# Chapter 5

# Analysis of $M^X/G_r^{(a,Y)}/1$ queue with second optional service and queue length dependent single and multiple vacation

## 5.1 Introduction

The concept of second optional service (SOS) was proposed by Madan [10]. He analyzed the $M/G/1$ queueing model using SVT. Later, many researchers have analyzed different queueing models with SOS, see, e.g., Medhi [111], Al-Jọrarha and Madan [112], Wang [113], and Choudhury and Tadj [114]. There are few literatures available on bulk queues with SOS, e.g., Ayyappan and Supraja [119], Singh et al. [120] , Ayyappan and Deepa [97], etc., and references therein. Ayyappan and Supraja [119] analyzed $M^X/G^{(a,b)}/1$ queue with unreliable server, second optional service, two different vacations, and restricted admissibility policy and obtained the queue length distribution at random and departure epoch using the SVT. Singh et al. [120] analyzed bulk arrival queue with different $m$-SOS, vacation, and unreliable server using SVT. Ayyappan and Deepa [97] considered $M^X/G^{(a,b)}/1$ queue with SOS, MV, and setup time. They obtained the PGF of the queue size at different epochs using SVT. To the best of the author's knowledge, the considered model, i.e., $M^X/G^{(a,Y)}/1$ queue with SOS and queue length dependent SV and MV, has

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not been analyzed so far in the literature that analyzes the joint probabilities of queue and server content for FES (SOS) at the service completion (arbitrary) epoch as well as joint probabilities of queue content and type of vacation at the vacation termination (arbitrary) epoch.

### 5.1.1 Practical motivation

Queueing models with SOS can be applied in many areas, *viz.*, barber shop, malls, etc. Considered model may be used to model blood sample testing in an epidemic situation such as COVID-19, as batch service queues have efficacious application in blood pooling, see, e.g., Abolnikov and Dukhovny [15], Bar-Lev et al. [13], and Claeys et al. [17]. In an epidemic (*viz.*, COVID-19), the health administration of any country wants to test more and more samples using less number of kits. Hence, a mixed sample is used for testing by taking a group of samples from the queue, see the references [124, 125, 126]. Further, in a pandemic situation handling the health workers' shortage is also a big challenge. To deal with such situation, the health administration may provide some additional work to the health workers (*viz.*, visiting the quarantine room, stocking the health care inventory, making people aware of the epidemic) when they have no primary work.

Suppose that a large number of samples arrive at the health department in bulk for testing from different sectors, then the health worker tests these samples in batches, termed as FES, according to the $(a, Y)$ rule with batch size dependent service. After FES, if the mixed sample diagnosed negative then the health worker decides how many samples will be mixed for the next test with a certain probability. For example, the health administration instructs the health worker that if the mixed sample is found negative, select the batch of maximum capacity for the next test, otherwise, choose the batch size of its minimum capacity. Therefore, $(a, Y)$ rule is justified here. After FES, if the mixed sample diagnosed positive then the sample go for the SOS to identify the infected sample.

Further, in the absence of primary work (which includes FES and SOS), the health worker does some additional work (*viz.*, stocking of health care inventory, increase people awareness, visiting the quarantine room, etc.). Before going for this additional work, the health worker always checks the queue size, and depending on the queue size, he fixed his returning time in the primary system. Hence, the QSDV policy rule may have a wide impact on the system's performance. The practical application discusses above motivate us to work on this problem.

Section 5.2 presents the model description of the considered model. In Section 5.3, joint probabilities of queue and server content as well as queue length and type of vacation obtained at different epoch. Some marginal distributions are presented in Section 5.4. The various important performance measures are presented in Section 5.5. The behavior of the system is discussed by means of tables and graphs in Section 5.6. Section 5.7 presents a cost model. The whole study ends with the conclusion (i.e., Section 5.8).

## 5.2 Model description

The present chapter investigates infinite capacity bulk arrival, batch size dependent bulk service queue with SOS, queue size dependent single (multiple) vacation. Here below is the detail mathematical description of the model.

The customers are coming in packets (groups) following the Poison distribution with rate $\lambda$. Let G be the size of the arriving group with probability mass function $P(X = m) = g_m$, $m \in \mathbb{N}$ associated with finite mean $E(X) = \tilde{g}$ and PGF $X(z) = \sum_{i=1}^{\infty} g_i z^i$. The customers are served in batches according to the VBS rule, i.e., $(a, Y)$ rule, where the random variable $Y$, denoting service capacity, has the following probability mass function,

$$Pr(Y = i) = \begin{cases} y_i, & a \le i \le B \\ 0, & otherwise. \end{cases}$$

Here $B$ is the maximum serving capacity of the server with $y_B > 0$ and $E(Y) = \tilde{y}$. At each service initiation epoch if the queue length lies in $[a, i)$ (where $i$ is the chosen service capacity at the service initiation epoch) then server does not wait for the queue length to reach $i$, but it takes entire customer for the service with probability $y_i$, and if the server finds the queue length $\ge i$ then it takes only $i$ customers for the service with probability $y_i$. The service (FES) time ($T_r$), of a batch of size $r$ ($a \le r \le B$) is generally distributed along with probability density function (pdf) $s_r(t)$, distribution function (DF) $S_r(t)$, the Laplace-Stieltjes transform (LST) $\tilde{S}_r(\theta)$ and the mean service time $\frac{1}{\mu_r} = s_r = -\tilde{S}_r^{(1)}(0)$ ($a \le r \le B$), where $\tilde{S}_r^{(1)}(0)$ is the derivative of $\tilde{S}_r(\theta)$ evaluated at $\theta$=0. After first essential service (FES) the served batch may choose second optional service (SOS) with probability $\alpha$. The optional service time ($\hat{T}$) of a batch is generally distributed along with probability density function (pdf) $s(t)$, distribution function (DF) $S(t)$, the Laplace-Stieltjes transform (LST) $\tilde{S}(\theta)$ and the mean service time $\frac{1}{\mu} = \varsigma = -\tilde{S}^{(1)}(0)$, where $\tilde{S}^{(1)}(0)$ is the derivative of $\tilde{S}(\theta)$ evaluated at $\theta = 0$. After FES if the queue length is found to be less than the minimum threshold limit $a$ and the batch served in FES does not choose SOS then the server goes for the $k^{th}$ type of vacation where $k$ $(0 \leq k \leq a-1)$ is the queue length at vacation initiation epoch, similarly, after SOS if the queue length is found to be $k < a$ then the server goes for $k^{th}$ type of vacation . At the end of the vacation if the queue length is $\geq a$ then it serves the customer as per the $(a, Y)$ rule, otherwise, depending on the vacation policy the server remains in the system at dormant state until queue length reaches at least the minimum threshold limit $a$ or takes repeated vacation until it finds queue length $\geq a$ at the end of the vacation. Vacation time $V_k$ of the $k^{th}$ type of vacation obeys general distribution with pdf $v_k(t)$, DF $V_k(t)$, LST $\tilde{V}_k(\theta)$. The mean vacation time $\frac{1}{\nu_k} = x_k = -\tilde{V}_k^{(1)}(0)$ where $\tilde{V}_k^{(1)}(0)$ is the derivative of $\tilde{V}_k(\theta)$ at $\theta = 0$. The traffic intensity of the system $\rho = \frac{\lambda\tilde{g}\sum\limits_{i=a}^{B}\frac{y_i}{\mu_i}+\lambda\tilde{g}\frac{\alpha}{\mu}}{\tilde{y}} (< 1)$ which ensures the stability of the system. In this chapter, SV and MV queues have been studied in an unified way by defining a variable $\delta$ as follows:

$$\delta = \begin{cases} 1, & \text{for MV}, \\ 0, & \text{for SV}. \end{cases}$$

## 5.3 System analysis

This section is devoted in obtaining the joint probabilities of the queue length and server content at the service (FES and SOS) completion epoch and the joint probabilities of the queue size and the type of vacation at the vacation termination epoch. Later the joint probabilities of the queue and server content are obtained during FES (SOS) and the joint probabilities of queue length and type of vacation at an arbitrary epoch by relating it to the joint probabilities at service completion and vacation termination epoch. From this perspective, the following random variables, at time $t$, are defined as follows:

- $N_q(t)$: be the number of customers in the queue.
- $S_1(t)$: be the number of customers with the server when the server is busy in FES.
- $S_2(t)$: be the number of customers with the server when the server is busy in SOS.
- $K(t)$: be the type of vacation taken by the server, when the server is on vacation.
- $U(t)$: remaining service (FES) time of the batch, if any.
- $\hat{U}(t)$: remaining service (SOS) time of the batch, if any.

- $V(t)$: remaining vacation time of the server, if any.

Point to be noted here that $S_1(t) = 0$ and $S_2(t) = 0$ will represent the server is in the dormant state at time $t$ (for the case of SV).

For SV, $\{(N_q(t), S_1(t) = 0, S_2(t) = 0)\} \cup \{(N_q(t), S_1(t), U(t))\} \cup \{(N_q(t), S_2(t), \hat{\boldsymbol{U}}(t))\} \cup \{(N_q(t), K(t), V(t))\}$ forms a Markov chain with state space $\{(n, 0, 0); 0 \leq n \leq a-1\} \bigcup \{(n, r, u); n \geq 0, a \leq r \leq B, u \geq 0\} \bigcup \{(n, k, u); 0 \leq k \leq a-1, n \geq k, u \geq 0\}$.

For MV, $\{(N_q(t), S_1(t), U(t))\} \cup \{(N_q(t), S_2(t), \hat{\boldsymbol{U}}(t))\} \cup \{(N_q(t), K(t), V(t))\}$ forms a Markov chain with state space $\{(n, r, u); n \geq 0, a \leq r \leq B, u \geq 0\} \bigcup \{(n, k, u); 0 \leq k \leq a-1, n \geq k, u \geq 0\}$.

Define the state probabilities, at time $t$, as

- $R_n(t) \equiv Pr\{N_q(t) = n, S_1(t) = 0, S_2(t) = 0\}, \quad 0 \leq n \leq a-1$ (exist only for SV).
- $P_{n,r}(u,t)du \equiv Pr\{N_q(t) = n, S_1(t) = r, u \leq U(t) \leq u + du\}, \quad n \geq 0 \ , \ a \leq r \leq B.$
- $W_{n,r}(u,t)du \equiv Pr\{N_q(t) = n, S_2(t) = r, u \leq \hat{U}(t) \leq u + du\}, \quad n \geq 0 \ , \ a \leq r \leq B.$
- $Q_n^{[k]}(u,t)du \equiv \ Pr\{N_q(t) = n, K(t) = k, u \leq V(t) \leq u + du\}, \ n \geq k \ , \ 0 \leq k \leq a-1.$

In steady state, as $t \to \infty$, the limiting probabilities are defined as follows.

$R_n = \lim_{t\to\infty} R_n(t)$ $(0 \leq n \leq a-1)$, (exist only for SV),
$P_{n,r}(u) = \lim_{t\to\infty} P_{n,r}(u,t), \quad n \geq 0, \quad a \leq r \leq B,$
$W_{n,r}(u) = \lim_{t\to\infty} W_{n,r}(u,t), \quad n \geq 0, \quad a \leq r \leq B,$
$Q_n^{[k]}(u) = \lim_{t\to\infty} Q_n^{[k]}(u,t), \quad n \geq k, \quad 0 \leq k \leq a-1.$
Now the system equation that governs the system behavior is obtained. Analyzing the system, at time $t$ and $t + dt$, in steady state, the Kolmogorov equations are obtained as follows:

$$0 = (1-\delta)\Big(-\lambda R_0 + Q_0^{[0]}(0)\Big), \tag{5.1}$$

$$0 = (1-\delta)\Big(-\lambda R_n + \lambda \sum_{i=1}^{n} g_i R_{n-i} + \sum_{k=0}^{n} Q_n^{[k]}(0)\Big), \quad 1 \leq n \leq a-1, \tag{5.2}$$

$$
\begin{aligned}
-\frac{d}{du}P_{0,r}(u) &= -\lambda P_{0,r}(u) + \Big(\sum_{k=0}^{a-1} Q_r^{[k]}(0) + \sum_{j=a}^{B} P_{r,j}(0)(1-\alpha) + \sum_{j=a}^{B} W_{r,j}(0)\Big)\sum_{i=r}^{B} y_i s_r(u) \\
&\quad +(1-\delta)\lambda\sum_{j=0}^{a-1} R_j g_{r-j}\sum_{i=r}^{B} y_i s_r(u), \quad a \le r \le B, \qquad (5.3) \\
-\frac{d}{du}P_{n,r}(u) &= -\lambda P_{n,r}(u) + \lambda\sum_{j=1}^{n} P_{n-j,r}(u) g_j \\
&\quad +\Big(\sum_{k=0}^{a-1} Q_{n+r}^{[k]}(0) + \sum_{j=a}^{B} P_{n+r,j}(0)(1-\alpha) + \sum_{j=a}^{B} W_{n+r,j}(0)\Big) y_r s_r(u) \\
&\quad +(1-\delta)\lambda\sum_{j=0}^{a-1} R_j g_{n+r-j} y_r s_r(u), \ a \le r \le B, \ n \ge 1, \qquad (5.4) \\
-\frac{d}{du}Q_k^{[k]}(u) &= -\lambda Q_k^{[k]}(u) + \Big(\sum_{r=a}^{B} P_{k,r}(0)(1-\alpha) + \sum_{r=a}^{B} W_{k,r}(0) \\
&\quad +\delta\sum_{j=0}^{k} Q_k^{[j]}(0)\Big) v_k(u), \quad 0 \le k \le a-1, \qquad (5.5) \\
-\frac{d}{du}Q_n^{[k]}(u) &= -\lambda Q_n^{[k]}(u) + \lambda\sum_{i=1}^{n-k} g_i Q_{n-i}^{[k]}(u), \quad n \ge k+1, \quad 0 \le k \le a-1, \qquad (5.6) \\
-\frac{d}{du}W_{0,r}(u) &= -\lambda W_{0,r}(u) + P_{0,r}(0)s(u)\alpha, \quad a \le r \le B, \qquad (5.7) \\
-\frac{d}{du}W_{n,r}(u) &= -\lambda W_{n,r}(u) + \lambda\sum_{j=1}^{n} W_{n-j,r}(u) g_j + P_{n,r}(0)s(u)\alpha, \quad n \ge 1, a \le r \le B. \qquad (5.8)
\end{aligned}
$$

Further, define for Re $\theta \ge 0$,

$$
\begin{aligned}
\tilde{S}_r(\theta) = \int_0^\infty e^{-\theta u} dS_r(u) &= \int_0^\infty e^{-\theta u} s_r(u) du, \quad a \le r \le B, \qquad (5.9) \\
\tilde{P}_{n,r}(\theta) &= \int_0^\infty e^{-\theta u} P_{n,r}(u) du, \quad a \le r \le B, \ n \ge 0, \qquad (5.10) \\
P_{n,r} \equiv \tilde{P}_{n,r}(0) &= \int_0^\infty P_{n,r}(u) du, \quad a \le r \le B, \ n \ge 0, \qquad (5.11) \\
\tilde{S}(\theta) = \int_0^\infty e^{-\theta u} dS(u) &= \int_0^\infty e^{-\theta u} s(u) du, \quad a \le r \le B, \qquad (5.12) \\
\tilde{W}_{n,r}(\theta) &= \int_0^\infty e^{-\theta u} W_{n,r}(u) du, \quad a \le r \le B, \ n \ge 0, \qquad (5.13) \\
W_{n,r} \equiv \tilde{W}_{n,r}(0) &= \int_0^\infty W_{n,r}(u) du, \quad a \le r \le B, \ n \ge 0, \qquad (5.14) \\
\tilde{V}_k(\theta) = \int_0^\infty e^{-\theta u} dV_k(u) &= \int_0^\infty e^{-\theta u} v_k(u) du, \quad 0 \le k \le a-1, \qquad (5.15)
\end{aligned}
$$

$$\tilde{Q}_n^{[k]}(\theta) = \int_0^\infty e^{-\theta u} Q_n^{[k]}(u)du, \quad 0 \le k \le a-1, \ n \ge k, \tag{5.16}$$

$$Q_n^{[k]} \equiv \tilde{Q}_n^{[k]}(0) = \int_0^\infty Q_n^{[k]}(u)du, \quad 0 \le k \le a-1, \quad n \ge k. \tag{5.17}$$

One may note here that $P_{n,r}$ $(W_{n,r})$ denotes the probability that there are $n$ $(n \ge 0)$ customers are in the queue and server is busy with $r$ $(a \le r \le B)$ customers during FES (SOS), at an arbitrary epoch. Also, $Q_n^{[k]}$ indicates the probability of $n$ $(n \ge k)$ customers in the queue and the server is on $k^{th}$ $(0 \le k \le a-1)$ type of vacation, at an arbitrary epoch.

Multiplying (5.3)-(5.8) by $e^{-\theta u}$ and integrating with respect to $u$ over 0 to $\infty$ one can obtain

$$\begin{aligned}
(\lambda-\theta)\tilde{P}_{0,r}(\theta) &= \left(\sum_{k=0}^{a-1} Q_r^{[k]}(0) + \sum_{j=a}^{B} P_{r,j}(0)(1-\alpha) + \sum_{j=a}^{B} W_{r,j}(0)\right)\sum_{i=r}^{B} y_i \tilde{S}_r(\theta) \\
&\quad +(1-\delta)\lambda\sum_{j=0}^{a-1} R_j g_{r-j}\sum_{i=r}^{B} y_i\tilde{S}_r(\theta) - P_{0,r}(0), \quad a \le r \le B, \qquad (5.18)\\
(\lambda-\theta)\tilde{P}_{n,r}(\theta) &= \lambda\sum_{j=1}^{n} g_j \tilde{P}_{n-j,r}(\theta) + \left(\sum_{k=0}^{a-1} Q_{n+r}^{[k]}(0) + \sum_{j=a}^{B} P_{n+r,j}(0)(1-\alpha)\right.\\
&\quad \left.+\sum_{j=a}^{B} W_{n+r,j}(0)\right) y_r\tilde{S}_r(\theta) + (1-\delta)\lambda\sum_{j=0}^{a-1} R_j g_{n+r-j} y_r \tilde{S}_r(\theta)\\
&\quad -P_{n,r}(0), \ a \le r \le B, \ n \ge 1, \qquad (5.19)\\
(\lambda-\theta)\tilde{Q}_k^{[k]}(\theta) &= \left(\sum_{r=a}^{B} P_{k,r}(0)(1-\alpha) + \sum_{r=a}^{B} W_{k,r}(0) + \delta\sum_{j=0}^{k} Q_k^{[j]}(0)\right)\tilde{V}_k(\theta)\\
&\quad -Q_k^{[k]}(0), \quad 0 \le k \le a-1, \qquad (5.20)\\
(\lambda-\theta)\tilde{Q}_n^{[k]}(\theta) &= \lambda\sum_{i=1}^{n-k} g_i \tilde{Q}_{n-i}^{[k]}(\theta) - Q_n^{[k]}(0), \ n \ge k+1, \quad 0 \le k \le a-1. \qquad (5.21)\\
(\lambda-\theta)\tilde{W}_{0,r}(\theta) &= P_{0,r}(0)\tilde{S}(\theta)\alpha - W_{0,r}(0), \quad a \le r \le B \qquad (5.22)\\
(\lambda-\theta)\tilde{W}_{n,r}(\theta) &= \lambda\sum_{j=1}^{n} \tilde{W}_{n-j,r}(\theta) g_j + P_{n,r}(0)\tilde{S}(\theta)\alpha\\
&\quad -W_{n,r}(0), \quad n \ge 1, a \le r \le B \qquad (5.23)
\end{aligned}$$

Now the main objective is to obtain the joint probabilities of the queue and server content during FES (SOS) as well as the joint probabilities of queue length and type of vacation at an arbitrary epoch, these arbitrary epoch joint probabilities are obtained by establishing a relationship between the joint probabilities of the queue length and server content at the service completion epoch, and the joint probabilities of the queue length and type of vacation at the vacation termination epoch. Towards this end, define,

$$
\begin{aligned}
P^+_{n,r} &= Pr\{n \text{ customers are in the queue at service (i.e., FES) completion epoch} \\
&\qquad \text{of a batch of size } r\}, \quad n \geq 0, \quad a \leq r \leq B, && (5.24)\\
P^+_{n} &= Pr\{n \text{ customers are in the queue at service (i.e., FES) completion epoch}\} \\
&= \sum_{r=a}^{B} P^+_{n,r}, \quad n \geq 0, && (5.25)\\
W^+_{n,r} &= Pr\{n \text{ customers are in the queue at service (i.e., SOS) completion epoch} \\
&\qquad \text{of a batch of size } r\}, \quad n \geq 0, \quad a \leq r \leq B, && (5.26)\\
W^+_{n} &= Pr\{n \text{ customers are in the queue at service (i.e., SOS) completion epoch}\} \\
&= \sum_{r=a}^{B} W^+_{n,r}, \quad n \geq 0, && (5.27)\\
Q^{[k]+}_{n} &= Pr\{n \text{ customers are in the queue at } k^{th} \text{ type of vacation termination epoch}\}, \\
&\qquad 0 \leq k \leq a-1, && (5.28)\\
Q^+_{n} &= Pr\{n \text{ customers are in the queue at the vacation termination epoch}\} \\
&= \sum_{k=0}^{min(n,a-1)} Q^{[k]+}_{n}, \quad n \geq 0. && (5.29)
\end{aligned}
$$

### 5.3.1 Joint probabilities at service (vacation) completion epoch

The primary objective of this section is to obtain $P^+_{n,r}$ ($W^+_{n,r}$) ($n \geq 0, a \leq r \leq B$) and $Q^{[k]+}_{n}$ ($0 \leq k \leq a-1, n \geq k$), i.e., the joint probabilities of the queue length and server content at service (FES and SOS) completion epoch and the joint probabilities of the queue size and the type of vacation at vacation termination epoch, in this connection the required bivariate generating functions are defined as follows:

$$
\begin{aligned}
P(z,y,\theta) &= \sum_{n=0}^{\infty}\sum_{r=a}^{B} \tilde{P}_{n,r}(\theta) z^n y^r, |z| \leq 1, \; |y| \leq 1, && (5.30)\\
P^+(z,y) &= \sum_{n=0}^{\infty}\sum_{r=a}^{B} P^+_{n,r} z^n y^r, |z| \leq 1, \; |y| \leq 1, && (5.31)\\
P^+(z,1) &= \sum_{n=0}^{\infty}\sum_{r=a}^{B} P^+_{n,r} z^n = \sum_{n=0}^{\infty} P^+_{n} z^n = P^+(z), \quad |z| \leq 1 && (5.32)\\
W(z,y,\theta) &= \sum_{n=0}^{\infty}\sum_{r=a}^{B} \tilde{W}_{n,r}(\theta) z^n y^r, |z| \leq 1, \; |y| \leq 1, && (5.33)
\end{aligned}
$$

$$W^+(z,y) = \sum_{n=0}^{\infty}\sum_{r=a}^{B} W_{n,r}^+ z^n y^r, |z| \leq 1, \ |y| \leq 1, \tag{5.34}$$

$$W^+(z,1) = \sum_{n=0}^{\infty}\sum_{r=a}^{B} W_{n,r}^+ z^n = \sum_{n=0}^{\infty} W_n^+ z^n = W^+(z), \quad |z| \leq 1 \tag{5.35}$$

$$Q(z,y,\theta) = \sum_{k=0}^{a-1}\sum_{n=k}^{\infty} \tilde{Q}_n^{[k]}(\theta) z^n y^k, |z| \leq 1, \ |y| \leq 1, \tag{5.36}$$

$$Q^+(z,y) = \sum_{k=0}^{a-1}\sum_{n=k}^{\infty} Q_n^{[k]+} z^n y^k, |z| \leq 1, \ |y| \leq 1, \tag{5.37}$$

$$Q^+(z,1) = \sum_{k=0}^{a-1}\sum_{n=k}^{\infty} Q_n^{[k]+} z^n = \sum_{n=0}^{\infty}\sum_{k=0}^{min(n,a-1)} Q_n^{[k]+} z^n$$
$$= \sum_{n=0}^{\infty} Q_n^+ z^n = Q^+(z), \quad |z| \leq 1. \tag{5.38}$$

Further, define

$$m_j^{(r)} = Pr\{j \text{ arrivals during the service (i.e., FES) time of a batch size } r\}, \quad a \leq r \leq B, j \geq 0$$
$$= \int_0^{\infty} \sum_{l=0}^{j} \frac{e^{-\lambda t}(\lambda t)^l}{l!} g_j^{l(*)} s_r(t) dt, \tag{5.39}$$
$$q_j = Pr\{j \text{ arrivals during the service (i.e., SOS) time}\}, \quad j \geq 0,$$
$$= \int_0^{\infty} \sum_{l=0}^{j} \frac{e^{-\lambda t}(\lambda t)^l}{l!} g_j^{l(*)} s(t) dt, \tag{5.40}$$
$$w_j^{(k)} = Pr\{j \text{ arrivals during the } k^{th} \text{ type of vacation }\}, \quad 0 \leq k \leq a-1, j \geq 0,$$
$$= \int_0^{\infty} \sum_{l=0}^{j} \frac{e^{-\lambda t}(\lambda t)^l}{l!} g_j^{l(*)} v_k(t) dt. \tag{5.41}$$

Where $g_j^{l(*)}$ is $l$-fold convolution function of $g_j$. Define the PGF (probability generating function) of $m_j^{(r)}$, $q_j$ and $w_j^{(k)}$ are as follows

$$M^{(r)}(z) = \sum_{j=0}^{\infty} m_j^{(r)} z^j = \tilde{S}_r(\lambda - \lambda X(z)), \quad a \leq r \leq B, \quad |z| \leq 1, \tag{5.42}$$

$$M_{os}(z) = \sum_{j=0}^{\infty} q_j z^j = \tilde{S}(\lambda - \lambda X(z)), \quad |z| \leq 1, \tag{5.43}$$

$$N^{(k)}(z) = \sum_{j=0}^{\infty} w_j^{(k)} z^j = \tilde{V}_k(\lambda - \lambda X(z)), \quad 0 \leq k \leq a-1, \quad |z| \leq 1. \tag{5.44}$$

**Lemma 5.1.** *The probabilities* $P^+_{n,r}$, $W^+_{n,r}$, $Q_n^{[k]+}$, $P_{n,r}(0)$, $W_{n,r}(0)$ *and* $Q_n^{[k]}(0)$ $(a \leq r \leq B, 0 \leq k \leq a-1)$ *are associated with the following relation*

$$P^+_{n,r} = \sigma P_{n,r}(0), \tag{5.45}$$

$$W^+_{n,r} = \sigma W_{n,r}(0), \tag{5.46}$$

$$Q_n^{[k]+} = \sigma Q_n^{[k]}(0), \tag{5.47}$$

*where* $\sigma^{-1} = \sum_{m=0}^{\infty}\sum_{r=a}^{B} P_{m,r}(0) + \sum_{m=0}^{\infty}\sum_{r=a}^{B} W_{m,r}(0) + \sum_{m=0}^{\infty}\sum_{k=0}^{min(m,a-1)} Q_m^{[k]}(0)$.

*Proof.* Since $P^+_{n,r}$, $W^+_{n,r}$, and $Q_n^{[k]+}$ are proportional to $P_{n,r}(0)$, $W_{n,r}(0)$ and $Q_n^{[k]}(0)$, respectively, applying the Bayes' theorem and $\sum_{n=0}^{\infty}\sum_{r=a}^{B}(P^+_{n,r} + W^+_{n,r}) + \sum_{n=0}^{\infty}\sum_{k=0}^{min(n,a-1)} Q_n^{[k]+}=1$ the desired outcome is obtained. ☐

**Lemma 5.2.** *The value* $\sigma^{-1}$ *is given by*

$$\sigma^{-1} = \frac{1-(1-\delta)\sum_{n=0}^{a-1} R_n}{\begin{aligned}&\sum_{n=B+1}^{\infty}\left(P_n^+(1-\alpha)+W_n^+ + Q_n^+\right)\sum_{r=a}^{B} y_r s_r + (P_a^+(1-\alpha)+W_a^+ + Q_a^+)s_a \\ &+ \sum_{n=a+1}^{B}\left(P_n^+(1-\alpha)+W_n^+ + Q_n^+\right)\Big(\sum_{i=a}^{n-1} y_i s_i + \sum_{i=n}^{B} y_i s_n\Big) \\ &+ \sum_{n=0}^{a-1}\Bigg[P_n^+ x_n(1-\alpha) + W_n^+ x_n + (1-\delta)\sum_{m=n}^{a-1} e_{m,n} Q_n^+\Bigg(\sum_{r=a}^{B} g_{r-m}\sum_{i=r}^{B} y_i\mu_r \\ &+ \sum_{l=1}^{\infty}\sum_{r=a}^{B} g_{r-m+l} y_r \mu_r\Bigg) + \delta Q_n^+ x_n\Bigg] \\ &+ \sum_{n=0}^{\infty} P_n^+ \varsigma\alpha\end{aligned}}. \tag{5.48}$$

*where* $e_{n,m} = \sum_{i=1}^{n-m} g_i e_{n-i,m}$, $1 \leq n \leq a-1$, $0 \leq m \leq n-1$ *and* $e_{n,n} = 1$ , $0 \leq n \leq a-1$

*Proof.* Using (5.1) and (5.2), the following expression is obtained

$$\lambda R_n = \sum_{m=0}^{n}\sum_{k=0}^{m} e_{n,m} Q_m^{[k]}(0), \quad 0 \leq n \leq a-1. \tag{5.49}$$

Using (5.49), summing (5.18)-(5.23), one have

$$\sum_{m=0}^{\infty}\sum_{r=a}^{B}(\tilde{P}_{m,r}(\theta)+\tilde{W}_{m,r}(\theta))+\sum_{m=0}^{\infty}\sum_{k=0}^{min(m,a-1)}\tilde{Q}_m^{[k]}(\theta)=\frac{A(\theta)}{\theta}, \tag{5.50}$$

where

$$\begin{aligned}A(\theta) &= \sum_{n=B+1}^{\infty}\left(\sum_{r=a}^{B}P_{n,r}(0)(1-\alpha)+\sum_{r=a}^{B}W_{n,r}(0)+\sum_{k=0}^{a-1}Q_n^{[k]}(0)\right)\sum_{r=a}^{B}y_r(1-\tilde{S}_r(\theta))\\
&+\sum_{n=a+1}^{B}\left(\sum_{r=a}^{B}P_{n,r}(0)(1-\alpha)+\sum_{r=a}^{B}W_{n,r}(0)+\sum_{k=0}^{a-1}Q_n^{[k]}(0)\right)\left(\sum_{i=a}^{n-1}y_i(1-\tilde{S}_i(\theta))\right.\\
&\left.+\sum_{i=n}^{B}y_i(1-\tilde{S}_n(\theta))\right)+\sum_{n=0}^{a-1}\left(\sum_{r=a}^{B}P_{n,r}(0)(1-\alpha)+\sum_{r=a}^{B}W_{n,r}(0)+\delta\sum_{k=0}^{n}Q_n^{[k]}(0)\right)\\
&(1-\tilde{V}_n(\theta))+\left(\sum_{j=a}^{B}P_{a,j}(0)(1-\alpha)+\sum_{j=a}^{B}W_{a,j}(0)+\sum_{k=0}^{a-1}Q_a^{[k]}(0)\right)(1-\tilde{S}_a(\theta))\\
&+\left(1-\sum_{m=n}^{a-1}e_{m,n}\left(\sum_{r=a}^{B}g_{r-m}\sum_{i=r}^{B}y_i\tilde{S}_r(\theta)+\sum_{l=1}^{\infty}\sum_{r=a}^{B}g_{r-m+l}y_r\tilde{S}_r(\theta)\right)\right)\\
&(1-\delta)\sum_{n=0}^{a-1}\sum_{k=0}^{n}Q_n^{[k]}(0)+\sum_{n=0}^{\infty}\sum_{r=a}^{B}P_{n,r}(0)\alpha(1-\tilde{S}(\theta)).\end{aligned}$$

Taking $\theta\to 0$ in (5.50) and using L'Hôspital's rule, the normalization condition $(1-\delta)\sum_{n=0}^{a-1}R_n+\sum_{n=0}^{\infty}\sum_{r=a}^{B}(P_{n,r}+W_{n,r})+\sum_{n=0}^{\infty}\sum_{k=0}^{min(n,a-1)}Q_n^{[k]}=1$, after few simplification desired outcome is obtained. □

**Lemma 5.3.**

$$W^+(z) \;=\; P^+(z)M_{os}(z)\alpha. \tag{5.51}$$

*Proof.* Multiplying (5.22)-(5.23) by proper power of $z$ and $y$ and summing over the range of $n$ and $r$ the following result is obtained

$$\begin{aligned}(\lambda-\theta-\lambda X(z))W(z,y,\theta) &= \sum_{n=0}^{\infty}\sum_{r=a}^{B}P_{n,r}(0)\alpha\tilde{S}(\theta)z^ny^r\\
&-\sum_{n=0}^{\infty}\sum_{r=a}^{B}W_{n,r}(0)z^ny^r.\end{aligned} \tag{5.52}$$

Substituting $\theta=\lambda-\lambda X(z)$ in the above expression and using Lemma 5.1 one have

$$W^+(z,y) \;=\; \sum_{n=0}^{\infty}\sum_{r=a}^{B}P_{n,r}^+\alpha M_{os}(z)z^ny^r. \tag{5.53}$$

Setting $y = 1$ in (5.53) and using (5.25) and (5.32) desired result (5.51) is obtained. □

**Lemma 5.4.**

$$W_n^+ = \alpha \sum_{i=0}^{n} P_i^+ q_{n-i}, \; n \geq 0, a \leq r \leq B \tag{5.54}$$

*Proof.* Using (5.34), (5.32), and (5.43) and then collecting the coefficients of $z^n$ $(n \geq 0)$ from both the side of (5.51) desired outcome (5.54) is obtained. □

**Lemma 5.5.**

$$Q^+(z) = \sum_{k=0}^{a-1}\sum_{n=k}^{\infty} Q_n^{[k]+} z^n = \sum_{k=0}^{a-1}(P_k^+(1-\alpha) + W_k^+ + \delta Q_k^+)N^{(k)}(z)z^k. \tag{5.55}$$

*Proof.* Multiplying (5.20) and (5.21) by proper power of $z$ and $y$ and summing them over the range of $n$ and $k$, one can get

$$\begin{aligned}(\lambda - \theta - \lambda X(z))Q(z, y, \theta) &= \sum_{k=0}^{a-1}\left(\sum_{r=a}^{B}(P_{k,r}(0)(1-\alpha) + W_{k,r}(0)) + \delta \sum_{j=0}^{k} Q_j^{[k]}(0)\right)\tilde{V}_k(\theta)z^k y^k \\ &\quad - \sum_{k=0}^{a-1}\sum_{n=k}^{\infty} Q_n^{[k]}(0)z^n y^k,\end{aligned} \tag{5.56}$$

Now substituting $\theta{=}\lambda - \lambda X(z)$ in (5.56) and using Lemma 5.1, (5.25), (5.27) and (5.29) one can obtain

$$\sum_{k=0}^{a-1}\sum_{n=k}^{\infty} Q_n^{[k]+} z^n y^k = \sum_{k=0}^{a-1}(P_k^+(1-\alpha) + W_k^+ + \delta Q_k^+)N^{(k)}(z)z^k y^k. \tag{5.57}$$

Substituting $y = 1$ in (5.57) desired result is obtained. □

**Lemma 5.6.**

$$Q_n^{[k]+} = \left(P_k^+(1-\alpha) + W_k^+ + \delta \sum_{j=0}^{k} Q_k^{[j]+}\right) w_{n-k}^{(k)}, \quad 0 \leq k \leq a-1, n \geq k. \tag{5.58}$$

*Proof.* From (5.57) collecting the coefficients of $y^k$ $(0 \leq k \leq a-1)$ one can obtain,

$$\sum_{n=k}^{\infty} Q_n^{[k]+} z^n = (P_k^+(1-\alpha) + W_k^+ + \delta Q_k^+)N^{(k)}(z)z^k. \tag{5.59}$$

Now using (5.44) and (5.29) in (5.59) and collecting the coefficients of $z^n$ $(n \geq k)$ the desired result (5.58) is obtained. □

Hence, from Lemma 5.6 it is clear that once $P_k^+$ $(0 \leq k \leq a-1)$ are known, the joint probabilities $Q_n^{[k]+}$ $(0 \leq k \leq a-1, n \geq k)$ are also known.

Multiplying (5.18)-(5.19) by proper power of $z$ and $y$ and summing over the range of $n$ and $r$ the following expression is obtained

$$\begin{aligned}(\lambda-\theta-\lambda X(z))P(z,y,\theta) &= \sum_{r=a}^{B}\left(\sum_{k=0}^{a-1}Q_r^{[k]}(0)+\sum_{j=a}^{B}(P_{r,j}(0)(1-\alpha)+W_{r,j}(0))\right)\sum_{i=r}^{B}y_i\tilde{S}_r(\theta)y^r \\ &+\sum_{n=1}^{\infty}\sum_{r=a}^{B}\left(\sum_{k=0}^{a-1}Q_{n+r}^{[k]}(0)+\sum_{j=a}^{B}P_{n+r,j}(0)(1-\alpha)\right. \\ &\left.+\sum_{j=a}^{B}W_{n+r,j}(0)\right)\tilde{S}_r(\theta)y_r z^n y^r \\ &+(1-\delta)\lambda\sum_{r=a}^{B}\sum_{j=0}^{a-1}R_j g_{r-j}\sum_{i=r}^{B}y_i\tilde{S}_r(\theta)y^r \\ &+(1-\delta)\lambda\sum_{n=1}^{\infty}\sum_{r=a}^{B}\sum_{j=0}^{a-1}R_j g_{n+r-j}y_r\tilde{S}_r(\theta)z^n y^r \\ &-\sum_{n=0}^{\infty}\sum_{r=a}^{B}P_{n,r}(0)z^n y^r.\end{aligned}$$

Substituting $\theta = \lambda - \lambda X(z)$ in the above expression and using Lemma 5.1, (5.25), (5.27), (5.29) and (5.31), one can get

$$\begin{aligned}P^+(z,y) &= \sum_{r=a}^{B}(Q_r^+ + P_r^+(1-\alpha)+W_r^+)\sum_{i=r}^{B}y_i M^{(r)}(z)y^r \\ &+\sum_{n=1}^{\infty}\sum_{r=a}^{B}(Q_{n+r}^+ + P_{n+r}^+(1-\alpha)+W_{n+r}^+)M^{(r)}(z)y_r z^n y^r \\ &+(1-\delta)\sum_{r=a}^{B}\sum_{j=0}^{a-1}\sum_{m=0}^{j}Q_m^+ e_{j,m}\left(g_{r-j}\sum_{i=r}^{B}y_i+\sum_{n=1}^{\infty}g_{n+r-j}y_r z^n\right)M^{(r)}(z)y^r \\ &. \end{aligned} \qquad (5.60)$$

Substituting $y = 1$ in (5.60) and using Lemma 5.5, (5.51) and (5.32) after some algebraic manipulation following expression is obtained.

$$P^+(z) = \frac{\begin{aligned}&\Bigg\{ z^B \sum_{r=a}^{B-1} (P_r^+(1-\alpha) + W_r^+ + Q_r^+) \sum_{i=r}^{B} y_i M^{(r)}(z) \\ &- \sum_{i=a}^{B-1} y_i M^{(i)}(z) \sum_{n=0}^{i} (P_n^+(1-\alpha) + W_n^+ + Q_n^+) z^{n+B-i} \\ &- \sum_{n=0}^{B-1} (P_n^+(1-\alpha) + W_n^+ + Q_n^+) z^n y_B M^{(B)}(z) \\ &+ \sum_{i=a}^{B} y_i M^{(i)}(z) \sum_{k=0}^{a-1} (P_k^+(1-\alpha) + W_k^+ + \delta Q_k^+) N^{(k)}(z) z^{B-i+k} \\ &+ (1-\delta) \sum_{n=0}^{a-1} Q_n^+ \sum_{m=n}^{a-1} e_{m,n} \Bigg( z^B \sum_{r=a}^{B} g_{r-m} \sum_{i=r}^{B} y_i M^{(r)}(z) \\ &+ \sum_{i=a}^{B} y_i M^{(i)}(z) z^{B-i+m} (X(z) - \sum_{n=1}^{i-m} g_n z^n) \Bigg) \Bigg\}\end{aligned}}{z^B - (1-\alpha+\alpha M_{os}(z)) \sum_{i=a}^{B} y_i M^{(i)}(z) z^{B-i}}. \tag{5.61}$$

Finally, using (5.61) in (5.60) after some algebraic manipulation one can get

$$P^+(z,y) = \frac{\bigwedge(z,y)}{z^B - (1-\alpha+\alpha M_{os}(z)) \sum_{i=a}^{B} y_i M^{(i)}(z) z^{B-i}} \tag{5.62}$$

where

$$\begin{aligned}\bigwedge(z,y) =& \sum_{r=a}^{B-1} (P_r^+(1-\alpha) + W_r^+ + Q_r^+) \sum_{i=r}^{B} y_i M^{(r)}(z) \Bigg\{ y^r \Bigg( z^B - (1-\alpha+\alpha M_{os}(z)) \\ &\sum_{j=a}^{B} y_j M^{(j)}(z) z^{B-j} \Bigg) + (1-\alpha+\alpha M_{os}(z)) \sum_{j=a}^{B} y_j M^{(j)}(z) z^{B-j} y^j \Bigg\} + \sum_{i=a}^{B-1} y_i M^{(i)}(z) z^{-i} \sum_{n=0}^{i} (P_n^+(1- \\ &\alpha) + W_n^+ + Q_n^+) z^n \Bigg\{ -y^i \Bigg( z^B - (1-\alpha+\alpha M_{os}(z)) \sum_{j=a}^{B} y_j M^{(j)}(z) z^{B-j} \Bigg) \\ &- (1-\alpha+\alpha M_{os}(z)) z^B \sum_{j=a}^{B} y_j M^{(j)}(z) z^{-j} y^j \Bigg\} + \sum_{n=0}^{B-1} (P_n^+(1-\alpha) + W_n^+ + Q_n^+) z^n y_B M^{(B)}(z) \\ &\Bigg\{ -z^{-B} y^B \Bigg( z^B - (1-\alpha+\alpha M_{os}(z)) \sum_{j=a}^{B} y_j M^{(j)}(z) z^{B-j} \Bigg) - \sum_{j=a}^{B} y_j M^{(j)}(z) z^{-j} y^j \Bigg\} + \sum_{k=0}^{a-1} (P_k^+(1- \\ &\alpha) + W_k^+ + \delta Q_k^+) N^{(k)}(z) z^k \sum_{j=a}^{B} y_j M^{(j)}(z) z^{B-j} y^j + (1-\delta) \sum_{n=0}^{a-1} Q_n^+ \sum_{m=n}^{a-1} e_{m,n} \\ &\Bigg( \Bigg( g_{r-m} \sum_{i=r}^{B} y_i M^{(r)}(z) y^r + \sum_{r=a}^{B} y_r M^{(r)}(z) z^{-r+m} y^r (X(z) - \sum_{n=1}^{i-m} g_n z^n) \Bigg) \Bigg( z^B - (1-\alpha+\alpha M_{os}(z))\end{aligned}$$

$$\sum_{j=a}^{B} y_j M^{(j)}(z) z^{B-j}\Big) + (1-\alpha+\alpha M_{os}(z)) \sum_{r=a}^{B} y_r M^{(r)}(z) z^{B-r} y^r \Big( \sum_{l=a}^{B} g_{l-m} \sum_{i=l}^{B} y_i M^{(l)}(z) + \sum_{i=a}^{B} y_i M^{(i)}(z) z^{-(i-m)} (X(z) - \sum_{n=1}^{i-m} g_n z^n)\Big)\Big)$$

It may be observed from (5.62) that the generating function $P^+(z,y)$ has been expressed in compact form, except for the $B$ unknowns $\{P_n^+\}_{n=0}^{B-1}$. One can further note that from Lemma 5.6 once $P_k^+$ $(0 \le k \le a-1)$ are known then the joint probabilities $Q_n^{[k]+}$ $(0 \le k \le a-1)$ are completely known. Hence, to find $P_{n,r}^+$ $(a \le r \le B, n \ge 0)$ and $Q_n^{[k]+}$ $(0 \le k \le a-1, n \ge k)$ one should find the unknowns $\{P_n^+\}_{n=0}^{B-1}$. Next section is dedicated in getting these unknowns $\{P_n^+\}_{n=0}^{B-1}$.

### 5.3.2 Procedure of getting the unknowns $P_n^+$ $(0 \le n \le B-1)$

It can be seen that the unknowns $P_n^+$ $(0 \le n \le B-1)$, as appeared in (5.62), are same as the unknowns which are appeared in (5.61). Using the result, given in Abolnikov and Dukhovny [129, Theorem 4.1 and Lemma 4.1, page 341], for $\frac{\lambda \bar{g} \sum_{i=a}^{B} \frac{y_i}{\mu_i} + \lambda \bar{g} \frac{\alpha}{\mu}}{\tilde{y}} < 1$, $z^B - (1-\alpha + \alpha M_{os}(z)) \sum_{i=a}^{B} y_i M^{(i)}(z) z^{B-i}$ has $(B-1)$ zeros, say, $x_1, x_2, ..., x_l$ with multiplicity $r_1, r_2, ..., r_l$, respectively, inside the unit circle $|z| = 1$ $\big($where $(l \le B-1)$ and $\sum_{i=1}^{l} r_i = (B-1)\big)$ and one simple zero, say, $z_B = 1$, on the boundary of unit circle $|z|$=1. Due to analyticity of (5.61) in $|z| \le 1$ these zeros are also the zeros of numerator of (5.61). Hence, from (5.61) set of $(B-1)$ linearly independent equations are obtained,

$$\Bigg[\frac{d^{i-1}}{dz^{i-1}} \Big\{ z^B \sum_{r=a}^{B-1} (P_r^+(1-\alpha) + W_r^+ + Q_r^+) \sum_{i=r}^{B} y_i M^{(r)}(z) - \sum_{i=a}^{B-1} y_i M^{(i)}(z) \sum_{n=0}^{i} (P_n^+(1-\alpha) + W_n^+ + Q_n^+) z^{n+B-i} - \sum_{n=0}^{B-1} (P_n^+(1-\alpha) + W_n^+ + Q_n^+) z^n y_B M^{(B)}(z) + \sum_{i=a}^{B} y_i M^{(i)}(z) \sum_{k=0}^{a-1} (P_k^+(1-\alpha) + W_k^+ + \delta Q_k^+) N^{(k)}(z) z^{B-i+k} + (1-\delta) \sum_{n=0}^{a-1} Q_n^+ \sum_{m=n}^{a-1} e_{m,n} \Big( z^B \sum_{r=a}^{B} g_{r-m} \sum_{i=r}^{B} y_i M^{(r)}(z) + \sum_{i=a}^{B} y_i M^{(i)}(z) z^{B-i+m} (X(z) - \sum_{n=1}^{i-m} g_n z^n)\Big)\Big\}\Bigg]_{z=x_j} = 0, \quad 1 \le j \le l \;\; \& \;\; 1 \le i \le r_j, \tag{5.63}$$

where $\frac{d^0}{dz^0}h(z) = h(z)$.
Now using (5.62), Lemma 5.5 and the normalization condition $(1+\alpha)P^+(1)+Q^+(1) = 1$, after applying L'Hôspital's rule, one can get

$$\begin{aligned}
&(1+\alpha)\Bigg(\sum_{r=a}^{B-1}(P_r^+(1-\alpha)+W_r^++Q_r^+)\sum_{i=r}^{B}y_i(B+\lambda\tilde{g}s_r) \\
&-\sum_{i=a}^{B-1}y_i\sum_{n=0}^{i}(P_n^+(1-\alpha)+W_n^++Q_n^+)(n+B-i+\lambda\tilde{g}s_i) \\
&-\sum_{n=0}^{B-1}(P_n^+(1-\alpha)+W_n^++Q_n^+)y_B(n+\lambda\tilde{g}s_B)) \\
&+(1-\delta)\sum_{n=0}^{a-1}Q_n^+\sum_{m=n}^{a-1}e_{m,n}\Big(\sum_{r=a}^{B}g_{r-m}\sum_{i=r}^{B}y_i(\lambda\tilde{g}s_r+B) \\
&+\sum_{i=a}^{B}y_i(\tilde{g}-\sum_{n=1}^{i-m}ng_n)+(1-\sum_{n=1}^{i-m}ng_n)(B-i+m+\lambda\tilde{g}s_i))\Bigg) \\
&+\sum_{i=a}^{B}\sum_{k=0}^{a-1}(P_k^+(1-\alpha)+W_k^++\delta Q_k^+)(\lambda\tilde{g}(s_i+x_k)+B-i+k) = \tilde{g}(1-\rho) \qquad (5.64)
\end{aligned}$$

Hence, (5.63) and (5.64) together forms non-homogenous system of $B$ linearly independent equations in $B$ unknowns $P_n^+$ $(0 \le n \le B-1)$, solving them $P_n^+$ $(0 \le n \le B-1)$ are uniquely determined.

Now using (5.31) in (5.62) and then collecting the coefficients of $y^r$ $(a \le r \le B)$ the following expression is obtained

$$\sum_{n=0}^{\infty}P_{n,r}^+z^n = \frac{\coprod(z,r)}{z^B-(1-\alpha+\alpha M_{os}(z))\sum_{i=a}^{B}y_iM^{(i)}(z)z^{B-i}}, a \le r \le B-1, \qquad (5.65)$$

where

$$\begin{aligned}
\coprod(z,r) &= (P_r^+(1-\alpha)+W_r^++Q_r^+)\sum_{i=r}^{B}y_iM^{(r)}(z)\Big(z^B-(1-\alpha+\alpha M_{os}(z))\sum_{j=a}^{B}y_jM^{(j)}(z)z^{B-j}\Big)+ \\
&(1-\alpha+\alpha M_{os}(z))y_rM^{(r)}(z)z^{B-r}\sum_{j=a}^{B-1}(P_j^+(1-\alpha)+W_j^++Q_j^+)\sum_{i=j}^{B}y_iM^{(i)}(z)-y_rM^{(r)}(z)z^{-r} \\
&\sum_{n=0}^{r}(P_n^+(1-\alpha)+W_n^++Q_n^+)z^n\Big(z^B-(1-\alpha+\alpha M_{os}(z))\sum_{j=a}^{B}y_jM^{(j)}(z)z^{B-j}\Big)-(1-\alpha+ \\
&\alpha M_{os}(z))\sum_{i=a}^{B-1}y_iM^{(i)}(z)\sum_{n=0}^{i}(P_n^+(1-\alpha)+W_n^++Q_n^+)z^{n+B-r-i}y_rM^{(r)}(z)-(1-\alpha+\alpha M_{os}(z))
\end{aligned}$$

$$\sum_{n=0}^{B-1}(P_n^+(1-\alpha)+W_n^++Q_n^+)z^ny_BM^{(B)}(z)y_rM^{(r)}(z)z^{-r}+\sum_{k=0}^{a-1}(P_k^+(1-\alpha)+W_k^++\delta Q_k^+)N^{(k)}(z)$$
$$z^ky_rM^{(r)}(z)z^{B-r}+(1-\delta)\sum_{n=0}^{a-1}Q_n^+\sum_{m=n}^{a-1}e_{m,n}\Bigg(\Bigg(g_{r-m}\sum_{i=r}^{B}y_iM^{(r)}(z)+y_rM^{(r)}(z)z^{-r+m}$$
$$(X(z)-\sum_{n=1}^{r-m}g_nz^n)\Bigg)\Bigg(z^B-(1-\alpha+\alpha M_{os}(z))\sum_{j=a}^{B}y_jM^{(j)}(z)z^{B-j}\Bigg)+(1-\alpha+\alpha M_{os}(z))y_rM^{(r)}(z)$$
$$z^{B-r}\Bigg(\sum_{l=a}^{B}g_{l-m}\sum_{i=l}^{B}y_iM^{(l)}(z)+\sum_{i=a}^{B}y_iM^{(i)}(z)z^{-(i-m)}(X(z)-\sum_{n=1}^{i-m}g_nz^n)\Bigg)\Bigg),\ a\le r\le B-1$$

and

$$\sum_{n=0}^{\infty}P_{n,B}^+z^n=\frac{\coprod(z,b)}{z^B-(1-\alpha+\alpha M_{os}(z))\sum_{i=a}^{B}y_iM^{(i)}(z)z^{B-i}}. \tag{5.66}$$

where

$$\coprod(z,b)=(1-\alpha+\alpha M_{os}(z))y_BM^{(B)}(z)\sum_{j=a}^{B-1}(P_j^+(1-\alpha)+W_j^++Q_j^+)\sum_{i=j}^{B}y_iM^{(i)}(z)-(1-$$
$$\alpha+\alpha M_{os}(z))\sum_{i=a}^{B-1}y_iM^{(i)}(z)\sum_{n=0}^{i}(P_n^+(1-\alpha)+W_n^++Q_n^+)z^{n-i}y_BM^{(B)}(z)-\sum_{n=0}^{B-1}(P_n^+(1-$$
$$\alpha)+W_n^++Q_n^+)z^ny_BM^B(z)z^{-B}\Bigg(z^B-(1-\alpha+\alpha M_{os}(z))\sum_{j=a}^{B}y_jM^{(j)}(z)z^{B-j}+(1-\alpha+$$
$$\alpha M_{os}(z))y_BM^{(B)}(z)\Bigg)+\sum_{k=0}^{a-1}(P_k^+(1-\alpha)+W_k^++\delta Q_k^+)N^{(k)}(z)z^ky_BM^{(B)}(z)$$
$$+(1-\delta)\sum_{n=0}^{a-1}Q_n^+\sum_{m=n}^{a-1}e_{m,n}\Bigg(g_{B-m}y_BM^{(B)}(z)+y_BM^{(B)}(z)z^{-B+m}(X(z)-\sum_{n=1}^{B-m}g_nz^n)\Bigg)$$
$$\Bigg(z^B-(1-\alpha+\alpha M_{os}(z))\sum_{j=a}^{B}y_jM^{(j)}(z)z^{B-j}\Bigg)+(1-\delta)\sum_{n=0}^{a-1}Q_n^+\sum_{m=n}^{a-1}e_{m,n}(1-\alpha+\alpha M_{os}(z))$$
$$y_BM^{(B)}(z)\Bigg(\sum_{l=a}^{B}g_{l-m}\sum_{i=l}^{B}y_iM^{(l)}(z)+\sum_{i=a}^{B}y_iM^{(i)}(z)z^{-(i-m)}(X(z)-\sum_{n=1}^{i-m}g_nz^n)\Bigg).$$

For further investigation, assume that the LST of the vacation time distribution and service time distribution, are rational function, i.e., $\tilde{V}_k(\theta)=\frac{X_k(\theta)}{Y_k(\theta)}$, $\tilde{S}_r(\theta)=\frac{X_r(\theta)}{Y_r(\theta)}$, and $\tilde{S}(\theta)=\frac{\hat{X}(\theta)}{\hat{Y}(\theta)}$ where $X_k(\theta), Y_k(\theta), X_r(\theta), Y_r(\theta), \hat{X}(\theta)$, and $\hat{Y}(\theta)$ are polynomials in $\theta$. Even, distribution functions having transcendental LST can also be rationalized by padé approximation. Substituting $\tilde{V}_k(\lambda-\lambda X(z))=\frac{X_k(\lambda-\lambda X(z))}{Y_k(\lambda-\lambda X(z))}$, $0\le k\le a-1$, $\tilde{S}_r(\lambda-\lambda X(z))=\frac{X_r(\lambda-\lambda X(z))}{Y_r(\lambda-\lambda X(z))}$, $a\le r\le B$, and $S(\lambda-\lambda X(z))=\frac{\hat{X}(\lambda-\lambda X(z))}{\hat{Y}(\lambda-\lambda X(z))}$ in (5.65)-(5.66) after some simplification one can get

$$\sum_{n=0}^{\infty}P_{n,r}^+z^n = \frac{L_r(z)}{D_r(z)},\ a\le r\le B. \tag{5.67}$$

Where $L_r(z)$ and $D_r(z)$ are polynomials of degree $u_r$ and $d_r$, respectively, and $D_r(z)$ is monic. Now partial fraction method is applied to (5.67) for obtaining the joint probabilities $P_{n,r}$ $(n \geq 0, a \leq r \leq B)$. Since, (5.67) is analytic in $|z| \leq 1$, therefore, the zeros of $D_r(z)$ $(a \leq r \leq B)$ lying inside and on the unit circle $|z| = 1$ do not play any role in getting $P_{n,r}$ $(n \geq 0, a \leq r \leq B)$. Hence, in order to obtain all the joint probabilities $P_{n,r}$ $(n \geq 0, a \leq r \leq B)$, it is necessary to know about all the zeros of $D_r(z)$ $(a \leq r \leq B)$ of modulus greater than one. Let $\gamma_{1,r}, \gamma_{2,r}, ..., \gamma_{l_r,r}$ be the zeros of $D_r(z)$ of modules greater than one with multiplicity $\tau_{1,r}, \tau_{2,r}, ..., \tau_{l_r,r}$, respectively, such that $\sum_{j=1}^{l_r} \tau_{j,r} \leq d_r$. Following two cases may arise now.

*Case I : $d_r \leq u_r$*

Applying the partial fraction method to the right hand side of (5.67) the following expression is obtained

$$\sum_{n=0}^{\infty} P^+_{n,r} z^n = \sum_{i=0}^{u_r-d_r} \varrho_i z^i + \sum_{j=1}^{l_r} \sum_{i=1}^{\tau_{j,r}} \frac{B_{i,j,r}}{(z-\gamma_{j,r})^{\tau_{j,r}-i+1}}, \tag{5.68}$$

where

$$B_{i,j,r} = \frac{1}{(i-1)!}\left[\frac{d^{i-1}}{dz^{i-1}}\left(\frac{L_r(z)\frac{d^{\tau_{j,r}}}{dz^{\tau_{j,r}}}(z-\gamma_{j,r})^{\tau_{j,r}}}{\frac{d^{\tau_{j,r}}}{dz^{\tau_{j,r}}}(D_r(z))}\right)\right]_{z=\gamma_{j,r}}, \tag{5.69}$$
$$a \leq r \leq B, j = 1, 2, ..., l_r,\ i = 1, 2, ..., \tau_{j,r}.$$

Accumulating the coefficients of $z^n$ $(n \geq 0)$ from both side of (5.68), one can get for $a \leq r \leq B$,

$$P^+_{n,r} = \begin{cases} \left(\varrho_n + \sum_{j=1}^{l_r}\sum_{i=1}^{\tau_{j,r}} \frac{B_{i,j,r}}{(-1)^{\tau_{j,r}-i+1}\gamma_{j,r}^{\tau_{j,r}+n-i+1}}\binom{\tau_{j,r}-i+n}{\tau_{j,r}-i}\right), & 0 \leq n \leq u_r - d_r, \\ \left(\sum_{j=1}^{l_r}\sum_{i=1}^{\tau_{j,r}} \frac{B_{i,j,r}}{(-1)^{\tau_{j,r}-i+1}\gamma_{j,r}^{\tau_{j,r}+n-i+1}}\binom{\tau_{j,r}-i+n}{\tau_{j,r}-i}\right), & n > u_r - d_r. \end{cases} \tag{5.70}$$

*Case II : $d_r > u_r$*

Removing the first summation term of the right hand side of (5.68), for $a \leq r \leq B$, one can obtain

$$P^+_{n,r} = \left(\sum_{j=1}^{l_r}\sum_{i=1}^{\tau_{j,r}} \frac{B_{i,j,r}}{(-1)^{\tau_{j,r}-i+1}\gamma_{j,r}^{\tau_{j,r}+n-i+1}}\binom{\tau_{j,r}-i+n}{\tau_{j,r}-i}\right), \quad n \geq 0. \tag{5.71}$$

Equation (5.70) [or (5.71)] gives the joint probabilities of queue and server content at service completion epoch.

**Theorem 5.7.**

$$W_{n,r}^{+} = \alpha \sum_{i=0}^{n} P_{i,r}^{+} q_{n-i}, \ n \geq 0, a \leq r \leq B. \tag{5.72}$$

*Proof.* Using (5.34), and collecting the coefficients of $y^r$ $(a \leq r \leq B)$ from both the side of (5.53) one can get

$$\sum_{n=0}^{\infty} W_{n,r}^{+} z^n = \sum_{n=0}^{\infty} P_{n,r}^{+} \alpha M_{os}(z) z^n. \tag{5.73}$$

Using (5.43) in the above expression and collecting the coefficients of $z^n$ $(n \geq 0)$ the desired outcome is obtained. □

Thus the evaluation of the joint probabilities of queue and server content at service (FES and SOS) completion epoch and the joint probabilities of queue length and type of vacation at vacation termination epoch complete here.

### 5.3.3 Joint probabilities at arbitrary epoch

In the previous section the joint probabilities of the queue and server content at service (FES and SOS) completion epoch, as well as the joint probabilities of the queue size and the type of vacation at vacation termination epoch have been successfully achieved. In this section, the main objective is focused for getting the joint probabilities at arbitrary epoch.

**Theorem 5.8.** *The probabilities* $R_n$ $(0 \le n \le a-1)$, $P_{n,r}$ $(W_{n,r})$ $(n \ge 0, a \le r \le B)$ *and* $Q_n^{[k]}$ $(n \ge k, 0 \le k \le a-1)$ *are given by,*

$$R_n = \frac{\sum\limits_{m=0}^{n} e_{n,m} Q_m^+}{E}, \quad 0 \le n \le a-1 \quad (exist\ only\ for\ SV), \tag{5.74}$$

$$P_{0,r} = \frac{(Q_r^+ + P_r^+(1-\alpha) + W_r^+)(\sum\limits_{i=r}^{B} y_i) + (1-\delta)\sum\limits_{n=0}^{a-1}\sum\limits_{m=n}^{a-1} e_{m,n} g_{r-m} Q_n^+ \sum\limits_{i=r}^{B} y_i - P_{0,r}^+}{E}, \ n \ge 0, \tag{5.75}$$

$$P_{n,r} = \sum_{j=1}^{n} P_{n-j,r} g_j + \frac{(Q_{n+r}^+ + P_{n+r}^+(1-\alpha) + W_{n+r}^+) y_r + (1-\delta)\sum\limits_{m=0}^{a-1} Q_m^+ \sum\limits_{j=m}^{a-1} e_{j,m} g_{n+r-j} y_r - P_{n,r}^+}{E},$$
$$n \ge 1, a \le r \le B, \tag{5.76}$$

$$Q_k^{[k]} = \frac{P_k^+(1-\alpha) + W_k^+ + \delta Q_k^+ - Q_k^{[k]+}}{E}, \quad 0 \le k \le a-1, \tag{5.77}$$

$$Q_n^{[k]} = \sum_{i=1}^{n-k} g_i Q_{n-i}^{[k]} - \frac{Q_n^{[k]+}}{E}, \quad n \ge k+1, \ 0 \le k \le a-1, \tag{5.78}$$

$$W_{0,r} = \frac{P_{0,r}^+ \alpha - W_{0,r}^+}{E}, \ a \le r \le B \tag{5.79}$$

$$W_{n,r} = \sum_{j=1}^{n} W_{n-j,r} g_j + \frac{P_{n,r}^+ \alpha - W_{n,r}^+}{E}, \ n \ge 1, \ a \le r \le B \tag{5.80}$$

where $E = \lambda f + (1-\delta)\sum\limits_{n=0}^{a-1}\sum\limits_{m=0}^{n} e_{n,m} Q_m^+$,

$$f = \sum_{n=B+1}^{\infty} \left(P_n^+(1-\alpha) + W_n^+ + Q_n^+\right) \sum_{r=a}^{B} y_r s_r + (P_a^+(1-\alpha) + W_a^+ + Q_a^+) s_a$$
$$+ \sum_{n=a+1}^{B} \left(P_n^+(1-\alpha) + W_n^+ + Q_n^+\right)\left(\sum_{i=a}^{n-1} y_i s_i + \sum_{i=n}^{B} y_i s_n\right)$$
$$+ \sum_{n=0}^{a-1}\Bigg[P_n^+ x_n(1-\alpha) + W_n^+ x_n + (1-\delta)\sum_{m=n}^{a-1} e_{m,n} Q_n^+ \Bigg(\sum_{r=a}^{B} g_{r-m} \sum_{i=r}^{B} y_i \mu_r$$
$$+ \sum_{l=1}^{\infty}\sum_{r=a}^{B} g_{r-m+l} y_r \mu_r\Bigg) + \delta Q_n^+ x_n\Bigg]$$
$$+ \sum_{n=0}^{\infty} P_n^+ \varsigma \alpha$$

*Proof.* Dividing (5.1) by $\sigma^{-1}$ and using Lemma 5.1, Lemma 5.2 and (5.29), one can obtain

$$R_0 = \frac{(1-\sum\limits_{n=0}^{a-1} R_n)Q_0^+}{\lambda f}. \tag{5.81}$$

Similarly, from (5.49), one have

$$R_n = \frac{(1-\sum\limits_{i=0}^{a-1} R_i)\sum\limits_{m=0}^{n}\sum\limits_{k=0}^{m} e_{n,m}Q_m^{[k]+}}{\lambda f}, \quad 0 \le n \le a-1. \tag{5.82}$$

Using (5.81) in (5.82), one can obtain

$$R_n = \frac{R_0}{Q_0^+}\sum_{m=0}^{n}\sum_{k=0}^{m} e_{n,m}Q_m^{[k]+}, \quad 0 \le n \le a-1. \tag{5.83}$$

Using (5.83) in (5.81) after some algebraic manipulation, one can obtain

$$R_0 = \frac{Q_0^+}{\lambda f + \sum\limits_{n=0}^{a-1}\sum\limits_{m=0}^{n}\sum\limits_{k=0}^{m} e_{n,m}Q_m^{[k]+}}. \tag{5.84}$$

Using (5.84) in (5.83), one can obtain

$$R_n = \frac{\sum\limits_{m=0}^{n} e_{n,m}Q_m^+}{\lambda f + \sum\limits_{j=0}^{a-1}\sum\limits_{l=j}^{a-1} e_{l,j}Q_j^+}, \quad 0 \le n \le a-1. \tag{5.85}$$

Setting $\theta$=0 in (5.18)-(5.23), one can get

$$\begin{aligned}
\lambda P_{0,r} &= \sum_{k=0}^{a-1} Q_r^{[k]}(0)\sum_{i=r}^{B} y_i + \sum_{j=a}^{B}(P_{r,j}(0)(1-\alpha)+W_{r,j}(0))\sum_{i=r}^{B} y_i \\
&\quad +(1-\delta)\lambda\sum_{j=0}^{a-1} R_j g_{r-j}\sum_{i=r}^{B} y_i - P_{0,r}(0), \quad a+1 \le r \le B, && (5.86)\\
\lambda P_{n,r} &= \lambda\sum_{j=1}^{n} P_{n-j,r}g_j + \sum_{j=a}^{B}(P_{n+r,j}(0)(1-\alpha)+W_{n+r,j}(0))y_r + \sum_{k=0}^{a-1} Q_{n+r}^{[k]}(0)y_r \\
&\quad +\lambda\sum_{j=0}^{a-1} R_j g_{n+r-j}y_r - P_{n,r}(0), \quad n \ge 1,\ a \le r \le B-1, && (5.87)
\end{aligned}$$

$$\lambda Q_k^{[k]} = \sum_{r=a}^{B}(P_{k,r}(0)(1-\alpha)+W_{k,r}(0))+\delta\sum_{j=0}^{k}Q_k^{[j]}(0)-Q_k^{[k]}(0),\quad 0\le k\le a-1, \tag{5.88}$$

$$\lambda Q_n^{[k]} = \lambda\sum_{i=1}^{n-k}g_iQ_{n-i}^{[k]}-Q_n^{[k]}(0),\quad n\ge k+1,\ 0\le k\le a-1. \tag{5.89}$$

$$\lambda W_{0,r} = P_{0,r}(0)\alpha-W_{0,r}(0),\quad a\le r\le B \tag{5.90}$$

$$\lambda W_{n,r} = \lambda\sum_{j=1}^{n}W_{n-j,r}g_j+P_{n,r}(0)\alpha-W_{n,r}(0),\quad n\ge 1, a\le r\le B \tag{5.91}$$

Dividing (5.86) by $\sigma^{-1}$, respectively, and then using Lemma 5.1, Lemma 5.2, (5.25) and (5.29), the following expression is obtained

$$P_{0,r} = \frac{\begin{array}{l}\big(1-(1-\delta)\sum_{i=0}^{a-1}R_i\big)\big((P_r^+(1-\alpha)+W_r^++Q_r^+)\sum_{i=r}^{B}y_i \\ +(1-\delta)\sum_{n=0}^{a-1}\sum_{m=n}^{a-1}g_{r-m}e_{m,n}Q_n^+\sum_{i=r}^{B}y_i-P_{0,r}^+\big)\end{array}}{\lambda f}. \tag{5.92}$$

Using (5.82) in (5.92), equation (5.75) is obtained.

Applying similar process to (5.87), (5.88), (5.89), (5.90), and (5.91), respectively, after some algebraic manipulation desired results (5.76), (5.77), (5.78), (5.79), and (5.80) are obtained. □

## 5.4 Marginal Probabilities

In this section, some important marginal probabilities are presented that can be derived from the steady state joint probabilities obtained in the previous section.

1. Queue length distribution is given by,

$$P_n^{queue} = \begin{cases}(1-\delta)R_n+\sum\limits_{r=a}^{B}(P_{n,r}+W_{n,r})+\sum\limits_{k=0}^{n}Q_n^{[k]},\ 0\le n\le a-1,\\ \sum\limits_{r=a}^{B}(P_{n,r}+W_{n,r})+\sum\limits_{k=0}^{a-1}Q_n^{[k]},\ n\ge a.\end{cases}$$

2. System length distribution is given by,

$$P_n^{system} = \begin{cases} R_n + \sum\limits_{k=0}^{n} Q_n^{[k]}, \ 0 \le n \le a-1, \\ \sum\limits_{m=a}^{n} (P_{n-m,m} + W_{n-m,m}) + \sum\limits_{k=0}^{a-1} Q_n^{[k]}, \ a \le n \le B, \\ \sum\limits_{r=a}^{B} (P_{n-r,r} + W_{n-r,r}) + \sum\limits_{k=0}^{a-1} Q_n^{[k]}, \ n \ge B+1. \end{cases}$$

3. The server content distribution, when the server is busy in FES, is given by,
$$FES_r^{ser} = \frac{\sum\limits_{n=0}^{\infty} P_{n,r}}{\sum\limits_{r=a}^{B}\sum\limits_{n=0}^{\infty} P_{n,r}}, \ (a \le r \le B) \ .$$

4. The server content distribution when the server is busy in SOS, is given by,
$$SOS_r^{ser} = \frac{\sum\limits_{n=0}^{\infty} W_{n,r}}{\sum\limits_{r=a}^{B}\sum\limits_{n=0}^{\infty} W_{n,r}}, \ (a \le r \le B) \ .$$

5. The server content distribution when the server is busy, is given by,
$$P_r^{ser} = \frac{\sum\limits_{n=0}^{\infty} (P_{n,r}+W_{n,r})}{\sum\limits_{j=a}^{B}\sum\limits_{n=0}^{\infty} (P_{n,j}+W_{n,j})}, \ (a \le r \le B).$$

6. The distribution of the type of vacation, when the server is on vacation, is given by,
$$Q_{vac}^{[k]} = \frac{\sum\limits_{n=k}^{\infty} Q_n^{[k]}}{\sum\limits_{l=0}^{a-1}\sum\limits_{n=l}^{\infty} Q_n^{[l]}}, \ 0 \le k \le a-1.$$

7. The probability that the server is in a dormant state, is given by, $P^{dor} = (1-\delta) \sum\limits_{n=0}^{a-1} R_n$.

8. The probability that the server is busy, is given by, $P_{busy} = \sum\limits_{n=0}^{\infty} \sum\limits_{r=a}^{B} (P_{n,r} + W_{n,r})$.

9. The probability that the server is on vacation, is given by, $Q_{vac} = \sum\limits_{n=0}^{\infty} \sum\limits_{k=0}^{min(n,a-1)} Q_n^{[k]}$.

10. The probability that the server is idle, is given by $P_{idle} = (1-\delta)P^{dor} + Q_{vac}$.

## 5.5 Performance measure

Performance measure is the values that collects the information of the system and helps the manager to run the system smoothly. Since all the steady state probabilities are known, the present section presents some important performance measures of the model under consideration.

1. The expected number of customers in the queue ($L_q$) is given by
$L_q = (1-\delta)\sum_{n=0}^{a-1} nR_n + \sum_{n=0}^{\infty}\sum_{r=a}^{B} n(P_{n,r}+W_{n,r}) + \sum_{k=0}^{a-1}\sum_{n=k}^{\infty} nQ_n^{[k]} = (1-\delta)\sum_{n=0}^{a-1} nP_n^{queue} + \sum_{n=a-\delta a}^{\infty} nP_n^{queue}$.

2. The expected number of customers in the system ($L_s$) is given by
$L_s = (1-\delta)\sum_{n=0}^{a-1} nR_n + \sum_{n=0}^{\infty}\sum_{r=a}^{B} (n+r)(P_{n,r}+W_{n,r}) + \sum_{k=0}^{a-1}\sum_{n=k}^{\infty} nQ_n^{[k]}$.

3. The expected waiting time of a customer in the queue ($W_q$) is given by
$W_q = \frac{L_q}{\lambda \bar{g}}$.

4. The expected waiting time of a customer in the system ($W_s$) is given by
$W_s = \frac{L_s}{\lambda \bar{g}}$.

5. Expected number of customers with the server when server is busy in FES ($L_{fes}^{ser}$) is given by
$L_{fes}^{ser} = \frac{\sum_{r=a}^{B}(rP_{n,r})}{\sum_{m=0}^{\infty}\sum_{j=a}^{b} P_{m,j}}$.

6. Expected number of customers with the server when server is busy in SOS ($L_{sos}^{ser}$) is given by
$L_{sos}^{ser} = \frac{\sum_{r=a}^{B}(rW_{n,r})}{\sum_{m=0}^{\infty}\sum_{j=a}^{b} W_{m,j}}$.

7. Expected number of customers with the server when server is busy ($L^{ser}$) is given by
$L^{ser} = \sum_{r=a}^{B}(rP_r^{ser})$.

8. Expected type of vacation taken by server when server is in vacation ( $L^{vac}$) is given by
$L^{vac} = \sum_{k=0}^{a-1}(kQ_{vac}^{[k]})$.

## 5.6 Numerical results

Table 5.1 and Table 5.2 present the steady state joint probabilities at service (FES and SOS) completion, vacation completion and arbitrary epoch for $M/G_r^{(a,Y)}/1$ queue with SOS and SV. Service time for both the FES and SOS follow the Erlang ($E_3$) distribution, and vacation time follows $E_2$ distribution. The other input parameters are taken as $\lambda$=26.098, $\mu_r = \frac{67.5}{r}$ $(4 \leq r \leq 7)$, $\mu = 17.5$, and $\nu_k = \nu_{k-1} + 2.35$ where $\nu_0 = 1.01$ $(1 \leq k \leq 3)$. $g_1 = 0.45$, $g_2 = 0.20$, $g_3 = 0.35$, $g_n = 0$ $(n \geq 4)$. $y_4 = 0.2$, $y_5 = 0$, $y_6 = 0$, $y_7 = 0.8$. The detail of Table 5.1 is given as follows:

- The 1st column presents the number of customers present in the queue (excluding the number in service).
- 2nd to 5th column present the joint probabilities of the queue and server content at service (FES) competition epoch.
- 6th to 9th column present the joint probabilities of the queue and server content at service (SOS) competition epoch.
- The 10th to 13th column presents the joint probabilities of the queue length and type of vacation at the vacation termination epoch.
- 14th column presents the queue length distribution at service or vacation completion epoch.

The detail of Table 5.2 is given as follows:

- The 1st column presents the number of customers present in the queue (excluding the number in service).
- 2nd column presents the queue length probability during server's dormant period.
- 3rd to 6th column present the joint probabilities of queue and server content during FES at arbitrary epoch.
- 7th to 10th column present the joint probabilities of queue and server content during SOS at arbitrary epoch.
- 11th to 14th column present the joint probabilities of queue length and type of vacation at arbitrary epoch.
- Last column presents the queue length distribution at arbitrary epoch.
- Performance measure (*viz.*, $L_q$, $L_s$, $W_q$, etc.) can be seen just below the table.

Similarly, Table 5.3 presents the steady state joint probabilities at service (FES and SOS) completion epoch and vacation completion epoch, and Table 5.4 presents the steady state joint probabilities at arbitrary epoch for $M/G_r^{(a,Y)}/1$ queue with SOS and MV. The input parameters, notations, and the service (vacation) time distribution are taken the same as taken for Table 5.1 and for Table 5.2.

Table 5.1: Steady state joint probabilities for $M^X/G_r^{(a,Y)}/1$ queue with SV and SOS at service (vacation) completion epoch

| $n$ | $P_{n,4}^+$ | $P_{n,5}^+$ | $P_{n,6}^+$ | $P_{n,7}^+$ | $W_{n,4}^+$ | $W_{n,5}^+$ | $W_{n,6}^+$ | $W_{n,7}^+$ | $Q_n^{[0]+}$ | $Q_n^{[1]+}$ | $Q_n^{[2]+}$ | $Q_n^{[3]+}$ | $P_n^+ + Q_n^+$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0.0038 | 0.0024 | 0.0021 | 0.0016 | 0.0003 | 0.0002 | 0.0002 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0109 |
| 1 | 0.0025 | 0.0013 | 0.0012 | 0.0027 | 0.0004 | 0.0002 | 0.0002 | 0.0003 | 0.0000 | 0.0003 | 0.0000 | 0.0000 | 0.0091 |
| 2 | 0.0025 | 0.0010 | 0.0010 | 0.0037 | 0.0004 | 0.0002 | 0.0002 | 0.0005 | 0.0000 | 0.0002 | 0.0007 | 0.0000 | 0.0105 |
| 3 | 0.0034 | 0.0015 | 0.0016 | 0.0051 | 0.0007 | 0.0003 | 0.0003 | 0.0008 | 0.0001 | 0.0002 | 0.0004 | 0.0015 | 0.0158 |
| 4 | 0.0030 | 0.0010 | 0.0011 | 0.0062 | 0.0007 | 0.0003 | 0.0003 | 0.0010 | 0.0001 | 0.0003 | 0.0004 | 0.0008 | 0.0152 |
| 5 | 0.0030 | 0.0008 | 0.0009 | 0.0071 | 0.0007 | 0.0003 | 0.0003 | 0.0013 | 0.0001 | 0.0003 | 0.0006 | 0.0007 | 0.0160 |
| 15 | 0.0029 | 0.0000 | 0.0001 | 0.0108 | 0.0009 | 0.0001 | 0.0001 | 0.0031 | 0.0001 | 0.0002 | 0.0002 | 0.0003 | 0.0186 |
| 16 | 0.0028 | 0.0000 | 0.0001 | 0.0107 | 0.0009 | 0.0000 | 0.0001 | 0.0031 | 0.0001 | 0.0002 | 0.0002 | 0.0002 | 0.0185 |
| 29 | 0.0022 | 0.0000 | 0.0000 | 0.0085 | 0.0007 | 0.0000 | 0.0000 | 0.0027 | 0.0001 | 0.0001 | 0.0000 | 0.0000 | 0.0143 |
| 30 | 0.0021 | 0.0000 | 0.0000 | 0.0083 | 0.0007 | 0.0000 | 0.0000 | 0.0027 | 0.0001 | 0.0001 | 0.0000 | 0.0000 | 0.0139 |
| 31 | 0.0021 | 0.0000 | 0.0000 | 0.0081 | 0.0007 | 0.0000 | 0.0000 | 0.0026 | 0.0001 | 0.0001 | 0.0000 | 0.0000 | 0.0136 |
| 51 | 0.0012 | 0.0000 | 0.0000 | 0.0047 | 0.0004 | 0.0000 | 0.0000 | 0.0015 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0080 |
| 52 | 0.0012 | 0.0000 | 0.0000 | 0.0046 | 0.0004 | 0.0000 | 0.0000 | 0.0015 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0077 |
| 75 | 0.0006 | 0.0000 | 0.0000 | 0.0024 | 0.0002 | 0.0000 | 0.0000 | 0.0008 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0041 |
| 76 | 0.0006 | 0.0000 | 0.0000 | 0.0023 | 0.0002 | 0.0000 | 0.0000 | 0.0008 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0039 |
| 77 | 0.0006 | 0.0000 | 0.0000 | 0.0023 | 0.0002 | 0.0000 | 0.0000 | 0.0007 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0038 |
| 135 | 0.0001 | 0.0000 | 0.0000 | 0.0004 | 0.0000 | 0.0000 | 0.0000 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0007 |
| 136 | 0.0001 | 0.0000 | 0.0000 | 0.0004 | 0.0000 | 0.0000 | 0.0000 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0006 |
| 155 | 0.0001 | 0.0000 | 0.0000 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0004 |
| 156 | 0.0001 | 0.0000 | 0.0000 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0003 |
| 197 | 0.0000 | 0.0000 | 0.0000 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0001 |
| 198 | 0.0000 | 0.0000 | 0.0000 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0001 |
| 212 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0001 |
| 213 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0001 |
| $\geq$ 214 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| sum | 0.1596 | 0.0107 | 0.0118 | 0.5625 | 0.0479 | 0.0032 | 0.0035 | 0.1687 | 0.0079 | 0.0065 | 0.0071 | 0.0102 | 0.9996 |

TABLE 5.2: Steady state joint probabilities for $M^X/G_r^{(a,Y)}/1$ queue with SV and SOS at arbitrary epoch

| $n$ | $R_n$ | $P_{n,4}$ | $P_{n,5}$ | $P_{n,6}$ | $P_{n,7}$ | $W_{n,4}$ | $W_{n,5}$ | $W_{n,6}$ | $W_{n,7}$ | $Q_n^{[0]}$ | $Q_n^{[1]}$ | $Q_n^{[2]}$ | $Q_n^{[3]}$ | $P_n^{queue}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0.00002 | 0.0038 | 0.0034 | 0.0039 | 0.0038 | 0.0003 | 0.0002 | 0.0002 | 0.0001 | 0.0032 | | | | 0.0190 |
| 1 | 0.00013 | 0.0018 | 0.0010 | 0.0013 | 0.0052 | 0.0003 | 0.0002 | 0.0002 | 0.0003 | 0.0014 | 0.0025 | | | 0.0142 |
| 2 | 0.00042 | 0.0017 | 0.0007 | 0.0009 | 0.0064 | 0.0003 | 0.0002 | 0.0001 | 0.0004 | 0.0013 | 0.0011 | 0.0026 | | 0.0161 |
| 3 | 0.00109 | 0.0022 | 0.0011 | 0.0014 | 0.0080 | 0.0005 | 0.0002 | 0.0002 | 0.0006 | 0.0019 | 0.0009 | 0.0010 | 0.0035 | 0.0226 |
| 4 | | 0.0019 | 0.0006 | 0.0008 | 0.0090 | 0.0005 | 0.0002 | 0.0002 | 0.0008 | 0.0016 | 0.0014 | 0.0008 | 0.0012 | 0.0189 |
| 5 | | 0.0019 | 0.0004 | 0.0006 | 0.0098 | 0.0005 | 0.0002 | 0.0002 | 0.0009 | 0.0015 | 0.0011 | 0.0013 | 0.0010 | 0.0192 |
| 21 | | 0.0016 | 0.0000 | 0.0000 | 0.0107 | 0.0005 | 0.0000 | 0.0000 | 0.0019 | 0.0013 | 0.0003 | 0.0001 | 0.0001 | 0.0164 |
| 22 | | 0.0016 | 0.0000 | 0.0000 | 0.0105 | 0.0005 | 0.0000 | 0.0000 | 0.0018 | 0.0012 | 0.0003 | 0.0001 | 0.0001 | 0.0161 |
| 51 | | 0.0007 | 0.0000 | 0.0000 | 0.0049 | 0.0002 | 0.0000 | 0.0000 | 0.0009 | 0.0006 | 0.0000 | 0.0000 | 0.0000 | 0.0074 |
| 52 | | 0.0007 | 0.0000 | 0.0000 | 0.0048 | 0.0002 | 0.0000 | 0.0000 | 0.0009 | 0.0006 | 0.0000 | 0.0000 | 0.0000 | 0.0072 |
| 64 | | 0.0005 | 0.0000 | 0.0000 | 0.0034 | 0.0002 | 0.0000 | 0.0000 | 0.0006 | 0.0004 | 0.0000 | 0.0000 | 0.0000 | 0.0052 |
| 65 | | 0.0005 | 0.0000 | 0.0000 | 0.0033 | 0.0002 | 0.0000 | 0.0000 | 0.0006 | 0.0004 | 0.0000 | 0.0000 | 0.0000 | 0.0050 |
| 88 | | 0.0003 | 0.0000 | 0.0000 | 0.0017 | 0.0001 | 0.0000 | 0.0000 | 0.0003 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0026 |
| 89 | | 0.0002 | 0.0000 | 0.0000 | 0.0017 | 0.0001 | 0.0000 | 0.0000 | 0.0003 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0025 |
| 126 | | 0.0001 | 0.0000 | 0.0000 | 0.0005 | 0.0000 | 0.0000 | 0.0000 | 0.0001 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0008 |
| 127 | | 0.0001 | 0.0000 | 0.0000 | 0.0005 | 0.0000 | 0.0000 | 0.0000 | 0.0001 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0008 |
| 147 | | 0.0000 | 0.0000 | 0.0000 | 0.0003 | 0.0000 | 0.0000 | 0.0000 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0004 |
| 148 | | 0.0000 | 0.0000 | 0.0000 | 0.0003 | 0.0000 | 0.0000 | 0.0000 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0004 |
| 198 | | 0.0000 | 0.0000 | 0.0000 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0001 |
| 199 | | 0.0000 | 0.0000 | 0.0000 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0001 |
| 209 | | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0001 |
| 210 | | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0001 |
| $\geq$ 211 | | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| sum | 0.0017 | 0.0998 | 0.0084 | 0.0111 | 0.6154 | 0.0289 | 0.0019 | 0.0021 | 0.1017 | 0.0822 | 0.0204 | 0.0131 | 0.0133 | 1.0000 |

$L_q$=37.016, $L_s$=42.681, $W_q$=0.747, $W_s$= 0.861, $L^{ser}$=6.517, $L^{vac}$= .671

$Q_{vac}$=0.129, $P_{idle}$=,0.131, $P_{busy}$=0.869

TABLE 5.3: Steady state joint probabilities for $M^X/G_r^{(a,Y)}/1$ queue with MV and SOS at service (vacation) completion epoch

| $n$ | $P_{n,4}^+$ | $P_{n,5}^+$ | $P_{n,6}^+$ | $P_{n,7}^+$ | $W_{n,4}^+$ | $W_{n,5}^+$ | $W_{n,6}^+$ | $W_{n,7}^+$ | $Q_n^{[0]+}$ | $Q_n^{[1]+}$ | $Q_n^{[2]+}$ | $Q_n^{[3]+}$ | $P_n^+ + Q_n^+$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0.0035 | 0.0023 | 0.0020 | 0.0017 | 0.0003 | 0.0002 | 0.0002 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0104 |
| 1 | 0.0023 | 0.0012 | 0.0012 | 0.0028 | 0.0003 | 0.0002 | 0.0002 | 0.0003 | 0.0000 | 0.0003 | 0.0000 | 0.0000 | 0.0089 |
| 2 | 0.0024 | 0.0010 | 0.0010 | 0.0038 | 0.0004 | 0.0002 | 0.0002 | 0.0005 | 0.0000 | 0.0002 | 0.0007 | 0.0000 | 0.0104 |
| 3 | 0.0032 | 0.0015 | 0.0015 | 0.0052 | 0.0006 | 0.0003 | 0.0003 | 0.0008 | 0.0001 | 0.0002 | 0.0005 | 0.0018 | 0.0160 |
| 4 | 0.0029 | 0.0010 | 0.0011 | 0.0064 | 0.0007 | 0.0003 | 0.0003 | 0.0011 | 0.0001 | 0.0003 | 0.0004 | 0.0010 | 0.0155 |
| 5 | 0.0029 | 0.0007 | 0.0009 | 0.0074 | 0.0007 | 0.0003 | 0.0003 | 0.0013 | 0.0001 | 0.0003 | 0.0006 | 0.0009 | 0.0164 |
| 15 | 0.0029 | 0.0000 | 0.0001 | 0.0111 | 0.0009 | 0.0000 | 0.0001 | 0.0032 | 0.0001 | 0.0002 | 0.0002 | 0.0003 | 0.0192 |
| 16 | 0.0029 | 0.0000 | 0.0001 | 0.0110 | 0.0009 | 0.0000 | 0.0001 | 0.0032 | 0.0001 | 0.0002 | 0.0002 | 0.0003 | 0.0190 |
| 29 | 0.0022 | 0.0000 | 0.0000 | 0.0085 | 0.0007 | 0.0000 | 0.0000 | 0.0027 | 0.0001 | 0.0001 | 0.0000 | 0.0000 | 0.0144 |
| 30 | 0.0021 | 0.0000 | 0.0000 | 0.0083 | 0.0007 | 0.0000 | 0.0000 | 0.0027 | 0.0001 | 0.0001 | 0.0000 | 0.0000 | 0.0140 |
| 31 | 0.0021 | 0.0000 | 0.0000 | 0.0081 | 0.0007 | 0.0000 | 0.0000 | 0.0026 | 0.0001 | 0.0001 | 0.0000 | 0.0000 | 0.0137 |
| 51 | 0.0012 | 0.0000 | 0.0000 | 0.0046 | 0.0004 | 0.0000 | 0.0000 | 0.0015 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0078 |
| 52 | 0.0011 | 0.0000 | 0.0000 | 0.0045 | 0.0004 | 0.0000 | 0.0000 | 0.0015 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0076 |
| 75 | 0.0006 | 0.0000 | 0.0000 | 0.0023 | 0.0002 | 0.0000 | 0.0000 | 0.0008 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0039 |
| 76 | 0.0006 | 0.0000 | 0.0000 | 0.0023 | 0.0002 | 0.0000 | 0.0000 | 0.0007 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0038 |
| 135 | 0.0001 | 0.0000 | 0.0000 | 0.0004 | 0.0000 | 0.0000 | 0.0000 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0006 |
| 136 | 0.0001 | 0.0000 | 0.0000 | 0.0004 | 0.0000 | 0.0000 | 0.0000 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0006 |
| 155 | 0.0001 | 0.0000 | 0.0000 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0003 |
| 156 | 0.0000 | 0.0000 | 0.0000 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0003 |
| 211 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0001 |
| 212 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0001 |
| $\geq$ 213 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| sum | 0.1581 | 0.0102 | 0.0114 | 0.5625 | 0.0474 | 0.0031 | 0.0034 | 0.1687 | 0.0075 | 0.0066 | 0.0080 | 0.0126 | 0.9996 |

TABLE 5.4: Steady state joint probabilities for $M^X/G_r^{(a,Y)}/1$ queue with MV and SOS at arbitrary epoch

| $n$ | $P_{n,4}$ | $P_{n,5}$ | $P_{n,6}$ | $P_{n,7}$ | $W_{n,4}$ | $W_{n,5}$ | $W_{n,6}$ | $W_{n,7}$ | $Q_n^{[0]}$ | $Q_n^{[1]}$ | $Q_n^{[2]}$ | $Q_n^{[3]}$ | $P_n^{queue}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0.0035 | 0.0032 | 0.0038 | 0.0040 | 0.0003 | 0.0002 | 0.0002 | 0.0001 | 0.0030 | 0.0000 | 0.0000 | 0.0000 | 0.0183 |
| 1 | 0.0017 | 0.0010 | 0.0012 | 0.0054 | 0.0003 | 0.0002 | 0.0001 | 0.0003 | 0.0014 | 0.0026 | 0.0000 | 0.0000 | 0.0140 |
| 2 | 0.0016 | 0.0007 | 0.0009 | 0.0066 | 0.0003 | 0.0001 | 0.0001 | 0.0004 | 0.0012 | 0.0011 | 0.0029 | 0.0000 | 0.0160 |
| 3 | 0.0021 | 0.0010 | 0.0014 | 0.0083 | 0.0004 | 0.0002 | 0.0002 | 0.0006 | 0.0019 | 0.0009 | 0.0011 | 0.0044 | 0.0225 |
| 4 | 0.0019 | 0.0005 | 0.0008 | 0.0093 | 0.0004 | 0.0002 | 0.0002 | 0.0008 | 0.0015 | 0.0014 | 0.0009 | 0.0015 | 0.0195 |
| 5 | 0.0019 | 0.0004 | 0.0006 | 0.0101 | 0.0005 | 0.0002 | 0.0002 | 0.0010 | 0.0015 | 0.0011 | 0.0014 | 0.0012 | 0.0199 |
| 21 | 0.0016 | 0.0000 | 0.0000 | 0.0109 | 0.0005 | 0.0000 | 0.0000 | 0.0019 | 0.0012 | 0.0003 | 0.0001 | 0.0001 | 0.0167 |
| 22 | 0.0016 | 0.0000 | 0.0000 | 0.0107 | 0.0005 | 0.0000 | 0.0000 | 0.0019 | 0.0012 | 0.0003 | 0.0001 | 0.0001 | 0.0163 |
| 51 | 0.0007 | 0.0000 | 0.0000 | 0.0048 | 0.0002 | 0.0000 | 0.0000 | 0.0009 | 0.0006 | 0.0000 | 0.0000 | 0.0000 | 0.0073 |
| 52 | 0.0007 | 0.0000 | 0.0000 | 0.0047 | 0.0002 | 0.0000 | 0.0000 | 0.0009 | 0.0006 | 0.0000 | 0.0000 | 0.0000 | 0.0071 |
| 64 | 0.0005 | 0.0000 | 0.0000 | 0.0033 | 0.0002 | 0.0000 | 0.0000 | 0.0006 | 0.0004 | 0.0000 | 0.0000 | 0.0000 | 0.0050 |
| 65 | 0.0005 | 0.0000 | 0.0000 | 0.0032 | 0.0002 | 0.0000 | 0.0000 | 0.0006 | 0.0004 | 0.0000 | 0.0000 | 0.0000 | 0.0049 |
| 88 | 0.0002 | 0.0000 | 0.0000 | 0.0017 | 0.0001 | 0.0000 | 0.0000 | 0.0003 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0025 |
| 89 | 0.0002 | 0.0000 | 0.0000 | 0.0016 | 0.0001 | 0.0000 | 0.0000 | 0.0003 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0024 |
| 126 | 0.0001 | 0.0000 | 0.0000 | 0.0005 | 0.0000 | 0.0000 | 0.0000 | 0.0001 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0008 |
| 127 | 0.0001 | 0.0000 | 0.0000 | 0.0005 | 0.0000 | 0.0000 | 0.0000 | 0.0001 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0008 |
| 147 | 0.0000 | 0.0000 | 0.0000 | 0.0003 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0004 |
| 148 | 0.0000 | 0.0000 | 0.0000 | 0.0003 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0004 |
| 149 | 0.0000 | 0.0000 | 0.0000 | 0.0003 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0004 |
| 208 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0001 |
| 209 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0001 |
| $\geq$ 210 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| sum | 0.0991 | 0.0080 | 0.0107 | 0.6168 | 0.0287 | 0.0019 | 0.0021 | 0.1020 | 0.0788 | 0.0209 | 0.0147 | 0.0165 | 1.0000 |

$L_q$=36.349, $L_s$= 42.017, $W_q$= 0.733, $W_s$= .847, $L^{ser}$= 6.522, $L^{vac}$= 0.763

$Q_{vac}$= 0.131, $P_{idle}$=.0.131, $P_{busy}$=0.869

### 5.6.1 Deduction of the results for $M/M/1$ queue

The model considered in this chapter reduces to the classical $M/M/1$ model when $a = 1$, $B = 1$, $X(z) = z$, i.e., $g_1 = 1$, $g_n = 0$ $(n \geq 2)$, $y_B = 1$, $\alpha = 0$, service time is exponentially distributed and vacation rate is taken considerably large. Table 5.5 and Table 5.6 give the values of $L_q$, $W_s$, $L^{ser}$ and $P_{idle}$ which are obtained for $M/M/1$ model for following two cases.

**Case I:** Results for $M/M/1$ model deduced from the analytical results presented in this chapter by considering $a = 1$, $B = 1$, $X(z) = z$, i.e., $g_1 = 1$, $g_n = 0$ $(n \geq 2)$, $y_B = 1$, $\alpha = 0$, exponential service time distribution and $\nu_0 \longrightarrow \infty$ $(\nu_0 = 200000)$.
**Case II:** Results for classical $M/M/1$ model, for which performance measures $L_q$, $W_s$ and probability $P_{idle}$ are calculated using standard formula $L_q = \frac{\lambda^2}{\mu_1(\mu_1-\lambda)}$, $W_s = \frac{1}{\mu_1-\lambda}$ and $P_{idle} = 1 - \rho$.

The following details are provided for Table 5.5 and Table 5.6.

- 1st and 2nd column present the values of the input parameters $\lambda$ and $\mu_1$, respectively, for which $\rho$ varies from 0.4375 to 0.875.
- 3rd, 4th, 5th and 6th column present the values of $L_q$, $W_s$, $L^{ser}$ and $P_{idle}$, respectively, for Case I.
- 7th, 8th and 9th column present the values of $L_q$, $W_s$ and $P_{idle}$, respectively, for Case II.

It is clearly observed from Table 5.5 and Table 5.6 that the results deduced from current study as a special case matches exactly with the results obtained from classical $M/M/1$ model. Also, the value of $L^{ser}$ calculated from the current study as a special case always gives the value 1 which is obvious and shows the correctness of present study.

TABLE 5.5: Table for Case I and Case II, for SV

| | | Case I | | | | Case II | | |
|---|---|---|---|---|---|---|---|---|
| $\lambda$ | $\mu_1$ | $L_q$ | $W_s$ | $L^{ser}$ | $P_{idle}$ | $L_q$ | $W_s$ | $P_{idle}$ |
| 3.5 | 4 | 6.1249997 | 1.9999999 | 1.0000000 | 0.1250000 | 6.1250000 | 2.0000000 | 0.1250000 |
| 3.5 | 6 | 0.8166667 | 0.4000000 | 1.0000000 | 0.4166667 | 0.8166667 | 0.4000000 | 0.4166667 |
| 3.5 | 8 | 0.3402778 | 0.2222222 | 1.0000000 | 0.5625000 | 0.3402778 | 0.2222222 | 0.5625000 |

TABLE 5.6: Table for Case I and Case II, for MV

| | | Case I | | | | Case II | | |
|---|---|---|---|---|---|---|---|---|
| $\lambda$ | $\mu_1$ | $L_q$ | $W_s$ | $L^{ser}$ | $P_{idle}$ | $L_q$ | $W_s$ | $P_{idle}$ |
| 3.5 | 4 | 6.1250000 | 2.0000000 | 1.0000000 | 0.1250000 | 6.1250000 | 2.0000000 | 0.1250000 |
| 3.5 | 6 | 0.8166700 | 0.4000000 | 1.0000000 | 0.4166700 | 0.8166667 | 0.4000000 | 0.4166667 |
| 3.5 | 8 | 0.3402800 | 0.2222200 | 1.0000000 | 0.5625000 | 0.3402778 | 0.2222222 | 0.5625000 |

## 5.7 Cost model

A cost model is also presented in this section which helps the manager to determine the optimal value of desired input parameters. The following cost parameters are taken for this purpose.

$C_{st}\equiv$ Startup cost per customer per unit time.

$C_b\equiv$ Holding cost per customer per unit time when the server is busy.

$C_v\equiv$ Holding cost per customer per unit time when the server is on vacation.

$C_d\equiv$ Holding cost per customer per unit time when the server is dormant (exists only for SV).

$C_{fes}\equiv$ Operating cost per customer per unit time when the server is busy in FES.

$C_{os}\equiv$ Operating cost per customer per unit time when the server is busy in SOS. Thus in long run

$$\begin{aligned}\text{Total System Cost (TSC)} &= \lambda C_{st} + C_b\sum_{n=0}^{\infty}\sum_{r=a}^{b} n\frac{(P_{n,r}+W_{n,r})}{P_{busy}} + C_v\sum_{n=0}^{\infty}\sum_{k=0}^{min(n,a-1)} n\frac{Q_n^{[k]}}{Q_{vac}} \\ &+(1-\delta)C_d\sum_{n=0}^{a-1} n\frac{R_n}{P^{dor}} + C_{fes}L_{fes}^{ser} + C_{os}L_{sos}^{ser}.\end{aligned}$$

Figure 5.1 reflects the behavior of TSC for different values of $a$ $(1 \le a \le 10)$ for SV and for $\lambda$=3.5 and 7.5. The maximum capacity is fixed at $B = 10$. Service time follows a 2 - stage hyper exponential distribution with service rate $\mu_{r,j} = \frac{r+j}{5}$, $(a \le r \le B,\ j = 1,2)$ and $\mu_r = \left(\sum\limits_{j=1}^{2}\frac{\tilde{\alpha}_{j,r}}{\mu_{j,r}}\right)^{-1}$ where $\tilde{\alpha}_{1,r} = 0.6$ and $\tilde{\alpha}_{2,r} = 0.4$, $(a \le r \le B)$. Vacation time follows $E_2$ distribution with vacation rate $\nu_k = \nu_{k-1} + 0.5$ $(1 \le k \le a-1)$ where $\nu_0 = 1.2$. Service time for SOS follows exponential distribution with rate $\mu = 3.5$. The other input parameters are considered as follows $g_1 = 0.65$, $g_2 = 0.25$, $g_3 = 0.10$, $g_n = 0$ $(n \ge 0)$, $y_r = 0$ $(a \le r \le B-1)$, $y_B = 1$ and $\alpha = 0.25$. The cost parameters are as follows: $C_{st} = 0.2$, $C_b = 2.0$, $C_v = 2.5$, $C_d = 1.0$, $C_{fes} = 4.2$ and $C_{os} = 3.2$. From this particular example, for $\lambda = 3.5$ (7.5) the optimum value for $a$ is 2 (3) and the corresponding minimum value of TSC is 59.462 (101.277). Similarely, Figure 5.2 depicts the behavior of TSC for different values of $a$ ($1 \leq a \leq 10$) for MV and for $\lambda$=3.5 and 7.5. The input parameters, cost parameters and the service (vacation) time distribution are taken the same as for Figure 5.1. For $\lambda$ = 0.5 (0.9), the optimum value for $a$ is 2 (4) and the corresponding minimum value of TSC is 59.482 (99.869). The minimum values of TSC, in each figure, are indicated by arrow sign.

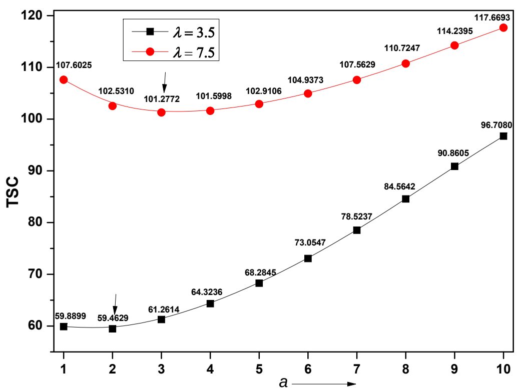


FIGURE 5.1: Effect of $a$ on TSC for SV

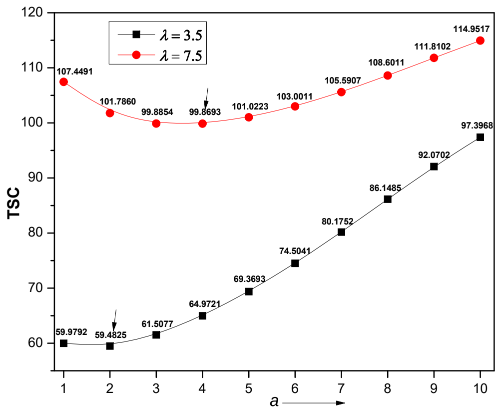


FIGURE 5.2: Effect of $a$ on TSC for MV

## 5.8 Conclusion

In this chapter, the infinite buffer bulk arrival batch size dependent bulk service queue and the queue length dependent SV (MV) with second optional service has been analyzed. The server operates the customer according to the $(a, Y)$ rule. The fundamental mathematical analysis of the model includes mainly the supplementary variable technique and bivariate generating function technique. The considered model analyzed for the joint probabilities of the queue and server content at service completion (arbitrary) epoch and the joint probabilities of queue length and type of vacation at vacation termination (arbitrary) epoch. Practical motivation and the numerical behavior of the considered model are also provided to validate the considered model in real life congestion control. The present model can be generalized with a more general arrival process (*viz.*, $BMAP$).