Chapter 3

An efficient robust domain decomposition method for singularly perturbed coupled systems of parabolic problems

In this chapter, we consider the following coupled system of parabolic singularly perturbed problems

$$\begin{cases} \boldsymbol{\mathcal{L}}\boldsymbol{u}(x,t) := (\partial_t \boldsymbol{u} + \boldsymbol{\mathcal{E}}\partial_x^2 \boldsymbol{u} + \boldsymbol{A}\boldsymbol{u})(x,t) = \boldsymbol{f}(x,t), \quad (x,t) \in \Omega = (0,1) \times (0,T], \\ \boldsymbol{u}(x,0) = \mathbf{0}, \; x \in [0,1], \quad \boldsymbol{u}(0,t) = \mathbf{g}_0(t), \boldsymbol{u}(1,t) = \mathbf{g}_1(t), \; t \in (0,T], \end{cases} \tag{3.1}$$

where

$$\boldsymbol{\mathcal{E}} = \begin{pmatrix} -\varepsilon_1 & 0 \\ 0 & -\varepsilon_2 \end{pmatrix}, \quad \boldsymbol{A} = \begin{pmatrix} a_{11}(x,t) & a_{12}(x,t) \\ a_{21}(x,t) & a_{22}(x,t) \end{pmatrix}, \; \boldsymbol{f} = (f_1, f_2)^T,$$

and $\varepsilon_1, \varepsilon_2$ are perturbation parameters such that $0 < \varepsilon_1 \le \varepsilon_2 \le 1$. We assume that the entries of the matrix $\boldsymbol{A}$ satisfy

$$\begin{cases} a_{ij}(x,t) \le 0, \; i \ne j; \quad a_{ij}(x,t) > 0, \; i = j, \\ \sum_{j=1}^{2} a_{ij}(x,t) \ge \alpha > 0, \; i = 1,2, (x,t) \in \overline{\Omega}. \end{cases} \tag{3.2}$$

Further, assume that the data of problem (3.1) is sufficiently regular and the compatibility conditions $\mathbf{g}_0^{(s)}(0) = \mathbf{g}_1^{(s)}(1) = \mathbf{0},\ s = 0, 1, 2;\ \partial_x^s \partial_t^q \boldsymbol{f}(0, 0) = \partial_x^s \partial_t^q \boldsymbol{f}(1, 0) = \mathbf{0},\ 0 \le s + 2q \le 2$, hold to ensure that $\boldsymbol{u} \in C^{4,2}(\overline{\Omega})^2$ (the time partial derivatives are continuous up to second order and the spatial partial derivatives are continuous up to fourth order) [54, 94]. Also, assume that $\Gamma_0 = \{(x, 0) \,|\, x \in [0, 1]\}$, and $\Gamma_1 = \{(x, t) \,|\, t \in (0, T],\ x = 0, 1\}$, $\Gamma = \Gamma_0 \cup \Gamma_1$. Here, perturbation parameters ε_1 and ε_2 can be of different magnitude and can take arbitrary small values. So, the solution of problem (3.1) exhibits multiscale character.

Over the last two decades, domain decomposition based numerical methods have attracted many researchers to find approximate solutions to partial differential equations (see [30, 31, 67, 90, 112–116] and the references therein). Particularly, for problem (3.1), we are aware of only one paper [53] in which a domain decomposition algorithm of SWR type is developed and analyzed. It is proved that the algorithm gives robust numerical approximations for the exact solution. However, the computational cost of this algorithm is high, since at each time level the components of the approximate solution are coupled. To decrease this computational cost, in this paper we consider two additive (or splitting) schemes [107, 117] for the discretization that allow the computation of the components of the approximate solution in a decoupled way at each time level. We provide convergence analysis of the algorithm using some auxiliary problems and the algorithm is shown to be robust convergent. In the fitted mesh framework, additive schemes for problem (3.1) are analyzed in [55]. Firstly, the time semidiscretization is introduced and some auxiliary semidiscrete problems are defined. Then the spatial discretization is introduced by discretizing these auxiliary semidiscrete problems. The time semidiscrete scheme and spatial discretization are analyzed separately, and to analyze the robust convergence of the spatial discretization a priori bounds on the solution of the semidiscrete auxiliary problems are derived, which is an additional task and require more regularity and higher order compatibility conditions. However, in the present paper, the totally discrete scheme is considered directly for analysis. Further, best to our knowledge, this is the first time additive (or splitting) schemes are analyzed in the domain decomposition framework for singularly perturbed systems.

The work in this chapter is arranged as follows. In Section 3.1, a priori bounds are given. In Section 3.2, we introduce a domain decomposition algorithm for problem (3.1). In Section 3.3, we analyze the algorithm using some auxiliary problems and establish the robust convergence of the algorithm. In Section 3.4, numerical results are given to validate the theoretical convergence and also to verify the efficiency of the algorithm. Further, some concluding remarks are given in Section 3.5.

3.1 Derivative bounds

The exact solution of (3.1) can be decomposed as $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}$ (see [54]), where $\boldsymbol{v}$ is the regular part satisfying

$$\boldsymbol{\mathcal{L}v} = \boldsymbol{f} \quad \text{in } \Omega, \quad \boldsymbol{v} = \boldsymbol{\zeta} \quad \text{on } \Gamma_1, \quad \boldsymbol{v} = \mathbf{0} \quad \text{on } \Gamma_0, \tag{3.3}$$

where $\boldsymbol{\zeta}$ satisfies

$$\partial_t \boldsymbol{\zeta} + \boldsymbol{A\zeta} = \boldsymbol{f}, \quad (x,t) \in \{0,1\} \times (0,T], \quad \boldsymbol{\zeta}(x,0) = \mathbf{0}, \quad x \in \{0,1\}, \tag{3.4}$$

and $\boldsymbol{w}$ is the singular part satisfying

$$\boldsymbol{\mathcal{L}w} = 0 \quad \text{in } \Omega, \quad \boldsymbol{w} = \boldsymbol{u} - \boldsymbol{v} \quad \text{on } \Gamma. \tag{3.5}$$

We next provide bounds on the derivatives of each part of the decomposition; for the regular part $\boldsymbol{v} = (v_1, v_2)^t$ it holds

$$||\partial_t^p \boldsymbol{v}||_{\overline{\Omega}} \leq C, \ ||\partial_x^p \boldsymbol{v}||_{\overline{\Omega}} \leq C, \quad p = 0, 1, 2,$$
$$||\partial_x^p v_1||_{\overline{\Omega}} \leq C(1 + \varepsilon_1^{(1-p/2)}), \ ||\partial_x^p v_2||_{\overline{\Omega}} \leq C(1 + \varepsilon_2^{(1-p/2)}), \quad p = 3, 4,$$

and for the singular part $\boldsymbol{w} = (w_1, w_2)^t$ it holds

$$|\partial_t^p w_n(x,t)| \leq C\mathcal{B}_{\varepsilon_2}(x), n = 1, 2, \ 0 \leq p \leq 2,$$
$$|w_1(x,t)| \leq C\mathcal{B}_{\varepsilon_2}(x), \quad |w_2(x,t)| \leq C\mathcal{B}_{\varepsilon_2}(x),$$
$$|\partial_x^p w_1(x,t)| \leq C(\varepsilon_1^{-p/2}\mathcal{B}_{\varepsilon_1}(x) + \varepsilon_2^{-p/2}\mathcal{B}_{\varepsilon_2}(x)),$$
$$|\partial_x^p w_2(x,t)| \leq C(\varepsilon_2^{-p/2}\mathcal{B}_{\varepsilon_2}(x)), \ p = 1, 2,$$
$$|\partial_x^p w_1(x,t)| \leq C(\varepsilon_1^{-p/2}\mathcal{B}_{\varepsilon_1}(x) + \varepsilon_2^{-p/2}\mathcal{B}_{\varepsilon_2}(x)), \quad p = 3, 4,$$
$$|\partial_x^p w_2(x,t)| \leq C\varepsilon_2^{-1}(\varepsilon_1^{-(p-2)/2}\mathcal{B}_{\varepsilon_1}(x) + \varepsilon_2^{-(s-2)/2}\mathcal{B}_{\varepsilon_2}(x)), \ p = 3, 4, \ \forall \ (x,t) \in \overline{\Omega}.$$

Further, for $\varepsilon_1 < \varepsilon_2$ and $\varepsilon_2 \leq \alpha/2$, the singular part $\boldsymbol{w} = (w_1, w_2)^T$ can be further decomposed as $w_1 = \hat{w}_{1,\varepsilon_1} + \hat{w}_{1,\varepsilon_2}, \ w_2 = \hat{w}_{2,\varepsilon_1} + \hat{w}_{2,\varepsilon_2}$, where

$$|\hat{w}_{1,\varepsilon_1}(x,t)| \leq \mathcal{B}_{\varepsilon_1}(x), \quad |\partial_x^2 \hat{w}_{1,\varepsilon_1}(x,t)| \leq \varepsilon_1^{-1}\mathcal{B}_{\varepsilon_1}(x), \quad |\partial_x^4 \hat{w}_{1,\varepsilon_2}(x,t)| \leq \varepsilon_2^{-2}\mathcal{B}_{\varepsilon_2}(x),$$
$$|\hat{w}_{2,\varepsilon_1}(x,t)| \leq \mathcal{B}_{\varepsilon_1}(x), \quad |\partial_x^2 \hat{w}_{2,\varepsilon_1}(x,t)| \leq \varepsilon_2^{-1}\mathcal{B}_{\varepsilon_1}(x), \quad |\partial_x^4 \hat{w}_{2,\varepsilon_2}(x,t)| \leq \varepsilon_2^{-2}\mathcal{B}_{\varepsilon_2}(x).$$

for $(x,t) \in \overline{\Omega}$. One can see a detailed proof of these bounds in [54].

3.2 Splitting schemes based domain decomposition method

From these bounds we observe that the solution $\boldsymbol{u}$ has overlapping layers near $x = 0, 1$. So, we divide the original domain into five overlapping subdomains. The decomposition of the domain is as follows: $\Omega_p = D_p \times (0, T], \ p = \ell\ell, \ell, m, r, rr$, where $D_{\ell\ell} = (0, 4\rho_1), \quad D_\ell = (\rho_1, 4\rho_2 - 3\rho_1), \quad D_m = (\rho_2, 1 - \rho_2), \quad D_r = (1 - 4\rho_2 +$

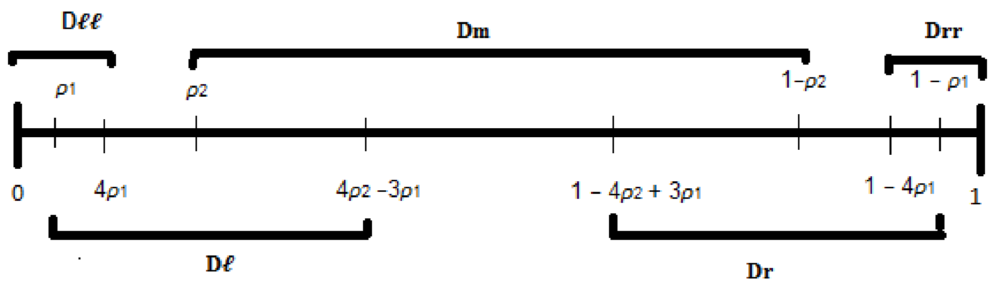

FIGURE 3.1: Decomposition in space.

$3\rho_1, 1-\rho_1)$, $D_{rr} = (1-4\rho_1, 1)$ with subdomain parameters ρ_1 and ρ_2 (see [118])

$$\rho_2 = \min\left\{\frac{4}{26},\ 2\sqrt{\frac{\varepsilon_2}{\alpha}}\ln N\right\},\quad \rho_1 = \min\left\{\frac{\rho_2}{4},\ 2\sqrt{\frac{\varepsilon_1}{\alpha}}\ln N\right\}, \tag{3.6}$$

where N is the spatial discretization parameter. Here, two are sublayer subdomains, one is a regular subdomain, and two subdomains are subdomains overlapping with sublayer subdomains and the regular subdomain. We remark that the parameters ρ_2 and ρ_1 considered in the paper are slightly different from than those used in [118]. The parameters are defined keeping in mind that the subdomains Ω_ℓ and Ω_r do not overlap, and further the subdomains $\Omega_{\ell\ell}$ and Ω_{rr} do not overlap with the subdomain Ω_m.

We consider a rectangular mesh $D_p^N \times \omega^M$ on each subdomain $\Omega_p = D_p \times (0, T]$. For $D_p = (c, d)$, we define a uniform mesh $\overline{D}_p^N = \{x_i\}_{i=0}^N$ with $h_p = (d-c)/N$ such that $c = x_0 < x_1 ... < x_{N-1} < x_N = d$. Let $\overline{\omega}^M = \{t_j\}_{j=0}^M$ be a uniform mesh in time direction with step length $\Delta t = T/M$. Suppose $D_p^N = \overline{D}_p^N \cap D_p$ and $\omega^M = \overline{\omega}^M \cap (0, T]$. On each subdomain $\Omega_p^{N,M}$, we consider the following discretization

$$\boldsymbol{\mathcal{L}}_p^{N,M}[\mathbf{U}_p]_{i,j} := \boldsymbol{L}_p^{N,M}[\mathbf{U}_p]_{i,j} - \boldsymbol{S}_{i,j}[\mathbf{U}_p]_{i,j-1} = \boldsymbol{f}_{i,j},$$

where

$$\boldsymbol{L}_p^{N,M}[\mathbf{U}_p]_{i,j} := [\delta_t \mathbf{U}_p]_{i,j} + \boldsymbol{\mathcal{E}}[\delta_x^2 \mathbf{U}_p]_{i,j} + \boldsymbol{P}_{i,j}[\mathbf{U}_p]_{i,j}$$

with

$$[\delta_t \mathbf{U}_p]_{i,j} = \frac{(\mathbf{U}_{p;i,j} - \mathbf{U}_{p;i,j-1})}{\Delta t}, \qquad [\delta_x^2 \mathbf{U}_p]_{i,j} = \frac{(\mathbf{U}_{p;i+1,j} - 2\mathbf{U}_{p;i,j} + \mathbf{U}_{p;i-1,j})}{h_p^2},$$

and $\boldsymbol{A}_{i,j} = \boldsymbol{P}_{i,j} - \boldsymbol{S}_{i,j}$. When $\boldsymbol{P}_{i,j} = \boldsymbol{A}_{i,j}$, the discretization reduces to the implicit Euler scheme in time and the central difference scheme in space, which is considered in [53]. Here, in order to decouple the system and reduce the computational time, we consider $\boldsymbol{P}_{i,j}$ as follows [55, 107, 117]

$$\boldsymbol{P}_{i,j} \in \left\{ \begin{pmatrix} a_{11}(x_i,t_j) & 0 \\ 0 & a_{22}(x_i,t_j) \end{pmatrix}, \begin{pmatrix} a_{11}(x_i,t_j) & 0 \\ a_{21}(x_i,t_j) & a_{22}(x_i,t_j) \end{pmatrix} \right\}.$$

The above two choices are considered for $\boldsymbol{P}_{i,j}$. We also use the notation $\boldsymbol{P}_{i,j} = diag(\boldsymbol{A}_{i,j})$ for the first matrix, and $\boldsymbol{P}_{i,j} = ltr(\boldsymbol{A}_{i,j})$ for the second matrix. Further, using $\boldsymbol{A}_{i,j} = \boldsymbol{P}_{i,j} - \boldsymbol{S}_{i,j}$, $\boldsymbol{S}_{i,j}$ is defined accordingly.

Setting $\overline{\Omega}^{N,M} := (\overline{\Omega}_{\ell\ell}^{N,M} \backslash \overline{\Omega}_{\ell}) \cup (\overline{\Omega}_{\ell}^{N,M} \backslash \overline{\Omega}_{m}) \cup \overline{\Omega}_{m}^{N,M} \cup (\overline{\Omega}_{r}^{N,M} \backslash \overline{\Omega}_{m}) \cup (\overline{\Omega}_{rr}^{N,M} \backslash \overline{\Omega}_{r})$, the iterative procedure to find the approximate solution of (3.1) is defined as follows: choosing the initial approximation $\mathbf{U}^{[0]}$ such that $\mathbf{U}^{[0]}(x_i,t_j) = \mathbf{0}$, $(x_i,t_j) \in (0,1) \times (0,T]$, with $\mathbf{U}^{[0]}(x_i,t_j) = \boldsymbol{u}(x_i,0)$, $(x_i,t_j) \in \overline{D}^N \times \{0\}$ and $\mathbf{U}^{[0]}(x_i,t_j) = \boldsymbol{u}(x_i,t_j)$, $(x_i,t_j) \in \{0,1\} \times \omega^M$, we compute $\mathbf{U}_p^{[k]}$, for each $k \geq 1$, by solving

$$\begin{cases} [\boldsymbol{L}_{\ell\ell}^{N,M} \mathbf{U}_{\ell\ell}^{[k]}]_{i,j} - \boldsymbol{S}_{i,j}[\mathbf{U}_{\ell\ell}^{[k]}]_{i,j-1} = \boldsymbol{f}_{i,j} & \text{for } (x_i,t_j) \in \Omega_{\ell\ell}^{N,M}, \\ \mathbf{U}_{\ell\ell}^{[k]}(x_i,0) = \mathbf{0} & \text{for } x_i \in \overline{D}_{\ell\ell}^N, \\ \mathbf{U}_{\ell\ell}^{[k]}(0,t_j) = \mathbf{g}_0(t_j) & \text{for } t_j \in \omega^M, \\ \mathbf{U}_{\ell\ell}^{[k]}(4\rho_1,t_j) = \mathcal{I}_j \mathbf{U}^{[k-1]}(4\rho_1,t_j) & \text{for } t_j \in \omega^M, \end{cases} \tag{3.7}$$

$$\begin{cases} [\boldsymbol{L}_{rr}^{N,M}\mathbf{U}_{rr}^{[k]}]_{i,j} - \boldsymbol{S}_{i,j}[\mathbf{U}_{rr}^{[k]}]_{i,j-1} = \boldsymbol{f}_{i,j} & \text{for } (x_i,t_j) \in \Omega_{rr}^{N,M}, \\ \mathbf{U}_{rr}^{[k]}(x_i,0) = \mathbf{0} & \text{for } x_i \in \overline{D}_{rr}^{N}, \\ \mathbf{U}_{rr}^{[k]}(1-4\rho_1,t_j) = \mathcal{I}_j\mathbf{U}^{[k-1]}(1-4\rho_1,t_j) & \text{for } t_j \in \omega^M, \\ \mathbf{U}_{rr}^{[k]}(1,t_j) = \mathbf{g}_1(t_j) & \text{for } t_j \in \omega^M, \end{cases} \tag{3.8}$$

$$\begin{cases} [\boldsymbol{L}_{r}^{N,M}\mathbf{U}_{r}^{[k]}]_{i,j} - \boldsymbol{S}_{i,j}[\mathbf{U}_{r}^{[k]}]_{i,j-1} = \boldsymbol{f}_{i,j} & \text{for } (x_i,t_j) \in \Omega_{r}^{N,M}, \\ \mathbf{U}_{r}^{[k]}(x_i,0) = \mathbf{0} & \text{for } x_i \in \overline{D}_{r}^{N}, \\ \mathbf{U}_{r}^{[k]}(1-4\rho_2+3\rho_1,t_j) = \mathcal{I}_j\mathbf{U}^{[k-1]}(1-4\rho_2+3\rho_1,t_j) & \text{for } t_j \in \omega^M, \\ \mathbf{U}_{r}^{[k]}(1-\rho_1,t_j) = \mathcal{I}_j\mathbf{U}_{rr}^{[k]}(1-\rho_1,t_j) & \text{for } t_j \in \omega^M, \end{cases} \tag{3.9}$$

$$\begin{cases} [\boldsymbol{L}_{\ell}^{N,M}\mathbf{U}_{\ell}^{[k]}]_{i,j} - \boldsymbol{S}_{i,j}[\mathbf{U}_{\ell}^{[k]}]_{i,j-1} = \boldsymbol{f}_{i,j} & \text{for } (x_i,t_j) \in \Omega_{\ell}^{N,M}, \\ \mathbf{U}_{\ell}^{[k]}(x_i,0) = \mathbf{0} & \text{for } x_i \in \overline{D}_{\ell}^{N}, \\ \mathbf{U}_{\ell}^{[k]}(\rho_1,t_j) = \mathcal{I}_j\mathbf{U}_{\ell\ell}^{[k]}(\rho_1,t_j) & \text{for } t_j \in \omega^M, \\ \mathbf{U}_{\ell}^{[k]}(4\rho_2-3\rho_1,t_j) = \mathcal{I}_j\mathbf{U}^{[k-1]}(4\rho_2-3\rho_1,t_j) & \text{for } t_j \in \omega^M, \end{cases} \tag{3.10}$$

$$\begin{cases} [\boldsymbol{L}_{m}^{N,M}\mathbf{U}_{m}^{[k]}]_{i,j} - \boldsymbol{S}_{i,j}[\mathbf{U}_{m}^{[k]}]_{i,j-1} = \boldsymbol{f}_{i,j} & \text{for } (x_i,t_j) \in \Omega_{m}^{N,M}, \\ \mathbf{U}_{m}^{[k]}(x_i,0) = \mathbf{0} & \text{for } x_i \in \overline{D}_{m}^{N}, \\ \mathbf{U}_{m}^{[k]}(\rho_2,t_j) = \mathcal{I}_j\mathbf{U}_{\ell}^{[k]}(\rho_2,t_j) & \text{for } t_j \in \omega^M, \\ \mathbf{U}_{m}^{[k]}(1-\rho_2,t_j) = \mathcal{I}_j\mathbf{U}_{r}^{[k]}(1-\rho_2,t_j) & \text{for } t_j \in \omega^M, \end{cases} \tag{3.11}$$

where $\mathcal{I}_j\boldsymbol{Z}$ is the piecewise linear interpolant of the mesh function $\boldsymbol{Z}$ at time level t_j. The above iterative process is repeated until $||\mathbf{U}^{[k]} - \mathbf{U}^{[k-1]}||_{\overline{\Omega}^{N,M}} \leq \gamma$ (user chosen parameter) is not satisfied, where the solution $\mathbf{U}^{[k]}$ is computed as follows

$$\mathbf{U}^{[k]} = \begin{cases} \mathbf{U}_{\ell\ell}^{[k]} & \text{in } \overline{\Omega}_{\ell\ell}^{N,M} \setminus \overline{\Omega}_{\ell}, \\ \mathbf{U}_{\ell}^{[k]} & \text{in } \overline{\Omega}_{\ell}^{N,M} \setminus \overline{\Omega}_{m}, \\ \mathbf{U}_{m}^{[k]} & \text{in } \overline{\Omega}_{m}^{N,M}, \\ \mathbf{U}_{r}^{[k]} & \text{in } \overline{\Omega}_{r}^{N,M} \setminus \overline{\Omega}_{m}, \\ \mathbf{U}_{rr}^{[k]} & \text{in } \in \overline{\Omega}_{rr}^{N,M} \setminus \overline{\Omega}_{r}. \end{cases} \tag{3.12}$$

3.3 Convergence analysis

For convergence analysis, firstly we split the global error into the discretization error $||\boldsymbol{u} - \widetilde{\mathbf{U}}||_{\overline{\Omega}^{N,M}}$ and the iterative error $||\widetilde{\mathbf{U}} - \mathbf{U}^{[k]}||_{\overline{\Omega}^{N,M}}$ using the following triangle inequality

$$||\boldsymbol{u} - \mathbf{U}^{[k]}||_{\overline{\Omega}^{N,M}} \leq ||\boldsymbol{u} - \widetilde{\mathbf{U}}||_{\overline{\Omega}^{N,M}} + ||\widetilde{\mathbf{U}} - \mathbf{U}^{[k]}||_{\overline{\Omega}^{N,M}}, \tag{3.13}$$

where $\widetilde{\mathbf{U}}$ is constructed from the following auxiliary problems in the similar way as (3.12):

$$\begin{cases} [\boldsymbol{L}_{\ell\ell}^{N,M}\widetilde{\mathbf{U}}_{\ell\ell}]_{i,j} - \boldsymbol{S}_{i,j}[\widetilde{\mathbf{U}}_{\ell\ell}]_{i,j-1} = \boldsymbol{f}_{i,j} & \text{for } (x_i, t_j) \in \Omega_{\ell\ell}^{N,M}, \\ \widetilde{\mathbf{U}}_{\ell\ell}(x_i, 0) = \mathbf{0} & \text{for } x_i \in \overline{D}_{\ell\ell}^{N}, \\ \widetilde{\mathbf{U}}_{\ell\ell}(\eta, t_j) = \boldsymbol{u}(\eta, t_j) & \text{for } (\eta, t_j) \in \{0, 4\rho_1\} \times \omega^M, \end{cases}$$

$$\begin{cases} [\boldsymbol{L}_{rr}^{N,M}\widetilde{\mathbf{U}}_{rr}]_{i,j} - \boldsymbol{S}_{i,j}[\widetilde{\mathbf{U}}_{rr}]_{i,j-1} = \boldsymbol{f}_{i,j} & \text{for } (x_i, t_j) \in \Omega_{rr}^{N,M}, \\ \widetilde{\mathbf{U}}_{rr}(x_i, 0) = \mathbf{0} & \text{for } x_i \in \overline{D}_{rr}^{N}, \\ \widetilde{\mathbf{U}}_{rr}(\eta, t_j) = \boldsymbol{u}(\eta, t_j) & \text{for } (\eta, t_j) \in \{1 - 4\rho_1, 1\} \times \omega^M, \end{cases}$$

$$\begin{cases} [\boldsymbol{L}_{r}^{N,M}\widetilde{\mathbf{U}}_{r}]_{i,j} - \boldsymbol{S}_{i,j}[\widetilde{\mathbf{U}}_{r}]_{i,j-1} = \boldsymbol{f}_{i,j} & \text{for } (x_i, t_j) \in \Omega_{r}^{N,M}, \\ \widetilde{\mathbf{U}}_{r}(x_i, 0) = \mathbf{0} & \text{for } x_i \in \overline{D}_{r}^{N}, \\ \widetilde{\mathbf{U}}_{r}(\eta, t_j) = \boldsymbol{u}(\eta, t_j) & \text{for } (\eta, t_j) \in \{1 - 4\rho_2 + 3\rho_1, 1 - \rho_1\} \times \omega^M, \end{cases}$$

$$\begin{cases} [\boldsymbol{L}_{\ell}^{N,M}\widetilde{\mathbf{U}}_{\ell}]_{i,j} - \boldsymbol{S}_{i,j}[\widetilde{\mathbf{U}}_{\ell}]_{i,j-1} = \boldsymbol{f}_{i,j} & \text{for } (x_i, t_j) \in \Omega_{\ell}^{N,M}, \\ \widetilde{\mathbf{U}}_{\ell}(x_i, 0) = \mathbf{0} & \text{for } x_i \in \overline{D}_{\ell}^{N}, \\ \widetilde{\mathbf{U}}_{\ell}(\eta, t_j) = \boldsymbol{u}(\eta, t_j) & \text{for } (\eta, t_j) \in \{\rho_1, 4\rho_2 - 3\rho_1\} \times \omega^M, \end{cases}$$

$$\begin{cases} [\boldsymbol{L}_{m}^{N,M}\widetilde{\mathbf{U}}_{m}]_{i,j} - \boldsymbol{S}_{i,j}[\widetilde{\mathbf{U}}_{m}]_{i,j-1} = \boldsymbol{f}_{i,j} & \text{for } (x_i, t_j) \in \Omega_{m}^{N,M}, \\ \widetilde{\mathbf{U}}_{m}(x_i, 0) = \mathbf{0} & \text{for } x_i \in \overline{D}_{m}^{N}, \\ \widetilde{\mathbf{U}}_{m}(\eta, t_j) = \boldsymbol{u}(\eta, t_j) & \text{for } (\eta, t_j) \in \{\rho_2, 1 - \rho_2\} \times \omega^M, \end{cases}$$

Next, we are going to introduce some notations and the discrete maximum principle which we shall use to establish convergence of the algorithm.

$$
\begin{aligned}
\mu_{\rho_1} &= \max\{\max_{t_j\in\omega^M}|(\widetilde{\mathbf{U}}_\ell-\widetilde{\mathbf{U}}_{\ell\ell})(\rho_1,t_j)|, \max_{t_j\in\omega^M}|(\widetilde{\mathbf{U}}_r-\widetilde{\mathbf{U}}_{rr})(1-\rho_1,t_j)\},\\
\mu_{\rho_2} &= \max\{\max_{t_j\in\omega^M}|(\widetilde{\mathbf{U}}_m-\widetilde{\mathbf{U}}_{\ell})(\rho_2,t_j)|, \max_{t_j\in\omega^M}|(\widetilde{\mathbf{U}}_m-\widetilde{\mathbf{U}}_{r})(1-\rho_2,t_j)\},\\
\mu_{4\rho_1} &= \max\{\max_{t_j\in\omega^M}|(\widetilde{\mathbf{U}}_{\ell\ell}-\mathcal{I}_j\widetilde{\mathbf{U}}_{\ell})(4\rho_1,t_j)|, \max_{t_j\in\omega^M}|(\widetilde{\mathbf{U}}_{rr}-\mathcal{I}_j\widetilde{\mathbf{U}}_{r})(1-4\rho_1,t_j)\},\\
\mu_{4\rho_2-3\rho_1} &= \max\{\max_{t_j\in\omega^M}|(\widetilde{\mathbf{U}}_{\ell}-\mathcal{I}_j\widetilde{\mathbf{U}}_{m})(4\rho_2-3\rho_1,t_j)|, \max_{t_j\in\omega^M}|(\widetilde{\mathbf{U}}_{r}-\mathcal{I}_j\widetilde{\mathbf{U}}_{r})(1-4\rho_2+3\rho_1,t_j)\},\\
\mu^{[k]} &= \max\{\max_{t_j\in\omega^M}|(\widetilde{\mathbf{U}}_{\ell\ell}-\mathcal{I}_j\mathbf{U}^{[k-1]})(4\rho_1,t_j)|, \max_{t_j\in\omega^M}|(\widetilde{\mathbf{U}}_{rr}-\mathcal{I}_j\mathbf{U}^{[k-1]})(1-4\rho_1,t_j)|,\\
&\quad \max_{t_j\in\omega^M}|(\widetilde{\mathbf{U}}_{\ell}-\mathcal{I}_j\mathbf{U}^{[k-1]})(4\rho_2-3\rho_1,t_j)|, \max_{t_j\in\omega^M}|(\widetilde{\mathbf{U}}_{r}-\mathcal{I}_j\mathbf{U}^{[k-1]})(1-4\rho_2+3\rho_1,t_j)|\}.
\end{aligned}
$$

Lemma 3.1. *Let $\boldsymbol{Z}_p$ be the mesh function such that $\boldsymbol{Z}_p(x_0,t_j)\geq \boldsymbol{0}$ and $\boldsymbol{Z}_p(x_N,t_j)\geq \boldsymbol{0}$ for $t_j\in\omega^M$, and $\boldsymbol{Z}_p(x_i,0)\geq\boldsymbol{0}$ for $x_i\in\overline{D}_p^N$. If $\boldsymbol{\mathcal{L}}_p^{N,M}\boldsymbol{Z}_p\geq\boldsymbol{0}$ in $\Omega_p^{N,M}$, then $\boldsymbol{Z}_p\geq\boldsymbol{0}$ in $\overline{\Omega}_p^{N,M}$.*

Lemma 3.2. *For $\boldsymbol{u}$ and $\widetilde{\mathbf{U}}_p$, respectively, the solution of (3.1) and auxiliary problems, we have*

$$||\boldsymbol{u}-\widetilde{\mathbf{U}}_p||_{\overline{\Omega}_p^{N,M}}\leq C(\Delta t+N^{-2}\ln^2 N).$$

Proof. Suppose $\boldsymbol{P}_{i,j}=\left\{\begin{pmatrix} a_{11}(x_i,t_j) & 0\\ 0 & a_{22}(x_i,t_j)\end{pmatrix}\right\}$, then for $(x_i,t_j)\in\Omega_p^{N,M}$, we have

$$[\boldsymbol{\mathcal{L}}_p^{N,M}(\boldsymbol{u}-\widetilde{\mathbf{U}}_p)]_{i,j}=[\boldsymbol{\mathcal{L}}_p^{N,M}\boldsymbol{u}-\boldsymbol{\mathcal{L}}\boldsymbol{u}]_{i,j}=[\delta_t\boldsymbol{u}-\boldsymbol{u}_t]_{i,j}+\boldsymbol{\mathcal{E}}[\delta_x^2\boldsymbol{u}-\boldsymbol{u}_{xx}]_{i,j}+\boldsymbol{S}_{i,j}[\boldsymbol{u}_{i,j}-\boldsymbol{u}_{i,j-1}].$$

Define $\boldsymbol{\mathcal{L}}^{N,M}=(\mathcal{L}_1^{N,M},\mathcal{L}_2^{N,M})^T$. For $n=1,2$, it holds

$$|\mathcal{L}_{p,n}^{N,M}(u-\widetilde{U}_p)_{i,j}|\leq|(\delta_t-\partial_t)\,u_{n;i,j}|+C|(u_{3-n;i,j}-u_{3-n;i,j-1})|+\varepsilon_n\left|\left(\delta_x^2-\partial_x^2\right)u_{n;i,j}\right|.$$

By Taylor expansions and bounds on the derivatives in Section 3.1, we get

$$\begin{aligned}|\left(\delta_t-\partial_t\right)u_{n;i,j}|+C|(u_{3-n;i,j}-u_{3-n;i,j-1})| &= C(t_j-t_{j-1})\left(\|\partial_t u_{3-n}(x_i,.)\|_{[t_{j-1},t_j]}\right)+\\ &\quad C(t_j-t_{j-1})\left(\|\partial_t^2 u_n(x_i,.)\|_{[t_{j-1},t_j]}\right)\\ &\le C\Delta t \ \text{ for } (x_i,t_j)\in\Omega_p^{N,M},\ n=1,2.\end{aligned}$$

Thus, we are left to obtain a bound for $|\left(\delta_x^2-\partial_x^2\right)u_{n;i,j}|$. For simplicity let us consider $\rho_2=(2\sqrt{\varepsilon_2}\ln N)/\sqrt{\alpha}$ and $\rho_1=(2\sqrt{\varepsilon_1}\ln N)/\sqrt{\alpha}$, that is ε_1 and ε_2 are small and of different magnitude. This is the most interesting case and other cases can be dealt accordingly. For $(x_i,t_j)\in\Omega_p^{N,M},\ p=\ell\ell,rr,\ n=1,2$, using Taylor expansions and bounds in Section 3.1 with $h_p\le C\sqrt{\varepsilon_1}N^{-1}\ln N$, we get

$$\varepsilon_n\left|\left(\delta_x^2-\partial_x^2\right)u_{n;i,j}\right|\le C\varepsilon_n h_p^2\left\|\partial_x^4 u_n(.,t_j)\right\|_{[x_{i-1},x_{i+1}]}\le CN^{-2}\ln^2 N.$$

For $(x_i,t_j)\in\Omega_p^{N,M},\ p=\ell,r$, use the solution decomposition $u_n=v_n+w_n,\ w_n=\hat{w}_{n,\varepsilon_1}+\hat{w}_{n,\varepsilon_2}$, Taylor expansions, bounds on the derivatives, and the mesh width to get

$$\begin{aligned}\varepsilon_n\left|\left(\delta_x^2-\partial_x^2\right)u_{n;i,j}\right| &\le C\varepsilon_n h_p^2\left\|\partial_x^4 v_n(.,t_j)\right\|_{[x_{i-1},x_{i+1}]}+C\varepsilon_n\left\|\partial_x^2\hat{w}_{n,\varepsilon_1}(.,t_j)\right\|_{[x_{i-1},x_{i+1}]}\\ &\quad +C\varepsilon_n h_p^2\left\|\partial_x^4\hat{w}_{n,\varepsilon_2}(.,t_j)\right\|_{[x_{i-1},x_{i+1}]}\\ &\le CN^{-2}\ln^2 N.\end{aligned}$$

Next, for $(x_i,t_j)\in\Omega_m^{N,M}$, we again use the decomposition $u_n=v_n+w_n$, Taylor expansions and bounds on derivatives in Section 3.1 to get

$$\begin{aligned}\varepsilon_n\left|\left(\delta_x^2-\partial_x^2\right)u_{n;i,j}\right| &\le \varepsilon_n\left|\left(\delta_x^2-\partial_x^2\right)v_{n;i,j}\right|+\varepsilon_n\left|\left(\delta_x^2-\partial_x^2\right)w_{n;i,j}\right|\\ &\le C\varepsilon_n h_m^2\left\|\partial_x^4 v_n(.,t_j)\right\|_{[x_{i-1},x_{i+1}]}+C\varepsilon_n\left\|\partial_x^2 w_n(.,t_j)\right\|_{[x_{i-1},x_{i+1}]}\\ &\le C(\Delta t+N^{-2}).\end{aligned}$$

Thus, we have $|[\boldsymbol{\mathcal{L}}_p^{N,M}(\boldsymbol{u}-\widetilde{\mathbf{U}}_p)]_{i,j}| \le C(\Delta t + N^{-2}\ln^2 N)$. Now, using Lemma 3.1 to the mesh function $C(\Delta t + N^{-2}\ln^2 N) \pm (\boldsymbol{u}-\widetilde{\mathbf{U}}_p)(x_i,t_j)$, we get

$$||\boldsymbol{u}-\widetilde{\mathbf{U}}_p||_{\overline{\Omega}_p^{N,M}} \le C(\Delta t + N^{-2}\ln^2 N).$$

Repeating the above arguments, we can obtain the same bound for the other choice of $\boldsymbol{P}_{i,j}$. □

Theorem 3.3. *Let $\mathbf{U}^{[k]}$ be the k^{th} iterate of the present algorithm and $\widetilde{\mathbf{U}}$ be the solution of the auxiliary problems. Then, it holds*

$$||\widetilde{\mathbf{U}}-\mathbf{U}^{[k]}||_{\overline{\Omega}^{N,M}} \le C2^{-k} + C(\Delta t + N^{-2}\ln^2 N). \tag{3.14}$$

Proof. For $(x_i,t_j) \in \Omega_{\ell\ell}^{N,M}$, $\widetilde{\mathbf{U}}_{\ell\ell}-\mathbf{U}_{\ell\ell}^{[1]}$ satisfies

$$\begin{cases} \boldsymbol{\mathcal{L}}_{\ell\ell}^{N,M}(\widetilde{\mathbf{U}}_{\ell\ell}-\mathbf{U}_{\ell\ell}^{[1]}) = \mathbf{0} & \text{in } \Omega_{\ell\ell}^{N,M}, \\ (\widetilde{\mathbf{U}}_{\ell\ell}-\mathbf{U}_{\ell\ell}^{[1]})(x_i,0) = \mathbf{0} & \text{for } x_i \in \overline{D}^N, \\ (\widetilde{\mathbf{U}}_{\ell\ell}-\mathbf{U}_{\ell\ell}^{[1]})(0,t_j) = \mathbf{0}, \quad |(\widetilde{\mathbf{U}}_{\ell\ell}-\mathbf{U}_{\ell\ell}^{[1]})(4\rho_1,t_j)| \le \mu^{[1]}\mathbf{1} & \text{for } t_j \in \omega^M. \end{cases}$$

Thus, using Lemma 3.1 to the mesh function $\frac{x_i}{4\rho_1}\mu^{[1]}\mathbf{1} \pm (\widetilde{\mathbf{U}}_{\ell\ell}-\mathbf{U}_{\ell\ell}^{[1]})(x_i,t_j)$, we get

$$|(\widetilde{\mathbf{U}}_{\ell\ell}-\mathbf{U}_{\ell\ell}^{[1]})| \le \frac{x_i}{4\rho_1}\mu^{[1]}\mathbf{1} \quad \text{for } (x_i,t_j) \in \overline{\Omega}_{\ell\ell}^{N,M}.$$

Hence,

$$||\widetilde{\mathbf{U}}_{\ell\ell}-\mathbf{U}_{\ell\ell}^{[1]}||_{\overline{\Omega}_{\ell\ell}^{N,M}\setminus\overline{\Omega}_\ell} \le 4^{-1}\mu^{[1]}. \tag{3.15}$$

Similarly,

$$||\widetilde{\mathbf{U}}_{rr}-\mathbf{U}_{rr}^{[1]}||_{\overline{\Omega}_{rr}^{N,M}\setminus\overline{\Omega}_r} \le 4^{-1}\mu^{[1]}. \tag{3.16}$$

Next, we have

$$\begin{cases} \boldsymbol{\mathcal{L}}_\ell^{N,M}(\widetilde{\mathbf{U}}_\ell - \mathbf{U}_\ell^{[1]}) = \mathbf{0} & \text{in } \Omega_\ell^{N,M}, \\ (\widetilde{\mathbf{U}}_\ell - \mathbf{U}_\ell^{[1]})(x_i, 0) = \mathbf{0} & \text{for } x_i \in \overline{D}^N, \\ |(\widetilde{\mathbf{U}}_\ell - \mathbf{U}_\ell^{[1]})(4\rho_2 - 3\rho_1, t_j)| \leq \mu^{[1]}\mathbf{1} & \text{for } t_j \in \omega^M, \\ |(\widetilde{\mathbf{U}}_\ell - \mathbf{U}_\ell^{[1]})(\rho_1, t_j)| \leq 4^{-1}\mu^{[1]}\mathbf{1} + \mu_{\rho_1}\mathbf{1} & \text{for } t_j \in \omega^M, \end{cases}$$

where the last bound follows by adding and subtracting $\widetilde{\mathbf{U}}_{\ell\ell}$. Now, using Lemma 3.1 for $\varphi(x_i)\mu^{[1]}\mathbf{1} + \mu_{\rho_1}\mathbf{1} \pm (\widetilde{\mathbf{U}}_\ell - \mathbf{U}_\ell^{[1]})(x_i, t_j)$, we have

$$|(\widetilde{\mathbf{U}}_\ell - \mathbf{U}_\ell^{[1]})(x_i, t_j)| \leq \varphi(x_i)\mu^{[1]}\mathbf{1} + \mu_{\rho_1}\mathbf{1}, \quad (x_i, t_j) \in \overline{\Omega}_\ell^{N,M},$$

where $\varphi(x) = \dfrac{-x^2 + (13\rho_2 - 11\rho_1)x + 12\rho_2^2 + 24\rho_1^2 - 37\rho_1\rho_2}{48(\rho_2 - \rho_1)^2}$, $x \in [\rho_1, 4\rho_2 - 3\rho_1]$. Further, since φ is the increasing function and $\varphi(\rho_2) = 1/2$, we have

$$||\widetilde{\mathbf{U}}_\ell - \mathbf{U}_\ell^{[1]}||_{\overline{\Omega}_\ell^{N,M} \setminus \overline{\Omega}_m} \leq 2^{-1}\mu^{[1]} + \mu_{\rho_1}. \tag{3.17}$$

Similarly,

$$||\widetilde{\mathbf{U}}_r - \mathbf{U}_r^{[1]}||_{\overline{\Omega}_r^{N,M} \setminus \overline{\Omega}_m} \leq 2^{-1}\mu^{[1]} + \mu_{\rho_1}. \tag{3.18}$$

Next, we have

$$\begin{aligned} \boldsymbol{\mathcal{L}}_m^{N,M}(\widetilde{\mathbf{U}}_m - \mathbf{U}_m^{[1]}) &= \mathbf{0} \quad \text{in } \Omega_m^{N,M}, \\ (\widetilde{\mathbf{U}}_m - \mathbf{U}_m^{[1]})(x_i, t_j) &= \mathbf{0} \quad \text{for } x_i \in \overline{D}^N, \\ |(\widetilde{\mathbf{U}}_m - \mathbf{U}_m^{[1]})(\eta, t_j)| &\leq 2^{-1}\mu^{[1]}\mathbf{1} + \mu_{\rho_1}\mathbf{1} + \mu_{\rho_2}\mathbf{1} \quad \text{for } (\eta, t_j) \in \{\rho_2, 1 - \rho_2\} \times \omega^M, \end{aligned}$$

where the last bound is obtained by adding and subtracting $\widetilde{\mathbf{U}}_\ell$ when $\eta = \rho_2$; and adding and subtracting $\widetilde{\mathbf{U}}_r$ when $\eta = 1-\rho_2$. Thus, by Lemma 3.1 we get

$$||\widetilde{\mathbf{U}}_m - \mathbf{U}_m^{[1]}||_{\overline{\Omega}_m^{N,M}} \leq 2^{-1}\mu^{[1]} + \mu_{\rho_1} + \mu_{\rho_2}. \tag{3.19}$$

On combining the previous bounds we have

$$||\widetilde{\mathbf{U}} - \mathbf{U}^{[1]}||_{\overline{\Omega}^{N,M}} \leq 2^{-1}\mu^{[1]} + \mu_{\rho_1} + \mu_{\rho_2}.$$

Next we require an estimate for $\mu^{[2]}$ to bound $||\widetilde{\mathbf{U}} - \mathbf{U}^{[2]}||$. Applying a triangle inequality, stability of $\mathcal{I}_j$ and (3.17), (3.18), (3.19), we get

$|(\widetilde{\mathbf{U}}_p - \mathcal{I}_j\mathbf{U}^{[1]})(\eta, t_j)| \leq \mu_{4\rho_1}\mathbf{1} + 2^{-1}\mu^{[1]}\mathbf{1} + \mu_{\rho_1}\mathbf{1}$ when $\{p = \ell\ell, \eta = 4\rho_1\}$ and $\{p = rr, \eta = 1-4\rho_1\}$,

$|(\widetilde{\mathbf{U}}_p - \mathcal{I}_j\mathbf{U}^{[1]})(\eta, t_j)| \leq \mu_{4\rho_2-3\rho_1}\mathbf{1} + 2^{-1}\mu^{[1]}\mathbf{1} + \mu_{\rho_1}\mathbf{1} + \mu_{\rho_2}\mathbf{1}$ when $\{p = \ell, \eta = 4\rho_2 - 3\rho_1\}$ and $\{p = r, \eta = 1 - 4\rho_2 + 3\rho_1\}$.

Therefore,

$$\mu^{[2]} \leq 2^{-1}\mu^{[1]} + \mu_{\rho_1} + \mu_{\rho_2} + \mu_{4\rho_1} + \mu_{4\rho_2-3\rho_1}.$$

Hence,

$$\max\left\{\mu^{[2]}, ||\widetilde{\mathbf{U}} - \mathbf{U}^{[1]}||_{\overline{\Omega}^{N,M}}\right\} \leq \lambda + 2^{-1}\mu^{[1]}, \quad \lambda = \mu_{\rho_1} + \mu_{\rho_2} + \mu_{4\rho_1} + \mu_{4\rho_2-3\rho_1}.$$

Repetition of the previous arguments give

$$\max\left\{\mu^{[k+1]}, ||\widetilde{\mathbf{U}} - \mathbf{U}^{[k]}||_{\overline{\Omega}^{N,M}}\right\} \leq \lambda + 2^{-1}\mu^{[k]}.$$

On simplifying we get $\mu^{[k]} \leq 2\lambda + 2^{-(k-1)}\mu^{[1]}$. Therefore

$$||\widetilde{\mathbf{U}} - \mathbf{U}^{[k]}||_{\overline{\Omega}^{N,M}} \leq 2\lambda + 2^{-k}\mu^{[1]}. \tag{3.20}$$

The bound on $\mu_{\rho_1} + \mu_{\rho_2}$ directly follows from Lemma 3.2. Note that $\mu_{4\rho_1} + \mu_{4\rho_2 - 3\rho_1}$ is the interpolation error. Hence, using the standard interpolation error bounds and the arguments in Lemma 3.2 we can bound $\mu_{4\rho_1} + \mu_{4\rho_2-3\rho_1}$. Thus, we have $\lambda \leq C(\Delta t + N^{-2}\ln^2 N)$. Futher, using Lemma 3.1 it follows that $\mu^{[1]} \leq C$. Hence, $||\widetilde{\mathbf{U}} - \mathbf{U}^{[k]}||_{\overline{\Omega}^{N,M}} \leq C2^{-k} + C(\Delta t + N^{-2}\ln^2 N)$. □

Theorem 3.4. *Let* $\mathbf{U}^{[k]}$ *be the* k^{th} *iterate of the present algorithm and* $\boldsymbol{u}$ *be the solution of (3.1). Then, it holds*

$$||\boldsymbol{u} - \mathbf{U}^{[k]}||_{\overline{\Omega}^{N,M}} \leq C2^{-k} + C(\Delta t + N^{-2}\ln^2 N). \tag{3.21}$$

Proof. Combining Lemma 3.2 and Theorem 3.3 together with (3.13), we have the proof of the theorem. □

3.4 Numerical results

In this section we shall consider two test problems and present the obtained numerical results.

Example 3.1. Consider the following problem

$$\begin{cases} \partial_t \boldsymbol{u} + \boldsymbol{\mathcal{E}}\partial_x^2 \boldsymbol{u} + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f} \quad in\ \Omega := (0,1)\times(0,1], \\ \boldsymbol{u}(x,0) = \mathbf{0}, \quad x \in [0,1], \quad \boldsymbol{u}(0,t) = \mathbf{g}_0(t), \boldsymbol{u}(1,t) = \mathbf{g}_1(t), \quad t \in (0,T], \end{cases}$$

where

$$\boldsymbol{A} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad \boldsymbol{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix},$$

and $f_1, f_2, \mathbf{g}_0$ and $\mathbf{g}_1$ are determined such that $\boldsymbol{u} = (u_1, u_2)^T$ is of the form

$$u_1(x,t) = t(\varphi_1(x) + \varphi_2(x) - 2) + (1+x)te^{-t},$$

$$u_2(x,t) = \varepsilon_1(1 - e^{-t})(\varphi_1(x) - 1) + t(1-t)(\varphi_2(x) - 1),$$

with

$$\varphi_n(x) = (e^{-x/\sqrt{\varepsilon_n}} + e^{-(1-x)/\sqrt{\varepsilon_n}})(1 + e^{-1/\sqrt{\varepsilon_n}})^{-1}, \quad n = 1, 2.$$

We compute the maximum pointwise errors $\mathbf{E}_{\varepsilon_1,\varepsilon_2}^{N,\Delta t}$ and the uniform errors $\mathbf{E}^{N,\Delta t}$ as follows

$$\mathbf{E}_{\varepsilon_1,\varepsilon_2}^{N,\Delta t} = ||\boldsymbol{u} - \mathbf{U}^{N,\Delta t}||_{\overline{\Omega}^{N,M}}, \quad \mathbf{E}^{N,\Delta t} = \max_{\varepsilon_1} \mathbf{E}_{\varepsilon_1}^{N,\Delta t},$$

where $\mathbf{E}_{\varepsilon_1}^{N,\Delta t} = \max\{\mathbf{E}_{\varepsilon_1,1}^{N,\Delta t}, \mathbf{E}_{\varepsilon_1,10^{-1}}^{N,\Delta t}, ..., \mathbf{E}_{\varepsilon_1,10^{-n}}^{N,\Delta t}\}$ is evaluated for a fixed value of $\varepsilon_1 = 10^{-q}$, q is a non-negative integer. We then define the uniform convergence rates as follows

$$\mathbf{R}^{N,\Delta t} = \log_2(\mathbf{E}^{N,\Delta t}/\mathbf{E}^{2N,\Delta t/4}).$$

TABLE 3.1: Present algorithm with $\boldsymbol{P}_{i,j} = \boldsymbol{A}_{i,j}$: Uniform errors $\mathbf{E}^{N,\Delta t}$ and rates of convergence $\mathbf{R}^{N,\Delta t}$ for Example 3.1.

	$N = 2^5$ $\Delta t = 1/4$	$N = 2^6$ $\Delta t = 1/4^2$	$N = 2^7$ $\Delta t = 1/4^3$	$N = 2^8$ $\Delta t = 1/4^4$	$N = 2^9$ $\Delta t = 1/4^5$
$E_1^{N,\Delta t}$	8.691e-02	2.481e-02	6.409e-03	1.616e-03	4.048e-04
$R_1^{N,\Delta t}$	1.808	1.953	1.988	1.997	
$E_2^{N,\Delta t}$	9.326e-02	2.411e-02	6.076e-03	1.522e-03	3.807e-04
$R_2^{N,\Delta t}$	1.951	1.989	1.997	1.999	

Tables 3.1, 3.2, and 3.3 display the componentwise uniform errors and rates of uniform convergence for Example 3.1, when $\boldsymbol{P}_{i,j} = \boldsymbol{A}_{i,j}$, $\boldsymbol{P}_{i,j} = diag(\boldsymbol{A}_{i,j})$ (the diagonal part of the matrix $\boldsymbol{A}_{i,j}$), and $\boldsymbol{P}_{i,j} = ltr(\boldsymbol{A}_{i,j})$ (the lower triangular part of the matrix $\boldsymbol{A}_{i,j}$), respectively. From these tables, we observe that the uniform errors are similar in size for all the choices. Tables 3.4 and 3.5 display the iteration counts

TABLE 3.2: Present algorithm with $\boldsymbol{P}_{i,j} = diag(\boldsymbol{A}_{i,j})$: Uniform errors $\mathbf{E}^{N,\Delta t}$ and rates of convergence $\mathbf{R}^{N,\Delta t}$ for Example 3.1.

	$N = 2^5$ $\Delta t = 1/4$	$N = 2^6$ $\Delta t = 1/4^2$	$N = 2^7$ $\Delta t = 1/4^3$	$N = 2^8$ $\Delta t = 1/4^4$	$N = 2^9$ $\Delta t = 1/4^5$
$E_1^{N,\Delta t}$	8.691e-02	2.481e-02	6.409e-03	1.616e-03	4.048e-04
$R_1^{N,\Delta t}$	1.808	1.953	1.988	1.997	
$E_2^{N,\Delta t}$	2.762e-01	7.497e-02	1.918e-02	4.822e-03	1.207e-03
$R_2^{N,\Delta t}$	1.881	1.967	1.992	1.998	

TABLE 3.3: Present algorithm with $\boldsymbol{P}_{i,j} = ltr(\boldsymbol{A}_{i,j})$: Uniform errors $\mathbf{E}^{N,\Delta t}$ and rates of convergence $\mathbf{R}^{N,\Delta t}$ for Example 3.1.

	$N = 2^5$ $\Delta t = 1/4$	$N = 2^6$ $\Delta t = 1/4^2$	$N = 2^7$ $\Delta t = 1/4^3$	$N = 2^8$ $\Delta t = 1/4^4$	$N = 2^9$ $\Delta t = 1/4^5$
$E_1^{N,\Delta t}$	8.691e-02	2.481e-02	6.409e-03	1.616e-03	4.048e-04
$R_1^{N,\Delta t}$	1.808	1.953	1.988	1.997	
$E_2^{N,\Delta t}$	9.160e-02	2.475e-02	6.320e-03	1.589e-03	3.977e-04
$R_2^{N,\Delta t}$	1.888	1.969	1.992	1.998	

TABLE 3.4: Present algorithm with $\boldsymbol{P}_{i,j} = \boldsymbol{A}_{i,j}$: Iteration counts taking $\varepsilon_1 = 10^{-9}$ in Example 3.1.

$\varepsilon_2 = 10^{-n}$	$N = 2^5$ $\Delta t = 1/4$	$N = 2^6$ $\Delta t = 1/4^2$	$N = 2^7$ $\Delta t = 1/4^3$	$N = 2^8$ $\Delta t = 1/4^4$	$N = 2^9$ $\Delta t = 1/4^5$
$n = 0$	2	3	4	5	6
1	2	2	3	3	5
2	1	2	3	4	5
3	1	3	4	5	6
4	2	3	4	5	6
5	3	4	4	5	6
6	3	3	3	3	3
7	2	2	2	2	2
8	1	1	1	1	1
9	1	1	1	1	1

that are required to achieve the stopping criterion of the algorithm with $\boldsymbol{P}_{i,j} = \boldsymbol{A}_{i,j}$ and $\boldsymbol{P}_{i,j} = diag(\boldsymbol{A}_{i,j})$ or $ltr(\boldsymbol{A}_{i,j})$, respectively. From these tables we note that the number of iterations required are same for the choices $\boldsymbol{P}_{i,j} = diag(\boldsymbol{A}_{i,j})$ and $\boldsymbol{P}_{i,j} = ltr(\boldsymbol{A}_{i,j})$, while the iterations differ slightly for the choice $\boldsymbol{P}_{i,j} = \boldsymbol{A}_{i,j}$.

TABLE 3.5: Present algorithm with $\boldsymbol{P}_{i,j} = diag(\boldsymbol{A}_{i,j})$ or $\boldsymbol{P}_{i,j} = ltr(\boldsymbol{A}_{i,j})$: Iteration counts taking $\varepsilon_1 = 10^{-9}$ in Example 3.1.

$\varepsilon_2 = 10^{-n}$	$N = 2^5$	$N = 2^6$	$N = 2^7$	$N = 2^8$	$N = 2^9$
	$\Delta t = 1/4$	$\Delta t = 1/4^2$	$\Delta t = 1/4^3$	$\Delta t = 1/4^4$	$\Delta t = 1/4^5$
$n = 0$	2	3	4	4	5
1	2	2	2	3	4
2	1	1	2	3	4
3	1	2	3	4	5
4	1	3	4	5	6
5	2	3	4	4	5
6	2	3	3	3	3
7	1	2	2	2	2
8	1	1	1	1	1
9	1	1	1	1	1

Example 3.2. Consider the following problem

$$\begin{cases} \partial_t \boldsymbol{u} + \boldsymbol{\mathcal{E}}\partial_x^2 \boldsymbol{u} + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f} \quad in\ \Omega := (0,1) \times (0,1], \\ \boldsymbol{u}(x,0) = \mathbf{0}, \quad x \in [0,1], \quad \boldsymbol{u}(0,t) = \mathbf{0}, \boldsymbol{u}(1,t) = \mathbf{0}, \quad t \in (0,T], \end{cases}$$

where

$$\boldsymbol{A} = \begin{pmatrix} 2(1+x)^2 & -(1+x^3) \\ -2\cos(\pi x/4) & 2.2e^{1-x} \end{pmatrix}, \quad \boldsymbol{f} = \begin{pmatrix} \cos(\pi x/2) \\ x \end{pmatrix}.$$

Since the exact solution of this test problem is not known, we apply the double mesh method to compute the maximum pointwise errors as follows

$$\mathbf{E}^{N,\Delta t}_{\varepsilon_1,\varepsilon_2} = ||\mathbf{U}^{N,\Delta t} - \mathbf{U}^{2N,\Delta t/4}||_{\overline{\Omega}^{N,M}},$$

where the approximate solution $\mathbf{U}^{2N,\Delta t/4}$ is obtained on a mesh that consists of same transition parameters ρ_1 and ρ_2 as for the solution $\mathbf{U}^{N,\Delta t}$, and contains $2N+1$ mesh points in space and step size $\Delta t/4$ in time. The uniform errors $\mathbf{E}^{N,\Delta t}$ and uniform convergence rates $\mathbf{R}^{N,\Delta t}$ are calculated in the same way as earlier.

Table 3.6: Present algorithm with $\boldsymbol{P}_{i,j} = \boldsymbol{A}_{i,j}$: Uniform errors $\mathbf{E}^{N,\Delta t}$ and rates of convergence $\mathbf{R}^{N,\Delta t}$ for Example 3.2.

	$N = 2^5$ $\Delta t = 1/4$	$N = 2^6$ $\Delta t = 1/4^2$	$N = 2^7$ $\Delta t = 1/4^3$	$N = 2^8$ $\Delta t = 1/4^4$	$N = 2^9$ $\Delta t = 1/4^5$
$E_1^{N,\Delta t}$	3.086e-02	1.072e-02	3.192e-03	9.070e-04	2.545e-04
$R_1^{N,\Delta t}$	1.526	1.748	1.815	1.833	
$E_2^{N,\Delta t}$	3.116e-02	1.094e-02	3.421e-03	1.080e-03	2.730e-04
$R_2^{N,\Delta t}$	1.511	1.677	1.664	1.984	

Table 3.7: Present algorithm with $\boldsymbol{P}_{i,j} = diag(\boldsymbol{A}_{i,j})$: Uniform errors $\mathbf{E}^{N,\Delta t}$ and rates of convergence $\mathbf{R}^{N,\Delta t}$ for Example 3.2.

	$N = 2^5$ $\Delta t = 1/4$	$N = 2^6$ $\Delta t = 1/4^2$	$N = 2^7$ $\Delta t = 1/4^3$	$N = 2^8$ $\Delta t = 1/4^4$	$N = 2^9$ $\Delta t = 1/4^5$
$E_1^{N,\Delta t}$	4.274e-02	1.318e-02	3.495e-03	9.540e-04	2.658e-04
$R_1^{N,\Delta t}$	1.697	1.915	1.873	1.843	
$E_2^{N,\Delta t}$	4.348e-02	1.472e-02	4.286e-03	1.266e-03	3.241e-04
$R_2^{N,\Delta t}$	1.562	1.780	1.759	1.966	

Table 3.8: Present algorithm with $\boldsymbol{P}_{i,j} = ltr(\boldsymbol{A}_{i,j})$: Uniform errors $\mathbf{E}^{N,\Delta t}$ and rates of convergence $\mathbf{R}^{N,\Delta t}$ for Example 3.2.

	$N = 2^5$ $\Delta t = 1/4$	$N = 2^6$ $\Delta t = 1/4^2$	$N = 2^7$ $\Delta t = 1/4^3$	$N = 2^8$ $\Delta t = 1/4^4$	$N = 2^9$ $\Delta t = 1/4^5$
$E_1^{N,\Delta t}$	3.620e-02	1.115e-02	3.293e-03	9.310e-04	2.616e-04
$R_1^{N,\Delta t}$	1.699	1.760	1.822	1.831	
$E_2^{N,\Delta t}$	3.742e-02	1.288e-02	3.813e-03	1.173e-03	2.980e-04
$R_2^{N,\Delta t}$	1.539	1.756	1.700	1.977	

Tables 3.6, 3.7, and 3.8 display the componentwise uniform errors and rates of uniform convergence for Example 3.2. Tables 3.9 and 3.10 give the number of iterations that are required to achieve the stopping criterion of the algorithm. From these tables, we have the same observations as we have had from tables corresponding to Example 3.1.

In order to compare the computational cost of the algorithm for different choices of $\boldsymbol{P}_{i,j}$ we include Tables 3.11 and 3.12, where we display the used CPU time (in seconds) for different values of N and Δt for Examples 3.1 and 3.2, respectively,

TABLE 3.9: Present algorithm with $\boldsymbol{P}_{i,j} = \boldsymbol{A}_{i,j}$: Iteration counts taking $\varepsilon_1 = 10^{-9}$ in Example 3.2.

$\varepsilon_2 = 10^{-n}$	$N = 2^5$ $\Delta t = 1/4$	$N = 2^6$ $\Delta t = 1/4^2$	$N = 2^7$ $\Delta t = 1/4^3$	$N = 2^8$ $\Delta t = 1/4^4$	$N = 2^9$ $\Delta t = 1/4^5$
$n = 0$	3	4	5	5	6
1	2	2	3	4	5
2	1	3	4	5	6
3	2	3	5	6	7
4	3	4	5	6	7
5	3	4	5	6	6
6	3	3	3	3	3
7	2	2	2	2	2
8	1	1	1	1	1
9	1	1	1	1	1

TABLE 3.10: Present algorithm with $\boldsymbol{P}_{i,j} = diag(\boldsymbol{A}_{i,j})$ or $\boldsymbol{P}_{i,j} = ltr(\boldsymbol{A}_{i,j})$: Iteration counts taking $\varepsilon_1 = 10^{-9}$ in Example 3.2.

$\varepsilon_2 = 10^{-n}$	$N = 2^5$ $\Delta t = 1/4$	$N = 2^6$ $\Delta t = 1/4^2$	$N = 2^7$ $\Delta t = 1/4^3$	$N = 2^8$ $\Delta t = 1/4^4$	$N = 2^9$ $\Delta t = 1/4^5$
$n = 0$	2	3	4	5	5
1	2	2	2	3	4
2	1	2	3	4	5
3	1	2	4	5	6
4	2	3	4	5	6
5	2	3	4	5	5
6	2	3	3	3	3
7	2	2	2	2	2
8	1	1	1	1	1
9	1	1	1	1	1

TABLE 3.11: The used CPU time in seconds for Example 3.1 with $\varepsilon_1 = 10^{-8}, \varepsilon_2 = 10^{-7}$.

Algorithm↓	$N = 2^6$ $\Delta t = 1/4^2$	$N = 2^7$ $\Delta t = 1/4^3$	$N = 2^8$ $\Delta t = 1/4^4$	$N = 2^9$ $\Delta t = 1/4^5$	$N = 2^{10}$ $\Delta t = 1/4^6$
$\boldsymbol{P}_{i,j} = diag(\boldsymbol{A}_{i,j})$	1.088	5.559	42.090	392.212	4818.501
$\boldsymbol{P}_{i,j} = ltr(\boldsymbol{A}_{i,j})$	1.141	5.526	43.136	393.257	5058.971
$\boldsymbol{P}_{i,j} = \boldsymbol{A}_{i,j}$	1.281	6.249	55.331	779.942	11143.219

taking $\varepsilon_1 = 10^{-8}$, $\varepsilon_2 = 10^{-7}$. From these tables, we observe that the algorithm is more efficient for the choices $\boldsymbol{P}_{i,j} = diag(\boldsymbol{A}_{i,j})$ and $\boldsymbol{P}_{i,j} = ltr(\boldsymbol{A}_{i,j})$ than the

TABLE 3.12: The used CPU time in seconds for Example 3.2 with $\varepsilon_1 = 10^{-8}, \varepsilon_2 = 10^{-7}$.

Algorithm↓	$N = 2^6$ $\Delta t = 1/4^2$	$N = 2^7$ $\Delta t = 1/4^3$	$N = 2^8$ $\Delta t = 1/4^4$	$N = 2^9$ $\Delta t = 1/4^5$	$N = 2^{10}$ $\Delta t = 1/4^6$
$\boldsymbol{P}_{i,j} = diag(\boldsymbol{A}_{i,j})$	0.387	1.530	12.547	157.318	2913.764
$\boldsymbol{P}_{i,j} = ltr(\boldsymbol{A}_{i,j})$	0.406	1.543	11.313	154.032	3170.091
$\boldsymbol{P}_{i,j} = \boldsymbol{A}_{i,j}$	0.463	2.236	41.814	917.521	19251.564

choice $\boldsymbol{P}_{i,j} = \boldsymbol{A}_{i,j}$. Further, we note that the used CPU time for the choices $\boldsymbol{P}_{i,j} = diag(\boldsymbol{A}_{i,j})$ and $\boldsymbol{P}_{i,j} = ltr(\boldsymbol{A}_{i,j})$ are similar. In summary, the uniform errors and the uniform rates are similar for all the choices, while the additive (or splitting) schemes are computationally more efficient than the standard discretization scheme.

3.5 Conclusions

A parabolic coupled system of singularly perturbed reaction-diffusion problems is considered in which perturbation parameters can be of distinct magnitude. We decomposed the original domain into five overlapping subdomains. On each subdomain, we consider two additive schemes on a uniform mesh in time and the standard central difference scheme on a uniform mesh in space. We provided convergence analysis of the algorithm using some auxiliary problems and the algorithm is shown to be uniformly convergent of order two in space and order one in time. Further, numerical results are given in support of the theoretical convergence result and as well as to illustrate the efficiency of the additive schemes.

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