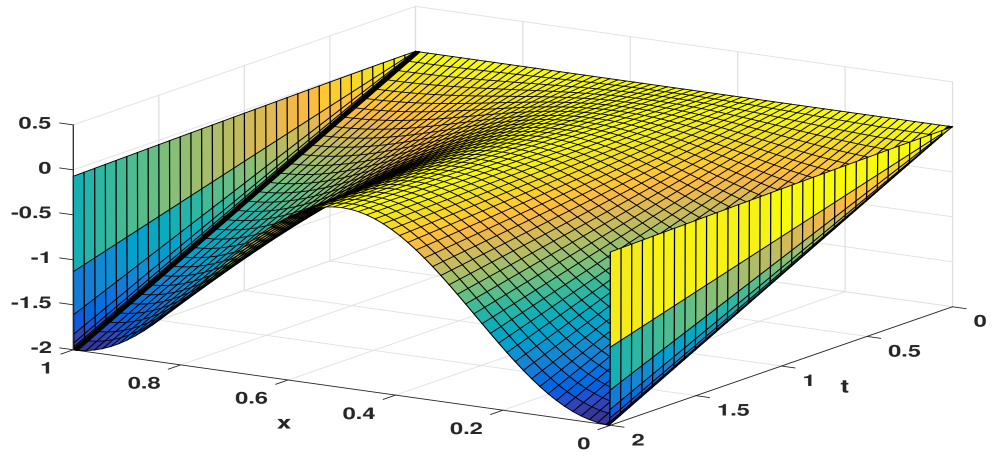

FIGURE 6.1: Solution plot for Example 6.1 with $\varepsilon = 10^{-7}$ and $N = 64, M = 32$.

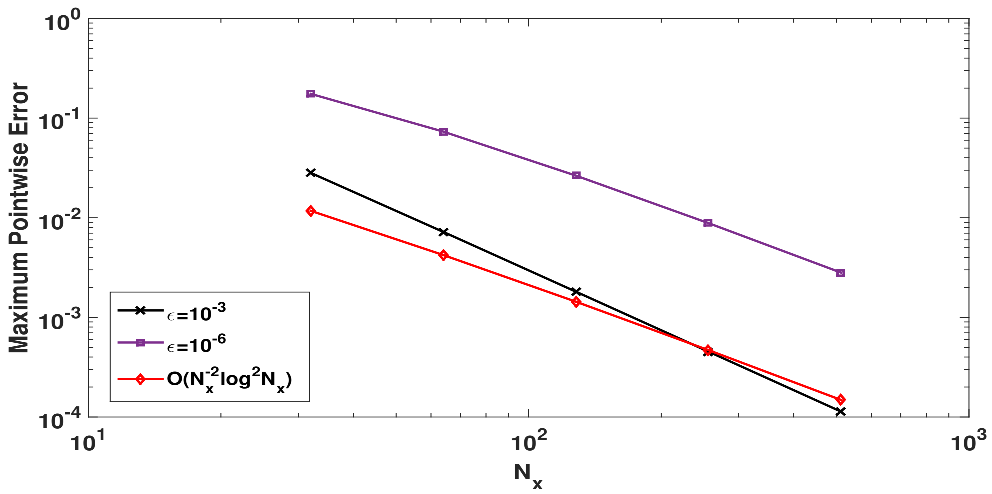

FIGURE 6.2: Error plot corresponding to Example 6.1.

small values of perturbation parameters, i.e. we can get the desired approximate solution only after one iteration of the algorithm.

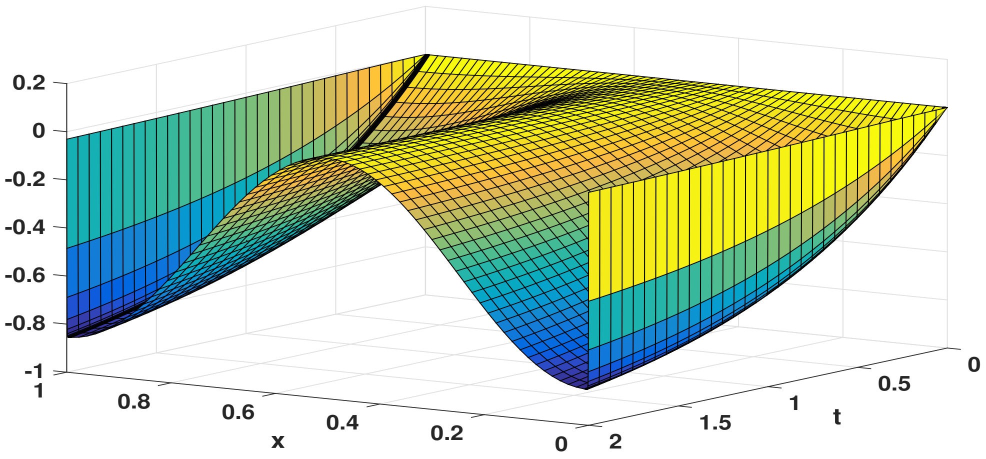

FIGURE 6.3: Solution plot for Example 6.2 with $\varepsilon = 10^{-7}$ and $N = 64, M = 32$.

Example 6.2. Consider

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} - \varepsilon \frac{\partial^2 u(x,t)}{\partial x^2} + (1 + xe^{-t})u(x,t) - 0.9u(x,t-1) = f(x,t) & (x,t) \in \Omega := D \times (0,2], \\ u(x,t) - \sqrt{\varepsilon}\frac{\partial u}{\partial x}(x,t) = g_\ell(t) & (x,t) \in \{0\} \times (0,2], \\ u(x,t) + \sqrt{\varepsilon}\frac{\partial u}{\partial x}(x,t) = g_r(t) & (x,t) \in \{1\} \times (0,2], \\ u(x,t) = g_b(x,t) & (x,t) \in [0,1] \times [-1,0]. \end{cases}$$

where g_b, f, g_ℓ and g_r are calculated by the following exact solution

$$u(x,t) = (1 - e^{-t}) \left[\frac{e^{-x/\sqrt{\varepsilon}} + e^{(x-1)/\sqrt{\varepsilon}}}{1 + e^{-1/\sqrt{\varepsilon}}} - \cos^2 \pi x \right].$$

The maximum pointwise errors, uniform errors and uniform rates of convergence for Example are evaluated as in Example (6.1).

Table 6.3 displays the numerical results for Example 6.2 that agree with the theoretical outcomes established in Theorems 6.6 and 6.7. From this, we can conclude that the scheme's rate of convergence is almost two. This is evident by the fact that in this case, the space discretization errors dominate the global errors.

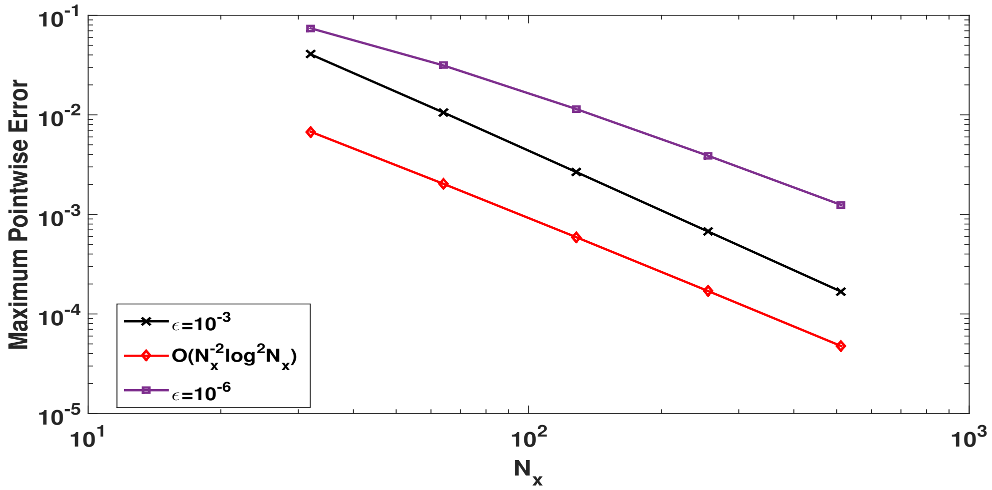

FIGURE 6.4: Error plot corresponding to Example 6.2.

To indicate the contribution of time discretization errors to global errors the following formula is used

$$R_*^{N,M} = \log_2\left(\frac{E^{N,M}}{E^{2N,2M}}\right).$$

The results are listed in Table 6.5. From this data, we can observe that the rate of convergence is one. Tables 6.4 and 6.6 show how many iterations the iterative procedure requires to reach the required accuracy.

TABLE 6.3: Maximum pointwise errors $E_\varepsilon^{N,M}$, uniform errors $E^{N,M}$ and uniform convergence rate $R^{N,M}$ for Example 6.2.

$\varepsilon = 10^{-p}$	$N = 2^6$, $M = 4^2$	$N = 2^7$, $M = 4^3$	$N = 2^8$, $M = 4^4$	$N = 2^9$, $M = 4^5$	$N = 2^{10}$, $M = 4^6$
$p = 1$	1.7501E-03	4.4629E-04	1.1214E-04	2.8072E-05	7.0201E-06
	5.0496E-03	1.2733E-03	3.1903E-04	7.9800E-05	1.9953E-05
	1.0580E-02	2.6743E-03	6.7038E-04	1.6771E-04	4.1935E-05
	2.8278E-02	7.3827E-03	1.8647E-03	4.6736E-04	1.1691E-04
	3.1430E-02	1.1480E-02	3.8707E-03	1.2453E-03	3.8822E-04
	3.1426E-02	1.1479E-02	3.8705E-03	1.2453E-03	3.8822E-04
	3.1425E-02	1.1479E-02	3.8705E-03	1.2453E-03	3.8822E-04
	3.1425E-02	1.1479E-02	3.8705E-03	1.2453E-03	3.8822E-04
$E^{N,M}$	3.1425E-02	1.1479E-02	3.8705E-03	1.2453E-03	3.8822E-04
$R^{N,M}$	1.453	1.568	1.636	1.682	

TABLE 6.4: Required iteration counts to attain the convergence rate $R^{N,M}$ for Example 6.2.

$\varepsilon = 10^{-p}$	$N = 2^6$ $M = 4^2$	$N = 2^7$ $M = 4^3$	$N = 2^8$ $M = 4^4$	$N = 2^9$ $M = 4^5$	$N = 2^{10}$ $M = 4^6$
$p = 1$	5	6	6	6	7
2	2	2	2	2	2
3	1	1	1	1	1
4	1	1	1	1	1
5	1	1	1	1	1
6	1	1	1	1	1
7	1	1	1	1	1
8	1	1	1	1	1

TABLE 6.5: Maximum pointwise errors $E_{\varepsilon}^{N,M}$, uniform errors $E^{N,M}$ and uniform convergence rate $R_*^{N,M}$ for Example 6.2.

$\varepsilon = 10^{-p}$	$N = 2^5$ $M = 2^3$	$N = 2^6$ $M = 2^4$	$N = 2^7$ $M = 2^5$	$N = 2^8$ $M = 2^6$	$N = 2^9$ $M = 2^7$
$p = 1$	3.4408E-03	1.7690E-03	8.9756E-04	4.5203E-04	2.2684E-04
	9.9801E-03	5.0470E-03	2.5377E-03	1.2725E-03	6.3713E-04
	2.0719E-02	1.0478E-02	5.2684E-03	2.6416E-03	1.3226E-03
	2.5421E-02	1.2764E-02	6.4047E-03	3.2086E-03	1.6060E-03
	2.8453E-02	1.3283E-02	6.6825E-03	3.3515E-03	1.6783E-03
	2.8455E-02	1.3394E-02	6.7437E-03	3.3836E-03	1.6948E-03
	2.8456E-02	1.3423E-02	6.7598E-03	3.3920E-03	1.6990E-03
	2.8456E-02	1.3432E-02	6.7646E-03	3.3945E-03	1.7003E-03
$E^{N,M}$	2.8456E-02	1.3432E-02	6.7646E-03	3.3945E-03	1.7003E-03
$R_*^{N,\Delta t}$	1.083	0.989	0.995	0.997	

TABLE 6.6: Required iteration counts to attain the convergence rate $R_*^{N,\Delta t}$ for Example 6.2.

$\varepsilon = 10^{-p}$	$N = 2^5$ $M = 2^3$	$N = 2^6$ $M = 2^4$	$N = 2^7$ $M = 2^5$	$N = 2^8$ $M = 2^6$	$N = 2^9$ $M = 2^7$
$p = 1$	5	6	6	7	7
2	2	2	2	2	2
3	1	1	1	1	1
4	1	1	1	1	1
5	1	1	1	1	1
6	1	1	1	1	1
7	1	1	1	1	1
8	1	1	1	1	1

The numerical solutions and corresponding error plots for Examples 6.1 and 6.2 are represented in Figures 6.1, 6.3 and 6.2, 6.4 respectively. From the Figures 6.1 and 6.3 we can clearly observe the multiscale character of the solutions near the boundary points and the slope obtained in error plots validate the convergence order.

6.5 Conclusions

We developed a Schwarz waveform relaxation approach for solving the time delayed singularly perturbed parabolic Robin boundary value problem in this work. The numerical algorithm is proved to be robust convergent, with accuracy of order one in time and almost two in space. Also, we observed that the numerical results obtained by proposed numerical algorithm corresponding to the test problems validate the theoretical outcomes. The results presented in Tables 6.2 and 6.4 demonstrate that only one iteration is sufficient to reach the given threshold for small values of perturbation parameters.

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