NUMERICAL APPROXIMATIONS FOR SINGULARLY PERTURBED PARABOLIC PROBLEMS BASED ON DOMAIN DECOMPOSITION AND FITTED MESHES

Thesis submitted in partial fulfillment

for the award of degree

DOCTOR OF PHILOSOPHY

by

Aakansha

DEPARTMENT OF MATHEMATICAL SCIENCES

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CERTIFICATE

It is certified that the work contained in the thesis titled ***"Numerical Approximations for Singularly Perturbed Parabolic Problems based on Domain Decomposition and Fitted Meshes"*** by ***Aakansha*** has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

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I, ***Aakansha***, certify that the work embodied in this thesis is my own bona fide work and carried out by me under the supervision of ***Dr. Sunil Kumar*** from ***July 2018*** to ***June 2023*** at the ***Department of Mathematical Sciences, Indian Institute of Technology (Banaras Hindu University), Varanasi.*** The matter embodied in this thesis has not been submitted for the award of any other degree/diploma. I declare that, I have faithfully acknowledged and given credits to the authors wherever their works have been cited in my work in this thesis. I further declare that I have not willfully copied any other's work, paragraphs, text, data, results, *etc.*, reported in journals, books, magazines, reports dissertations, theses, *etc.*, or available at websites and have not included them in this thesis and have not cited as my own work.

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Varanasi-221005 **Aakansha**

Dedicated

to

My Parents

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Symbols

$\varepsilon,\ \varepsilon_1,\ \varepsilon_2$	Perturbation parameters
$N,\ M,\ m_\tau$	Discretization parameters
k	Iteration number
O	Landau symbol
Ω	Domain of the problem
D	Domain of space variable
$\rho,\ \rho_1,\ \rho_2$	Subdomain parameters
$\Omega_{\ell\ell},\ \Omega_\ell,\ \Omega_m,\ \Omega_r,\ \Omega_{rr}$	Subdomains of Ω
$D_{\ell\ell},\ D_\ell,\ D_m,\ D_r,\ D_{rr}$	Subdomains of D
$C^{r,s}(\Omega), C^r(D)$	Function spaces
$g_{i,j}$	$g(x_i, t_j)$
$g_{p;i,j}$	$g_p(x_i, t_j)$
$\boldsymbol{g}$	$(g_1, g_2)^T$
$\boldsymbol{g}_{i,j}$	$\boldsymbol{g}(x_i, t_j) = (g_{1;i,j}, g_{2;i,j})^T$
$\boldsymbol{g}(a, t_j)$	$(g_1(a, t_j), g_2(a, t_j))^T$
$\boldsymbol{g}_{p;i,j}$	$\boldsymbol{g}_p(x_i, t_j) = (g_{p,1;i,j}, g_{p,2;i,j})^T$
$\boldsymbol{g} \le \boldsymbol{h}$	If $g_n \le h_n,\ n = 1, 2$
$\lvert\boldsymbol{g}\rvert$	$(\lvert g_1\rvert, \lvert g_2\rvert)^T$
$\lvert\lvert g\rvert\rvert_{\overline{\Omega}^{N,M}}$	$\max_{(x_i,t_j)\in\overline{\Omega}^{N,M}} \lvert g(x_i, t_j)\rvert$
$\lvert\lvert \boldsymbol{g}\rvert\rvert_{\overline{\Omega}^{N,M}}$	$\max\{\lvert\lvert g_1\rvert\rvert_{\overline{\Omega}^{N,M}}, \lvert\lvert g_2\rvert\rvert_{\overline{\Omega}^{N,M}}\}$

$\|\|g\|\|_{\overline{\Omega}}$	$\max_{x\in\overline{\Omega}} \|g(x)\|$
$\|\|\boldsymbol{g}(a,t_j)\|\|_\infty$	$\max_{i=1,2} \max_{t_j} \|g_i(a,t_j)\|$
C	Generic positive constant, independent of ε, N, k and M
$\boldsymbol{C}$	Generic positive constant vector

PREFACE

This thesis is concerned with design, analysis, and implementation of robust numerical methods using domain decomposition and fitted meshes for singularly perturbed parabolic partial differential equations. These problems are characterized by the presence of boundary layers in their solutions. Although the layer regions are small, they have a considerable impact on the final result. Thus, solving these problems accurately and efficiently using standard numerical methods is inappropriate. Therefore, special numerical methods are needed to provide a numerical approximation that converges to the exact solution independent of the perturbation parameters. The developed numerical methods for singular perturbation problems are referred to as parameter-robust, parameter-uniform, or uniformly convergent.

First, we consider a coupled system of two singularly perturbed delay parabolic reaction–diffusion problems. In each equation, the diffusion term is multiplied by a small parameter. The parameters can be of different magnitudes, due to which the problem exhibits overlapping layers. To approximate the solution, we consider two splitting (or additive) schemes on uniform meshes for time discretization and the central difference scheme on Shishkin and generalized Shishkin meshes for space discretization. We prove that the proposed numerical method is uniformly convergent having convergence of order one in time and almost two in space. The splitting schemes reduced the computational time by decoupling the solution components at each time level. Further, we present numerical results for two test problems to support our theoretical findings and demonstrate that the splitting schemes are more efficient than the classical scheme.

Next, we consider a coupled system of singularly perturbed reaction-diffusion problems with perturbation parameters of different magnitudes. To solve this system numerically, we develop additive schemes based domain decomposition algorithm of Schwarz waveform relaxation type. On each subdomain, we consider two additive schemes on a uniform mesh in time and the standard central difference scheme on a uniform mesh in space. We provide a convergence analysis of the algorithm using some auxiliary problems, and the algorithm is shown to be uniformly convergent. The additive schemes make the computation more efficient as they decouple the components of the approximate solution at each time level. Numerical results for two test problems are given to support the theoretical convergence result and illustrate the efficiency of additive schemes.

After that, a coupled system of singularly perturbed semilinear parabolic problems is considered, where the diffusion term in each equation is multiplied by the distinct perturbation parameters. An overlapping domain decomposition algorithm is proposed to solve this system numerically. On each subdomain, a classical central difference scheme in space along with splitting of components technique in time are employed. In this manner, the solution is computed by decoupling the components of the solution, which results in a significantly lower computational cost for the method than for the classical method. We introduce an iterative process to solve the semilinear coupled system where the Dirichlet boundaries are used to exchange the information between the subdomains. The method is proved to be parameter uniform by including some auxiliary problems. To support the theoretical findings, we consider two test problems. Moreover, to show the efficiency of the proposed method, we compare the CPU time (in seconds) for the proposed method and the classical Euler method.

Then we focus on a class of singularly perturbed partial differential equations with Robin type boundary conditions. The method considers three subdomains, of which two are finely meshed, and the other is coarsely meshed. The partial differential equation associated with the problem is discretized using the finite difference scheme on each subdomain, while the Robin boundary conditions related to the problem are approximated using a special finite difference scheme to maintain accuracy. Then, an iterative method is introduced, where information is transmitted to the neighbors using a piecewise linear interpolation. It is proved that the resulting numerical approximations are parameter-uniform and, more interestingly, the convergence of the iterates is optimal for small values of the perturbation parameters. The numerical results support the theoretical results about convergence.

In the end, we extend the previously introduced approach to time delayed singularly perturbed reaction-diffusion parabolic problems with Robin type boundary conditions. In this method, the computational domain is decomposed into three overlapping subdomains (one is a regular domain, and the other two are layer subdomains) to adapt the singular behavior in thin regions near the boundaries. On each subdomain, the backward Euler scheme is used for time discretization, while the standard central difference scheme is used for space discretization. It is proved that the proposed method is robust convergent with order one in time and almost two in space using the barrier function approach. We also demonstrate that for small values of the perturbation parameter, one iteration is required to attain the desired accuracy. Further, some numerical results are included to illustrate the efficiency of the method.