

COLLOCATION APPROXIMATION FOR FRACTIONAL INTEGRO-DIFFERENTIAL AND SUB-DIFFUSION EQUATIONS



Thesis submitted in partial fulfillment for the
award of degree

Doctor of Philosophy

By

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Contents

List of Figures	xi
List of Tables	xv
Preface	xviii
1 Introduction	1
1.1 Definition of Fractional Integral and Fractional Derivative	2
1.2 Fractional Integro-Differential Equations	4
1.3 Fractional Sub-Diffusion Equations	5
1.4 Motivation	6
1.5 Literature Review	7
1.5.1 Literature Review on the Fractional Integro-Differential Equations	7
1.5.2 Literature Review on the Time Fractional Diffusion Equations	9
1.6 Problem Statement and Thesis Objectives	11
1.7 Outline of the Thesis	12
2 A Convergent Collocation Approach for Generalized Fractional Integro-Differential Equations Using Jacobi Poly-Fractonomials	15
2.1 Introduction	15
2.2 Generalized Fractional Integro-Differential Equations	16
2.2.1 Preliminaries. Jacobi Poly-Fractonomials and Function Approximation	19
2.2.1.1 Definition and properties of shifted Jacobi poly-fractonomials	19
2.2.2 Function Approximation Using Jacobi Poly-fractonomials	21
2.3 Collocation Method for GFIDEs	23
2.4 Convergence Analysis	25
2.5 Error Analysis	27
2.6 Numerical Examples	30
2.7 Conclusions	42

3	Two-Dimensional Collocation Method for Generalized Partial Integro-Differential Equations of Fractional Order with Applications	43
3.1	Introduction	43
3.2	Preliminaries	47
3.3	Two-Dimensional Collocation Method for PIDEFO	48
3.4	Convergence Analysis	50
3.5	Error Analysis	54
3.6	Applications of The PIDEFO	55
3.6.1	Linear Space-Time Fractional Reaction-Diffusion Equations in Sense of B Operator	56
3.6.2	Space-Time Fractional Diffusion Equation in Sense of B Operator	57
3.6.3	Time Fractional Telegraph Equation	58
3.7	Numerical Examples	59
3.7.1	Effect of Collocating at Equally-Spaced Points	61
3.8	Conclusions	76
4	Finite Difference-Collocation Method for Generalized Fractional Diffusion Equation	77
4.1	Introduction	77
4.2	Numerical Scheme and Stability Analysis	78
4.2.1	Discretization in Time Direction	78
4.2.2	Approximation in Space Direction	80
4.3	Error and Convergence Analysis	81
4.4	Numerical Results	86
4.5	Conclusion	95
5	A Legendre Collocation Method for New Generalized Fractional Diffusion Equation	97
5.1	Introduction	97
5.2	Preliminaries	99
5.3	Function Approximation Using Legendre Polynomials	100
5.3.1	Approximation of a Two Variable Function Using Legendre Polynomials	100
5.3.2	Approximation of a Three Variable Function Using Legendre Polynomials	101
5.4	Numerical Algorithm	101
5.4.1	Collocation Method for One-Dimensional GFADE	102
5.4.2	Collocation Method for Two-Dimensional GFADE	103
5.5	Convergence Analysis	106
5.6	Numerical Results	113
5.7	Conclusion	129

6 Conclusion and Future Work	131
6.1 Conclusion	131
6.2 Future Work	133
Bibliography	135
LIST OF PUBLICATIONS	153

List of Figures

2.1	Plot of approximate solution for different values of a and $N = 2$ for Examples (2.6.1).	32
2.2	Plot of MAE of Examples (2.6.1) for $N = 3$, and $\gamma = 2/3$	32
2.3	Plot of MAE of Examples (2.6.2) for $\gamma = 1/4$ and $R = 3$	34
2.4	Plot of the approximate solution for different values of γ for Examples (2.6.2).	34
2.5	Plot of the exact versus approximate solution of Examples (2.6.2) and (2.6.4).	35
2.6	Plot of MAE of Example (2.6.3) for different values of N	37
2.7	Plot of the exact versus approximate solution of Examples (2.6.5) and (2.6.3).	37
2.8	Plot of MAE of Example (2.6.4) for $\gamma = 2/3$, and N	39
2.9	The numerical solution of Example (2.6.4) different values of d	39
2.10	Plot of MAE for different values of N	41
2.11	Plot of MAE Example (2.6.5) for $N = 12$	41
3.1	Plot of absolute numerical error for Eq. (3.56) for $N = 2$, $M = 3$	62
3.2	Numerical solution for different values of γ_1 for Equation Eq. (3.56) at $t = 1$	64
3.3	Numerical solution for different values of γ_1 for Equation Eq. (3.56) at $t = 0.5$	64
3.4	Numerical solution for different values of γ_1 for Equation Eq. (3.56) at $t = 0$	65
3.5	Numerical solution for different values of γ_2 for Equation Eq. (3.56) at $t = 1$	65
3.6	Numerical solution for different values of γ_2 for Eq. (3.56) at $t = 0.5$	66
3.7	Numerical solution for different values of γ_2 for Eq. (3.56) at $t = 0$	66
3.8	Numerical solution with different noise term for Eq. (3.56) at $t = 1$	67
3.9	Numerical solution with different noise term for Eq. (3.56) at $t = 0.5$	68
3.10	Numerical solution with different noise term for Eq. (3.56) at $t = 0$	68
3.11	Plot of absolute numerical error for $N = 2$, $M = 3$ for Example (3.7.2).	69
3.12	Plot of absolute numerical error for $N = M = 3$ for Example (3.7.2).	70

3.13	Numerical solution for different values of γ_1 for Example (3.7.2) at $t = 1$.	72
3.14	Numerical solution for different values of γ_1 for Example (3.7.2) at $t = 0.5$.	72
3.15	Numerical solution for different values of γ_1 for Example (3.7.2) at $t = 0$.	73
3.16	Numerical solution for different values of γ_2 for Example (3.7.2) at $t = 1$.	73
3.17	Numerical solution for different values of γ_2 for Example (3.7.2) at $t = 0.5$.	74
3.18	Numerical solution for different values of γ_2 for Example (3.7.2) at $t = 0$.	74
3.19	Numerical solution for different noise terms for Example (3.7.2) at $t = 1$.	75
3.20	Numerical solution for different noise terms for Example (3.7.2) at $t = 0.5$.	75
4.1	Comparison of AE at $t = 0.5$, $\mathcal{N} = 5$, $z(t) = t$ and different values of γ for Example 4.4.1.	90
4.2	Plot of AE for different values of \mathcal{N} at $t = 0.1$ and $z(t) = t$ for Example 4.4.1.	90
4.3	Comparison of the numerical solution for different values of γ at $t = 1$ and $z(t) = t^2$ for Example 4.4.1.	91
4.4	Comparison of the numerical and the exact solution at $t = 0.1$ and different values of γ for Example 4.4.2.	94
4.5	Plot of the numerical solutions for different values of \mathcal{N} at $t = 0.1$ for Example 4.4.2.	94
5.1	The numerical solution of Example 5.6.1 for different values of $z(\tau)$ and $w(\tau)$ at $\tau = 1$.	114
5.2	The numerical solution for different $\mathcal{N} = \mathcal{K} = 4(left), 6(right)$, $\gamma = 0.2$ and $z(\tau) = \tau^3$ for Example 5.6.1.	115
5.3	AE function at different values of t with $\mathcal{N} = \mathcal{M} = \mathcal{K} = 10$ and $\gamma = 0.2$ for Example 5.6.2.	117
5.4	$\omega(\tau) = e^{4\tau}$, $\gamma = 0.2$	118
5.5	Plot of the approximate solution for different choices of weight functions $w(\tau)$ and $z(\tau) = \tau$ for Example 5.6.2.	118
5.6	The approximate solution for different choice of scale functions $z(\tau)$ and $\omega(\tau) = 1$ for Example 5.6.2.	119
5.7	(a) AE for $\mathcal{N} = \mathcal{M} = \mathcal{K} = 10$ with x axis, (b) The comparison of the approximate solution to exact solution for different choice of x , $w \in \mathcal{I}_1 \times \mathcal{I}_2$ with τ of Example 5.6.2.	121

5.8	The approximate solution and the exact solution for different choice of x , w with spatial axis x of Example 5.6.2.	121
5.9	Plot of the AE function at different values of β with $\mathcal{N} = \mathcal{M} = \mathcal{K} = 10$, $w(\tau) = 1$ and $\gamma = 0.2$ for Example 5.6.3.	123
5.10	The exact and the approximate solution for different value of $\mathcal{N} = \mathcal{M} = \mathcal{K}$ and $\beta = 0.5, 1, 1.5$ and 2 respectively with x axis of Example 5.6.3.	124
5.11	Plot of the exact and the approximate solutions at a particular point for different value of $\beta = 0.5, 1, 1.5$ and 2 respectively for Example 5.6.3.	127
5.12	The comparison of AE at different values of $\beta = 1, 2, 1.5$ and 0.5 respectively with various value of $\mathcal{N} = \mathcal{M} = \mathcal{K}$ and $\gamma = 0.2$ for Example 5.6.3.	129

List of Tables

2.1	MAE of Example 2.6.1 and Example 2.6.2 for different value of N . . .	31
2.2	Comparison of MAEs for Example 2.6.2 with the method proposed in [1] for different value of N	35
2.3	Comparison of MAEs for Example 2.6.3 with the method given in [1] for different value of N	36
2.4	Numerical solution for different values of M and N for Example (2.6.5).	40
3.1	Numerical solution for different values of M and N for Example (3.7.1).	61
3.2	AE for Example (3.7.1).	61
3.3	L_2 and L_∞ errors for Example (3.7.1).	62
3.4	Comparison of different collocation points with $N = M = 3$ for Example (3.7.1).	63
3.5	AE for Example (3.7.2).	71
3.6	L_2 and L_∞ errors for Example (3.7.2).	71
4.1	Comparison of MAE for Example 4.4.1 with $\gamma = 0.75$, $z(t) = t$	88
4.2	Comparison of MAE for Example 4.4.1 with $\gamma = 0.9$, $z(t) = t$	89
4.3	CO and MAE for Example 4.4.1 with various values of γ and $z(t) = t^2$	89
4.4	Comparison of AE for Example 4.4.2 at $\gamma = 0.1, 0.3$ and various values of \mathcal{N}	92
4.5	Comparison of AE for Example 4.4.2 at $\gamma = 0.5, 0.7$ and various values of \mathcal{N}	93
4.6	MAE and CO for Example 4.4.2 for various values of γ and \mathcal{M}	93
4.7	Comparison of MAE of the Example 4.4.2 with $\gamma = 0.5$	95
5.1	MAE of the Example 5.6.2 with $\gamma = 0.2$	116
5.2	AE of the Example 5.6.2 with $\gamma = 0.2$	117
5.3	MAE of the Example 5.6.3 with $\gamma = 0.2$, $\tau = 0.5$	123
5.4	AE of the Example 5.6.3 with $\gamma = 0.2$, $\mathcal{N} = \mathcal{M} = \mathcal{K} = 10$ and various values of β	125

List of Abbreviations

FC	Fractional Calculus
FIDE(s)	Fractional Integro-Differential Equation(s)
FADE(s)	Fractional Advection Diffusion Equation(s)
GFD(s)	Generalized Fractional Derivative(s)
FPDE(s)	Fractional Partial Diffusion Equation(s)
FPIDE(s)	Fractional Partial Integro-Differential Equation(s)
ADE(s)	Advection Diffusion Equation(s)
FDE(s)	Fractional Diffusion Equation(s)
MAE	Maximum Absolute Error
STFDE(s)	Space-Time Fractional Diffusion Equation(s)
FRDE(s)	Fractional Reaction Diffusion Equation(s)
CO	Order of Convergence
AE	Absolute Error
GFIDE(s)	Generalized Fractional Integro-Differential Equation(s)
GCD(s)	Generalized Caputo Derivative(s)
PIDEFO	Partial Integro-Differential Equations of Fractional Order
STFADE(s)	Space Time Fractional Advection Diffusion Equation(s)
STFTE(s)	Space-Time Fractional Telegraph Equation(s)
FDM	Finite Difference Method
TFDE(s)	Time Fractional Diffusion Equation(s)

PREFACE

Fractional integro-differential equations (FIDE) are integro-differential equations in which we deal with the derivative and integration of non-integer order of a function. The origin of FIDEs may be traced from the work of Abel, Fredholm, Volterra, Mathus, Verholst in mechanics, mathematical biology, economics etc. The use of fractional differentiation and integration becomes important due to fact that it provides a more accurate model of the system under consideration as the classical order derivatives fail to do. Over the past few decades, FIDEs reflected in a large number of research papers and books covering some of these areas.

The applications of the fractional order derivatives and integration were seen around 19th century. But its origin was considered around the 17th century by L'Hôpital. Nowadays fractional calculus (FC) has a wide range of applications in different fields like study of several natural and real-world problems such as frequency-dependent-damping problems arise in viscoelastic materials [3], continuum and statistical mechanics [4], and dynamics of interfaces between nanoparticles and substrates [5]. Fractional derivatives and integration which mostly used in this thesis are defined using Riemann Liouville and Caputo derivatives. Recently, the generalization of Riemann- Liouville and Caputo derivatives are presented in [6, 30] and its few applications were discussed. In this thesis, we have derived a generalized method to find the solution of FIDEs in terms of the B operator. In general, finding the solution to FIDEs is more complex as it contains both derivative and integration of functions. Here, an attempt is made to solve FIDEs defined in terms of the B -operator using collocation method approach in Chapter 2. Further, the collocation approach is extended for two dimensional generalized fractional partial integro-differential equation in Chapter 3.

Chapter 4 and 5 are based on solving fractional diffusion equations. Fractional diffusion equations are used to study the anomalous behavior of diffusion equations which is derived from Fick's second law. Anomalous behavior is the classification of the diffusion process when there is no linear relationship between the mean square displacement (MSD) with time. In Chapter 4, we study fractional diffusion equations where derivatives on the time variable are defined in the Caputo sense and diffusion terms on space variables are taken as classical derivatives. Further, we apply the finite difference method to the time variable and the collocation approach to space variables. In this way, we get a fully discretized scheme of the considered problem and further we use it for the numerical approximations. Chapter 5 deals with the fractional advection diffusion equation (FADE) in three variables. FADE given in Chapter 5, is the extension of the problem defined in Chapter 4. Finally the collocation method is studied for solving the FADE in one and two dimensions.

