COLLOCATION APPROXIMATION FOR FRACTIONAL INTEGRO-DIFFERENTIAL AND SUB-DIFFUSION EQUATIONS

Thesis submitted in partial fulfillment for the award of degree

Doctor of Philosophy

By

Sandeep Kumar

DEPARTMENT OF MATHEMATICAL SCIENCES
INDIAN INSTITUTE OF TECHNOLOGY
(BANARAS HINDU UNIVERSITY)
VARANASI– 221005
INDIA

Roll No. 18121006

June 2023

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01.06.2023

Dr. Rajesh Kumar Pandey
(Supervisor)
Associate Professor
Department of Mathematical Sciences
Indian Institute of Technology
(Banaras Hindu University)
Varanasi-221005

पर्यवेक्षक/Supervisor
गणितीय विज्ञान विभाग
Department of Mathematical Sciences
भारतीय प्रौद्योगिकी संस्थान
Indian Institute of Technology
(काशी हिन्दू विश्वविद्यालय)
Banaras Hindu University)
वाराणसी/Varanasi-221005

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I, ***Sandeep Kumar***, certify that the work embodied in this thesis is my own bonafide work and carried out by me under the supervision of ***Dr. Rajesh Kumar Pandey*** from ***July, 2018*** to ***May, 2023*** at the ***Department of Mathematical Sciences, Indian Institute of Technology (Banaras Hindu University), Varanasi***. The matter embodied in this thesis has not been submitted for the award of any other degree/diploma. I declare that I have faithfully acknowledged and given credits to the research workers wherever their works have been cited in my work in this thesis. I further declare that I have not willfully copied any other's work, paragraphs, text, data, results, *etc.*, reported in journals, books, magazines, reports dissertations, theses, *etc.*, or available at websites and have not included them in this thesis and have not cited as my own work.

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Sandeep Kumar
(Sandeep Kumar)

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It is certified that the above statement made by the student is correct to the best of my/our knowledge.

01.06.2023
(Dr. Rajesh Kumar Pandey)
Associate Professor
Department of Mathematical Sciences
Indian Institute of Technology
(Banaras Hindu University)
Varanasi-221005

पर्यवेक्षक /Supervisor
गणितीय विज्ञान विभाग
Department of Mathematical Sciences
भारतीय प्रौद्योगिकी संस्थान
Indian Institute of Technology
(काशी हिन्दू विश्वविद्यालय)
Banaras Hindu University)
वाराणसी /Varanasi-221005

01.06.2023
(Prof. S. K. Pandey)
Professor and Head
Department of Mathematical Sciences
Indian Institute of Technology
(Banaras Hindu University)
Varanasi-221005

विभागाध्यक्ष /HEAD
गणितीय विज्ञान विभाग
Department of Mathematical Sciences
भारतीय प्रौद्योगिकी संस्थान
Indian Institute of Technology
काशी हिन्दू विश्वविद्यालय)
Banaras Hindu University)
वाराणसी /Varanasi-221005

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Sandeep Kumar

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FC	Fractional Calculus
FIDE(s)	Fractional Integro-Differential Equation(s)
FADE(s)	Fractional Advection Diffusion Equation(s)
GFD(s)	Generalized Fractional Derivative(s)
FPDE(s)	Fractional Partial Diffusion Equation(s)
FPIDE(s)	Fractional Partial Integro-Differential Equation(s)
ADE(s)	Advection Diffusion Equation(s)
FDE(s)	Fractional Diffusion Equation(s)
MAE	Maximum Absolute Error
STFDE(s)	Space-Time Fractional Diffusion Equation(s)
FRDE(s)	Fractional Reaction Diffusion Equation(s)
CO	Order of Convergence
AE	Absolute Error
GFIDE(s)	Generalized Fractional Integro-Differential Equation(s)
GCD(s)	Generalized Caputo Derivative(s)
PIDEFO	Partial Integro-Differential Equations of Fractional Order
STFADE(s)	Space Time Fractional Advection Diffusion Equation(s)
STFTE(s)	Space-Time Fractional Telegraph Equation(s)
FDM	Finite Difference Method
TFDE(s)	Time Fractional Diffusion Equation(s)

PREFACE

Fractional integro-differential equations (FIDE) are integro-differential equations in which we deal with the derivative and integration of non-integer order of a function. The origin of FIDEs may be traced from the work of Abel, Fredholm, Volterra, Mathus, Verholst in mechanics, mathematical biology, economics etc. The use of fractional differentiation and integration becomes important due to fact that it provides a more accurate model of the system under consideration as the classical order derivatives fail to do. Over the past few decades, FIDEs reflected in a large number of research papers and books covering some of these areas.

The applications of the fractional order derivatives and integration were seen around 19th century. But its origin was considered around the 17^{th} century by L'Hôpital. Nowadays fractional calculus (FC) has a wide range of applications in different fields like study of several natural and real-world problems such as frequency-dependent-damping problems arise in viscoelastic materials [3], continuum and statistical mechanics [4], and dynamics of interfaces between nanoparticles and substrates [5]. Fractional derivatives and integration which mostly used in this thesis are defined using Riemann Liouville and Caputo derivatives. Recently, the generalization of Riemann- Liouville and Caputo derivatives are presented in [6, 30] and its few applications were discussed. In this thesis, we have derived a generalized method to find the solution of FIDEs in terms of the B operator. In general, finding the solution to FIDEs is more complex as it contains both derivative and integration of functions. Here, an attempt is made to solve FIDEs defined in terms of the B-operator using collocation method approach in Chapter 2. Further, the collocation approach is extended for two dimensional generalized fractional partial integro-differential equation in Chapter 3.

Chapter 4 and 5 are based on solving fractional difffusion equations. Fractional diffusion equations are used to study the anomalous behavior of diffusion equations which is derived from Fick's second law. Anomalous behavior is the classification of the diffusion process when there is no linear relationship between the mean square displacement (MSD) with time. In Chapter 4, we study fractional diffusion equations where derivatives on the time variable are defined in the Caputo sense and diffusion terms on space variables are taken as classical derivatives. Further, we apply the finite difference method to the time variable and the collocation approach to space variables. In this way, we get a fully discretized scheme of the considered problem and further we use it for the numerical approximations. Chapter 5 deals with the fractional advection diffusion equation (FADE) in three variables. FADE given in Chapter 5, is the extension of the problem defined in Chapter 4. Finally the collocation method is studied for solving the FADE in one and two dimensions.