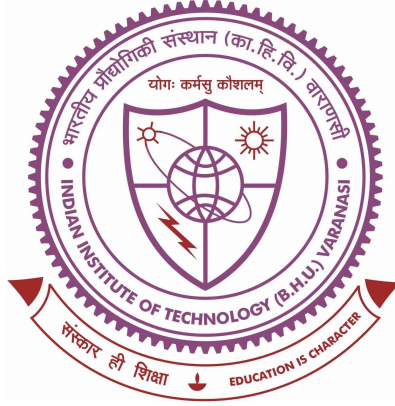


NUMERICAL APPROXIMATIONS FOR SINGULARLY
PERTURBED PARABOLIC PROBLEMS BASED ON
DOMAIN DECOMPOSITION AND FITTED MESHES



Thesis submitted in partial fulfillment

for the award of degree

DOCTOR OF PHILOSOPHY

by

Aakansha

DEPARTMENT OF MATHEMATICAL SCIENCES

INDIAN INSTITUTE OF TECHNOLOGY

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It is certified that the work contained in the thesis titled "*Numerical Approximations for Singularly Perturbed Parabolic Problems based on Domain Decomposition and Fitted Meshes*" by *Aakansha* has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

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ACKNOWLEDGEMENTS

I would like to start by expressing my sincere gratitude to my Ph.D. supervisor, Dr. Sunil Kumar, Associate Professor, Department of Mathematical Sciences, Indian Institute of Technology (BHU), Varanasi, for his unwavering support of this research work as well as for his patience, motivation, inspiration, and comprehensive understanding. His advice was helpful to me throughout the entire research and thesis-writing process. I am thankful for his dedicated efforts, without which this research work would not have been possible to complete.

I must also express my gratitude to my Research Progress Evaluation Committee (RPEC) members Dr. Vineet Kumar Singh, Associate Professor, Department of Mathematical Sciences, Indian Institute of Technology (BHU), Varanasi and Dr. Amitesh Kumar, Assistant Professor, Department of Mechanical Engineering, Indian Institute of Technology (BHU), Varanasi. Their insightful comments and suggestions, helped me to improve my work. I also want to thank my SRF committee members Prof. Arvind Kumar Mishra, Department of Mathematics, Banaras Hindu University, Varanasi and Prof. Tanmay Som, former Head of the Department for their valuable inputs during my research. I owe my special thanks to Prof. Sanjay Kumar Pandey, the Head of the Department, and Dr. Ashok Gupta, Convener, Department Post Graduate Committee (DPGC), for their encouragement and support throughout during my research work.

I am grateful to the Council of Scientific and Industrial Research (CSIR), Govt. of India, New Delhi, for providing me the Junior and Senior Research Fellowship for my Ph.D. program. I am also thankful to my institute, IIT(BHU), for providing the necessary resources that I required for my Ph.D. program.

I extend my deepest gratitude to my parents, Mr. Devendra Kumar and Mrs. Santosh Devi, for their unwavering support, constant encouragement, and unending

patience. When I grew exhausted, their motivation and support helped me to get back. Last but not least, I want to thank my elder sister Mrs. Shalu Singh for her love, support, and encouragement throughout my academic and personal life.

I sincerely thank my friend Mr. Nikhil Srivastava for his constant encouragement and support during my Ph.D. Also, I would like to thank my colleagues Pankhuri Jain, Yashveer Kumar, and Priyanka Rajput for their support during this journey.

I also want to thank my seniors, Dr. Joginder Singh and Dr. Sumit Saini, and the lab members Kuldeep, Shashikant, Anshima, Aishwarya, Richa, and Priyanka for always being there for me. In addition, I also want to thank my friends and other people whose names are not mentioned here but for supporting me in good and bad times.

This acknowledgment would be incomplete if the name of great visionary [Pt. Madan Mohan Malaviya](#) is not mentioned who made this divine centre of knowledge. I express my deepest regards to him.

Above all, I would like to thank the Almighty, Shri Kashi Vishwanath for his showers of blessings to successfully complete my Ph.D.

Varanasi-221005

Aakansha

Dedicated
to
My Parents

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Symbols

$\varepsilon, \varepsilon_1, \varepsilon_2$	Perturbation parameters
N, M, m_τ	Discretization parameters
k	Iteration number
O	Landau symbol
Ω	Domain of the problem
D	Domain of space variable
ρ, ρ_1, ρ_2	Subdomain parameters
$\Omega_{\ell\ell}, \Omega_\ell, \Omega_m, \Omega_r, \Omega_{rr}$	Subdomains of Ω
$D_{\ell\ell}, D_\ell, D_m, D_r, D_{rr}$	Subdomains of D
$C^{r,s}(\Omega), C^r(D)$	Function spaces
$g_{i,j}$	$g(x_i, t_j)$
$g_{p;i,j}$	$g_p(x_i, t_j)$
\mathbf{g}	$(g_1, g_2)^T$
$\mathbf{g}_{i,j}$	$\mathbf{g}(x_i, t_j) = (g_{1;i,j}, g_{2;i,j})^T$
$\mathbf{g}(a, t_j)$	$(g_1(a, t_j), g_2(a, t_j))^T$
$\mathbf{g}_{p;i,j}$	$\mathbf{g}_p(x_i, t_j) = (g_{p,1;i,j}, g_{p,2;i,j})^T$
$\mathbf{g} \leq \mathbf{h}$	If $g_n \leq h_n, n = 1, 2$
$ \mathbf{g} $	$(g_1 , g_2)^T$
$\ \mathbf{g}\ _{\overline{\Omega}^{N,M}}$	$\max_{(x_i, t_j) \in \overline{\Omega}^{N,M}} g(x_i, t_j) $
$\ \mathbf{g}\ _{\overline{\Omega}^{N,M}}$	$\max\{\ \mathbf{g}_1\ _{\overline{\Omega}^{N,M}}, \ \mathbf{g}_2\ _{\overline{\Omega}^{N,M}}\}$

$$\|g\|_{\overline{\Omega}} \qquad \max_{x \in \overline{\Omega}} |g(x)|$$

$$\|\mathbf{g}(a, t_j)\|_{\infty} \qquad \max_{i=1,2} \max_{t_j} |g_i(a, t_j)|$$

C Generic positive constant, independent of ε , N , k and M

\mathbf{C} Generic positive constant vector

PREFACE

This thesis is concerned with design, analysis, and implementation of robust numerical methods using domain decomposition and fitted meshes for singularly perturbed parabolic partial differential equations. These problems are characterized by the presence of boundary layers in their solutions. Although the layer regions are small, they have a considerable impact on the final result. Thus, solving these problems accurately and efficiently using standard numerical methods is inappropriate. Therefore, special numerical methods are needed to provide a numerical approximation that converges to the exact solution independent of the perturbation parameters. The developed numerical methods for singular perturbation problems are referred to as parameter-robust, parameter-uniform, or uniformly convergent.

First, we consider a coupled system of two singularly perturbed delay parabolic reaction–diffusion problems. In each equation, the diffusion term is multiplied by a small parameter. The parameters can be of different magnitudes, due to which the problem exhibits overlapping layers. To approximate the solution, we consider two splitting (or additive) schemes on uniform meshes for time discretization and the central difference scheme on Shishkin and generalized Shishkin meshes for space discretization. We prove that the proposed numerical method is uniformly convergent having convergence of order one in time and almost two in space. The splitting schemes reduced the computational time by decoupling the solution components at each time level. Further, we present numerical results for two test problems to support our theoretical findings and demonstrate that the splitting schemes are more efficient than the classical scheme.

Next, we consider a coupled system of singularly perturbed reaction-diffusion problems with perturbation parameters of different magnitudes. To solve this system

numerically, we develop additive schemes based domain decomposition algorithm of Schwarz waveform relaxation type. On each subdomain, we consider two additive schemes on a uniform mesh in time and the standard central difference scheme on a uniform mesh in space. We provide a convergence analysis of the algorithm using some auxiliary problems, and the algorithm is shown to be uniformly convergent. The additive schemes make the computation more efficient as they decouple the components of the approximate solution at each time level. Numerical results for two test problems are given to support the theoretical convergence result and illustrate the efficiency of additive schemes.

After that, a coupled system of singularly perturbed semilinear parabolic problems is considered, where the diffusion term in each equation is multiplied by the distinct perturbation parameters. An overlapping domain decomposition algorithm is proposed to solve this system numerically. On each subdomain, a classical central difference scheme in space along with splitting of components technique in time are employed. In this manner, the solution is computed by decoupling the components of the solution, which results in a significantly lower computational cost for the method than for the classical method. We introduce an iterative process to solve the semilinear coupled system where the Dirichlet boundaries are used to exchange the information between the subdomains. The method is proved to be parameter uniform by including some auxiliary problems. To support the theoretical findings, we consider two test problems. Moreover, to show the efficiency of the proposed method, we compare the CPU time (in seconds) for the proposed method and the classical Euler method.

Then we focus on a class of singularly perturbed partial differential equations with Robin type boundary conditions. The method considers three subdomains, of which two are finely meshed, and the other is coarsely meshed. The partial differential equation associated with the problem is discretized using the finite difference scheme

on each subdomain, while the Robin boundary conditions related to the problem are approximated using a special finite difference scheme to maintain accuracy. Then, an iterative method is introduced, where information is transmitted to the neighbors using a piecewise linear interpolation. It is proved that the resulting numerical approximations are parameter-uniform and, more interestingly, the convergence of the iterates is optimal for small values of the perturbation parameters. The numerical results support the theoretical results about convergence.

In the end, we extend the previously introduced approach to time delayed singularly perturbed reaction-diffusion parabolic problems with Robin type boundary conditions. In this method, the computational domain is decomposed into three overlapping subdomains (one is a regular domain, and the other two are layer subdomains) to adapt the singular behavior in thin regions near the boundaries. On each subdomain, the backward Euler scheme is used for time discretization, while the standard central difference scheme is used for space discretization. It is proved that the proposed method is robust convergent with order one in time and almost two in space using the barrier function approach. We also demonstrate that for small values of the perturbation parameter, one iteration is required to attain the desired accuracy. Further, some numerical results are included to illustrate the efficiency of the method.

