Chapter 1

Introduction

1.1 Singularly perturbed problems

Singular perturbation problems (SPPs) play an essential role in understanding the hydrodynamical phenomenon and have many applications known to the real world. They account for many branches of applied mathematics and engineering, e.g. computational fluid dynamics, financial modeling, elasticity, gas porous electrodes theory, chemical reactors, fluid mechanics. These problems are usually characterized by an arbitrarily small parameter or perturbation parameter 'ε' multiplied with the highest order derivative involved in the differential equation. Singularity is observed in the solution when the perturbation parameter $\varepsilon \to 0$, as the problem becomes ill-posed because the order of the differential equation is reduced although the conditions on the boundary remain the same in number. The solution here reaches a discontinuous limit as $\varepsilon \to 0$, which is not in the case of regular perturbation problems. As a result, the solution of SPPs exhibits a multiscale phenomenon, i.e. the solution changes rapidly in some regions of the domain and shows smooth behavior away from them. The region where this rapid change occurs is termed as the layer region (boundary or interior layers) and where the solution behaves smoothly as the regular region.

SPPs drew the attention of researchers in the late twentieth century, when Prandtl gave his seminal paper [1] in the third International Congress of Mathematics held at Heidelberg in 1904. It revolutionalized the area of modern fluid dynamics, explaining the crucial role played by boundary layers in determining the fluid flow pattern.

In his work, he explained that the properties of the dependent variable may be significantly different inside the boundary layer than outside of boundary layer. He added that the frictional forces near the boundary layers are dominant over those of the viscous forces on the body. Thus, making the study of singular perturbation problems more exciting and useful to work upon. Moreover, the numerical treatment of these problems includes exponential fitting, adaptive grids (meshes), and ideas based on the method of asymptotic expansions. Their occurrence in the areas of science and engineering is quite frequent. The most striking example we can think of is the Navier-Stokes equation with a large Reynolds number, i.e.

$$\begin{cases} \dfrac{\partial \mathbf{v}}{\partial t} + \mathbf{v}.\nabla \mathbf{v} + \nabla p = \dfrac{1}{Re}\nabla^2 \mathbf{v}, \\ \nabla.\mathbf{v} = 0, \end{cases} \tag{1.1}$$

along with suitable initial and boundary conditions, where p represents the pressure and $v = (v_1, v_2)$ is the velocity field with components v_1 and v_1 along the axes respectively. The parameter $Re = |v|L/\mu$ represents the *Reynolds number* with L being the characteristic length and μ the kinematic viscosity of the fluid. For large value of Re, the modeled equation (1.1) behaves like a singularly perturbed differential equation. The solution to such problems generally exhibits layers, i.e. boundary layers near the boundary and interior layers near the point of discontinuity in the domain. Friedrichs and Wasow were the first to coin the term '*Singular perturbation*' in their paper [2]. Though, it was Prandtl who made it possible for the substantial work of Wasow to gain more generality by introducing the terminology 'boundary layers.' Depending on the characteristics, a boundary layer can be of either regular or parabolic type. It is said to be a *parabolic boundary layer* if corresponding to $\varepsilon \to 0$, the characteristics of the reduced equation are parallel to the boundary, and a *regular boundary layer* if the characteristics are not parallel to the boundary. The layer near the corner is termed the corner layer. For a detailed description of the definitions described above, we refer the reader to [3].

The singular perturbation theory gained its greatest significance through Freidrichs and Wasow's groundbreaking works [2, 4]. Numerous mathematicians then began to concentrate on this area of mathematics. This area of mathematics has flourished to a remarkable level over the previous few decades. Although various beneficial methods have been developed, significant advancements are still being made, and valuable research is ongoing. Below we provide a formal definition of SPPs.

Definition 1.1.1. [5] Let $\mathcal{P}_\varepsilon$ be a problem with solution $u_\varepsilon \in S$ for all $\varepsilon \in G$, where S a is function space with norm $\|.\|_S$ and $G \subset \mathbb{R}^n$ is a parameter domain. The continuous function $u : G \to S, \varepsilon \mapsto u_\varepsilon$ is said to be regular for $\varepsilon \to \varepsilon^\star \in \partial G$ if there exists a function $u^\star \in S$ such that:

$$\lim_{\varepsilon \to \varepsilon^\star} \|u_\varepsilon - u^\star\|_S = 0,$$

otherwise we say u_ε is **singular** and $\mathcal{P}_\varepsilon$ is **singularly perturbed**.

1.2 Numerical methods for singularly perturbed differential equations

Singularly perturbed differential equations with arbitrarily small perturbation parameter ε such that $0 < \varepsilon \ll 1$ attracted both mathematicians and engineers to understand the boundary layer theory of fluid dynamics. Asymptotic expansion technique was mainly used to approximate the solution in the initial development of this area. The asymptotic expansion utilizes the asymptotic sequence of the perturbation parameter ε leading to a differential equation of lower order than the original one. Consequently, this asymptotic expansion doesn't allow the solution to satisfy all the initial and the boundary conditions. To understand and answer these questions, several mathematicians and physicians worked in this field, making the boundary layer theory the foundation for modern fluid dynamics. Several numerical approaches can be listed for SPPs, from classical to robust numerical methods to capture their multiscale behavior. The classical methods (standard finite difference methods) are proven inefficient on uniform meshes as they fail to capture the layer behavior of the solution unless extremely large number of mesh points are chosen. This is due to the presence of steep gradients in the exact solution.

Classical numerical methods are not good enough to decrease the maximum pointwise error unless the perturbation parameter is of the magnitude of the mesh size [6]. The result motivated the search for such a numerical method that satisfies our natural expectation of reduced error when the mesh is refined. In turn, researchers found it exciting to develop an ε-uniform/parameter-uniform/ parameter-robust/uniformly convergent/robust convergent numerical method for which the numerical error and the order of convergence do not depend on the perturbation parameter ε.

Definition 1.2.1. [5] Let $\mathcal{P}_\varepsilon$ be a problem with solution u_ε and let $U_\varepsilon^{N,M}$ be its approximation obtained by some numerical method. The method is said to be uniformly convergent or robust with respect to the perturbation parameter ε in a given norm $\|.\|_\star$ if there exists N_0 and M_0 independent of ε such that

$$\|u_\varepsilon - U_\varepsilon^{N,M}\|_\star \leq C\vartheta(N,M) \quad \text{for } N \geq N_0,\ M \geq M_0,$$

with a function ϑ that is independent of ε and $\lim_{N\to\infty,\ M\to\infty} \nu(N,M) = 0$; and a constant $C > 0$ that is independent of ε and $N,\ M$.

Mainly, numerical methods are classified into three categories, i.e. *fitted operator method* (FOM), *fitted mesh method* (FMM), and *domain decomposition method* (DDM). The fitted operator method is defined on a standard mesh, where the exponential fitting factors control the numerical solution's rapid growth or decay in the boundary layers. These operators are constructed by choosing the difference coefficients so that some or all the exponential functions lie in the null space of the difference operator. This method was first suggested for solving the problem of a viscous fluid past a cylinder by Allen and Southwell [7] in 1955. More insights about fitted operator methods can be found in [7–13].

On the contrary, *fitted mesh method* utilizes standard finite difference operators on special meshes that are fine near the boundary or interior layers and coarse in the region away from it. The well-known layer resolving fitted meshes are Shishkin mesh [14] (which is piecewise uniform and easy to construct) and Bakhvalov mesh [15] (which is constructed using a non-linear mesh generating function). They require prior information about the location and width of the boundary layers. FOMs are difficult to implement on higher order differential equations or sometimes impossible. FMMs are more popular than FOMs because they can be easily implemented on the standard finite difference operators and can be extended for solving problems of higher dimensions and non-linear problems with complicated domain structures. Some applications of Shishkin meshes are given in [6, 9, 16–28].

Domain decomposition is an effective numerical approach for solving partial differential equations. It has gained popularity and become essential in simulations of fluid flows, particularly those that originate from models of the real world and demand extensive parallel computation. More than a century ago, Schwarz established the concept of domain decomposition to solve partial differential equations (PDEs) [29]. Today, it is an effective computing method for simulating fluid flows. In the past few decades, computational mathematics, sciences, and engineering fields have witnessed a rapid and steady rise in the amount of study devoted to the theory and application of DDMs. DDMs were initially developed to solve the problem with complex geometries. Essentially, the concept is to divide a big, complicated problem into smaller, simpler sub-problems, solve them separately, and then combine their solutions to provide a global solution. The motivation behind the use of these methods is encouraged by their potential for effective parallelization through data locality, their adaptability to PDEs posed on geometrically challenging domains, their ability to handle PDEs with varying behavior across the domain, and superior convergence properties of the iterative method. We refer the reader to [30, 31] for a detailed overview of domain decomposition methods.

DDMs have also been developed and investigated for time-dependent problems. One standard method to solve time dependent problems is to apply the Domain decomposition to semi-discretized problems in time and solve the resulting spatial problems [32]. An improved approach is the Schwarz waveform relaxation (SWR) that directly addresses the time-dependent problem to provide an iterative solution [33]. The method divides the spatial domain, but after that, it solves the time-dependent problems on each subdomain across a time interval. The solution is then updated by sharing traces of the solution over time from each subdomain with its neighbors. The Schwarz waveform relaxation method converges depending on the amount of overlap (if overlapping partitions are utilized), the characteristics of the transmission conditions in space-time, and the size of the time window. Some optimized Schwarz waveform relaxation methods are suggested in [34, 35] by adapting better transmission conditions for faster convergence. Next, we define the definition of parameter uniform numerical method in a domain decomposition environment.

Definition 1.2.2. Let u_ε be the exact solution of a singularly perturbed problem and let U_ε^k be an approximation to u_ε generated by kth step of some iterative numerical method. The numerical method is said to be uniformly convergent with respect to ε in a given norm $||.||_*$ if there exists $N_0 > 0,\ M_0 > 0$ independent of ε such that for all $N \geq N_0,\ M \geq M_0$

$$||u_\varepsilon - U_\varepsilon^k||_* \leq C\rho^k + C\nu(N, M), \qquad 0 \leqslant \rho < 1, \qquad \lim_{N\to\infty,\ M\to\infty} \nu(N, M) = 0,$$

with a function ν that is independent of ε, and constant $C > 0$ that is independent of $\varepsilon,\ N, M$ and k.

1.3 Literature review

Singularly perturbed problems (SPPs) arise in mathematical modeling of several practical problems in engineering and applied mathematics, for example, in describing the theory of gyroscopes [36], in studying linear spring-mass system without damping but with forcing and a small spring constant [37], in variational problems in control theory [38], etc. We refer the reader to [5, 6, 9, 39] for a detailed description. Now, we provide a brief overview of the relevant literature.

1.3.1 Singularly perturbed parabolic coupled system of reaction-diffusion delay problems

There are several practical phenomena in which delay differential equations frequently arise, e.g., in modeling of the human pupil–light reflex [40], population dynamics [41], physiological processes [42], and in control theory [43] etc. In the past two decades, many researchers have paid attention to delay differential equations and developed various numerical methods, see e.g., [44–48] and the references therein. In [49], a fitted operator finite difference method is developed for a partial delay differential equations system. An initial value method is proposed to solve a weakly coupled system of singularly perturbed reaction-diffusion delay problems in [50]. Further, an adaptive mesh selection method is developed for a coupled system of singularly perturbed convection-diffusion problems in [51], where the mesh is generated with the help of an entropy production operator. Recently, in [52], a robust numerical method for a singularly perturbed coupled system of parabolic delay problems is constructed based on the backward Euler discretization in time and the central difference discretization in space. However, the computational cost of this methods is high because a coupled system needs to be solved at each time level.

1.3.2 Singularly perturbed parabolic coupled system of reaction-diffusion problems

The robust convergent numerical methods for a singularly perturbed parabolic coupled system of reaction-diffusion problems are developed using fitted mesh, domain decomposition, and fitted operator approaches (see [25, 53–58] and the references therein). The asymptotic analysis of the solution and its derivatives is given in [54]. The authors used the Euler scheme on a uniform mesh in the time variable and the central difference scheme on a piecewise uniform Shishkin mesh in the space variable to discretize the problem. In [59], the authors developed a numerical method combining the Crank-Nicolson scheme with a central finite difference scheme defined on a piecewise uniform Shishkin mesh for space discretization to obtain a higher order convergence for the time variable. A parameter uniform convergent method is developed in [60] based on the mesh equidistribution approach. In [55], the authors proposed a uniform convergent numerical method where two splitting schemes on uniform meshes are used for time discretization and the central difference scheme on Shishkin mesh for space discretization. A fitted operator method on uniform meshes is studied in [58]. Further, in [53], a domain decomposition method of SWR type is developed and analyzed. They used the backward Euler scheme on uniform mesh in the time variable and the standard central difference scheme on uniform mesh in the space variable. It is proved that the method gives robust numerical approximations for the exact solution. However, the computational cost of this algorithm is high, since the components of the approximate solution are coupled at each time level.

1.3.3 Singularly perturbed parabolic coupled system of reaction-diffusion semilinear problems

Scalar singularly perturbed semilinear problems are studied extensively in the literature. But very little research is available for solving system of singularly perturbed parabolic semilinear reaction-diffusion problems. For a scalar problem, on layer-adapted meshes of the Bakhvalov and Shishkin types, the authors established the existence and investigated the accuracy of discrete solutions in [61]. In [62], authors considered semi and fully discretization of the backward Euler and Crank-Nicolson methods and gave a posteriori estimates. A monotone iterative method is proposed by the author in [63]. Further, uniform convergence of the monotone methods is investigated. Also, a domain decomposition method of Schwarz waveform relaxation type is analyzed in [64]. The above mentioned studies mainly looked at scalar problems. Later, the idea was extended for coupled systems of semilinear reaction-diffusion equations [65] based on the fitted mesh method.

1.3.4 Singularly perturbed parabolic reaction-diffusion problems with Robin boundary conditions (RBCs)

Singularly perturbed problems with Dirichlet-type boundary conditions have been studied extensively in the literature (see [16, 66–71] and the references therein). However, there are few studies of such problems with Robin boundary conditions. In [72], the authors developed a finite difference method for a Neumann problem on a special piecewise uniform mesh. Then, a higher order time accurate finite difference method is analyzed in [73] with Robin type boundaries. In [74], the authors considered stationary reaction-diffusion problems with RBCs and proposed a uniform parameter numerical method combining the cubic spline and classical finite difference schemes. In [75], a numerical method involving the cubic spline scheme for boundary conditions and the central difference scheme at the interior points is given and analyzed. The mesh is constructed by the equidistribution of a positive monitor function. A parameter uniform finite difference scheme is analyzed in [76] where the central difference scheme is used to discretize the space variable, and the backward Euler scheme is used for the time variable. Recently, in [77], the authors have proposed a parameter-uniform numerical method on equidistributed meshes. Moreover, we are not aware of any study involving the domain decomposition approach for SPPs with Robin boundary conditions.

1.3.5 Singularly perturbed time delayed parabolic reaction-diffusion problems with Robin boundary conditions

In mathematical modeling of several physical phenomena of bioscience and control theory, delay differential equations frequently arise, for instance, in population dynamics [41], immunology [78], physiology [79], and neural networks [80]. The delay term in these models occurs due to feedback control because a finite time is required to sense and react to the data like the immune period, the time between a cell is infected and the new virus is produced, the stages of the life cycle, the duration of the infectious period, etc. [81]. Singularly perturbed delay differential equations generally arise when the highest order derivative term is multiplied with a small parameter, e.g., in modeling of the human pupil–light reflex [82], variational problems in control theory [83], the study of bistable devices [84] and chemical processes [85]. The numerical solution of a delay partial differential equation depends on the current stage and some previous stages of the solution. Researchers have proposed various numerical methods to solve singularly perturbed time delay parabolic reaction-diffusion equations [86–91]. All of the noted authors concentrate on Dirichlet boundary conditions. However, only a few research papers study these problems with Robin boundary conditions. In [76], the authors proposed and analyzed a fitted mesh method. They used the central difference scheme on a piecewise uniform Shishkin mesh for the space variable and the backward Euler scheme on a uniform mesh for the time variable. A uniformly convergent numerical method based on the collocation method is analyzed in [92]. Some recent studies examined the singularly perturbed reaction-diffusion problem with RBCs. In [77], a parameter uniform numerical method is given based on equidistributed meshes, and in [93], a domain decomposition method is developed. No study is available in the literature that analyzed singularly perturbed delayed reaction-diffusion problems in domain decomposition framework.

1.4 Objective of the thesis

The main objectives of the thesis are to develop parameter uniform numerical approximation for the following class of singularly perturbed partial differential equations based on domain decomposition and fitted mesh methods.

- Coupled systems of singularly perturbed time delayed parabolic reaction-diffusion problems having two perturbation parameters of different magnitudes.
- Coupled systems of singularly perturbed parabolic reaction-diffusion problems having two perturbation parameters of different magnitudes.
- Coupled systems of singularly perturbed semilinear parabolic reaction-diffusion problems having two perturbation parameters of different magnitudes.
- Singularly perturbed parabolic reaction-diffusion problems with Robin boundary conditions.
- Singularly perturbed parabolic time delayed reaction-diffusion problems with Robin boundary conditions.

Further, we provide the theoretical convergence analysis for the proposed numerical methods. Also, some test problems are considered to validate the theoretical outcomes.

1.5 Outline of the thesis

This thesis focuses on developing and analyzing robust numerical methods for singularly perturbed partial differential equations. The work presented in the thesis is divided into six chapters.

Chapter 1 introduces singularly perturbed problems, parameter uniform numerical methods, and domain decomposition methods. After that, a brief overview of the literature and an outline of the thesis is given.

In Chapter 2, we develop an adaptive numerical method for a coupled system of two singularly perturbed delay parabolic reaction-diffusion problems, where a perturbation parameter of different magnitude is multiplied with the diffusion term in each equation. The problem is discretized by splitting schemes on uniform meshes in the time direction and the central difference scheme on Shishkin and generalized Shishkin meshes in the space direction. We discuss error analysis of the method for both Shishkin and generalized Shishkin meshes and prove that the method is robustly convergent of order almost two in space and one in time. Numerical results are provided in support of theoretical findings.

In Chapter 3, we design and analyze additive schemes based domain decomposition method of SWR type for a parabolic coupled system of singularly perturbed reaction-diffusion problems involving perturbation parameters of different magnitudes. The original domain is divided into five overlapping subdomains. We consider two additive schemes on a uniform mesh in time on each subdomain and the standard central difference scheme on a uniform mesh in space. We prove that the proposed method yields uniformly convergent numerical approximations of almost second order in space and first order in time. Further, we have shown that the proposed method is computationally more efficient than the standard Euler scheme. Some numerical results are presented in support of the theory.

In Chapter 4, a domain decomposition method of SWR type is presented for a coupled system of singularly perturbed semilinear parabolic problems involving two perturbation parameters of different magnitudes. The computational domain is divided into five overlapping subdomains. On each subdomain, a classical central difference scheme in space, an Euler scheme, and splitting of components approach in time are employed. We prove that the method is uniformly convergent, having an order of almost two in space and one in time. Numerical results are given in support of the theoretical convergence result and to illustrate the efficiency of additive schemes.

In Chapter 5, we develop and analyze a domain decomposition method of SWR type for singularly perturbed partial differential equations with Robin type boundary conditions. The original domain is divided into three overlapping subdomains, and the problems are discretized on each subdomain using the backward Euler scheme in the time direction and the central difference scheme in the spatial direction, while the Robin boundary conditions associated with the problem are approximated using a special finite difference scheme to maintain the accuracy. The numerical method is proved to be unconditionally stable and uniformly convergent of order almost two in space and one in time. More interestingly, the convergence of the iterates is optimal for small values of the perturbation parameters. Numerical experiments are presented to confirm the theoretically proven convergence result.

In Chapter 6, a time delayed singularly perturbed reaction-diffusion parabolic problem with Robin type boundary conditions is considered. We divide the computational domain into three overlapping subdomains and discretize the problem by employing a central difference scheme in space direction and the Euler scheme in time direction on uniform mesh while a specific finite difference scheme is used to approximate the boundary conditions. We prove that the proposed method is robust convergent having the accuracy of order one in time and almost two in space. More importantly, we showed that only one iteration is needed for small values of the perturbation parameter. To support the theoretical findings, we include two test problems.

****