Chapter 4

Analysis of $MAP/G_r^{(a,b)}/1$ queue with queue length dependent single and multiple vacation

4.1 Introduction

Bulk service queueing system with vacation where the authors considered Poisson/renewal arrival process, can be found in the literatures [88, 93, 99, 100, 101, 102, 103, 127, 128] and the references therein, however, in most of the real life queues (e.g., telecommunication, computer network, etc.) the Poisson/renewal arrival process does not fit due to highly irregular traffic. A good representation for analyzing such bursty and correlated traffic is a non-renewal arrival process, i.e., the Markovian arrival process (MAP) proposed by Lucantoni et al. [4]. Some other input processes are also included in MAP, *viz.*, Markov modulated Poisson process (MMPP), the phase (PH)-type renewal process, the interrupted Poisson process (IPP), Poisson process. Gupta and Sikdar [85] and Sikdar and Samanta [100], respectively, discussed $MAP/G^{(a,b)}/1/N$ queue with SV and $BMAP/G^Y/1/N$ queue with SV (MV), respectively, and obtained queue length distribution at various epoch using embedded Markov chain technique (EMCT) and supplementary variable technique (SVT). In discrete time set up, Nandy and Pradhan [103] considered discrete time batch size dependent batch service queue with SV and MV. They carried out their analysis for queue and

The content of this chapter is accepted for publication in International Journal of Operational Research, INDERSCIENCE.

server content during busy period and queue length probabilities during vacation period using the SVT and bivariate probability generating function technique.

In most of the vacation queueing models, authors considered the length of vacation of the server as random and unaffected by the queue size at the vacation initiation epoch. The queueing model with vacation where the length of vacation depends on the queue size (length) at the vacation initiation epoch is known as the queue size dependent (QSD) vacation model. Such QSD vacation models have been analyzed by few researchers, see the references [74, 101, 108, 109]. Gupta et al. [101] considered $M/G_r^{(a,b)}/1/N$ queue with QSD SV (MV) and obtained the joint distribution of queue and server content and the joint distribution of queue content and type of vacation using the SVT. In Chapter 3 author analyzed the same model as Gupta et al. [101] for infinite buffer queue using the SVT and bivariate generating function approach. To the extent of the author's knowledge, an infinite capacity queueing system ($MAP/G_r^{(a,b)}/1$ queue) with MAP and queue size dependent SV and MV has not been investigated previously in the literature. Further, it is observed that the queue size dependent vacation policy remarkably reduces congestion.

In computer network with highly irregular traffic, the proposed model is applicable. A desktop computer system connects to a local area network (LAN) via Ethernet (IEEE802.11h) link. Digital signals are transmitted over Ethernet in the form of a group (packet), with the transmission rate varying with the packet under transmission. Power utility (power utility increases with transmission rate) depends on transmission rate. The medium access control (MAC) handshake protocol is helpful in decreasing the average power utility. It is achieved by measuring the queue size (signals) waiting for transmission. After a transmission, if the number of signals are lower than the previously established lower threshold, the handshake mechanism (vacation period) activates, and it depends on the queue size at the vacation initiation epoch.

A model description of the considered model can be found in Section 4.2. In Section 4.3, the model has been analyzed mathematically. Section 4.4 presents some marginal distributions. Section 4.5 contains the performance measures. Numerical results are presented in Section 4.6. Section 4.7 presents a cost model, and for the conclusion, readers are invoked to see Section 4.8.

4.2 Model description

In this section, single server infinite capacity, batch size dependent bulk service queue with queue size dependent vacation (single and multiple) is introduced, in which Markovian arrival process (MAP) governs the customer's arrival to the system. The MAP is governed by the underlying Markov chain (UMC). In the UMC, there are transition from state i to state j $(1 \leq i, j \leq m)$. Assume that $d_{i,j}$ be the transition rate from state i to j with an arrival and $c_{i,j}$ be the transition rate from state i to j without an arrival. The $m \times m$ matrix $C = (c_{i,j})$ has non-negative off diagonal and negative diagonal members, whereas, the $m \times m$ matrix $D = (d_{i,j})$ has non-negative elements. The infinitesimal generator of the UMC is presented by the matrix $(C + D)$. Assume that $\xi = (\xi_1, \xi_2, ..., \xi_m)$ is a stationary probability vector such that ξ(C+D) $= \mathbf{0}$, $\xi\mathbf{e}$=1, where $\mathbf{e}$ is $m \times 1$ column matrix with each element 1 and $\mathbf{0}$ is $1 \times m$ zero matrix. The fundamental arrival rate is determined by $\lambda = \xi D\mathbf{e}$. Assume that I refers to an identity matrix with an appropriate dimension. According to GBS rule, the costomers are served in batches (groups). The service time (T_r) of a batch of size r $(h \leq r \leq H)$ is generally distributed with probability density function (pdf) $s_r(t)$, distribution function (DF) $S_r(t)$, the Laplace-Stieltjes transform (LST) $\tilde{S}_r(\theta)$ and the mean service time $\frac{1}{\mu_r} = s_r = -\tilde{S}_r^{(1)}(0)$ $(h \leq r \leq H)$, where $\tilde{S}_r^{(1)}(0)$ is the derivative of $\tilde{S}_r(\theta)$ evaluated at θ=0. When a service is finished, and if the server determines that the queue length is l $(\geq h)$, it begins service in accordance with the GBS rule, i.e., it serves a batch of size $\min(l, H)$, where H is the server's maximum capacity. If the queue length is k $(< h)$ after a service, the server begins the vacation, which has a random length and is dependent on the queue length k $(0 \leq k \leq h-1)$. Let $V_k(t)$ $\{v_k(t)\}$ $[\tilde{V}_k(\theta)]$ be the DF {pdf} [LST] of a typical vacation time V_k $(0 \leq k \leq h-1)$ which is generally distributed. The mean vacation time $\frac{1}{\nu_k} = x_k = -\tilde{V}_k^{(1)}(0)$ where $\tilde{V}_k^{(1)}(0)$ is the derivative of $\tilde{V}_k(\theta)$ at θ=0. If the server finds at least h waiting customers at the end of the vacation, it operates those customers in accordance with GBS rule, otherwise, it enters a state of dormancy until the queue length reaches the minimum threshold h, or it takes another vacation depending on the vacation policy being considered, namely either single vacation (SV) or multiple vacation (MV). The system's stability is ensured by the traffic intensity, which is $\frac{\lambda s_H}{H}(< 1)$. Using the following definition of the variable δ, single vacation (SV) and multiple vacation (MV) queues are examined in this chapter in a unified manner.

$$\delta = \begin{cases} 1, & \text{for MV}, \\ 0, & \text{for SV}. \end{cases}$$

4.3 System analysis

The following random variables, at time t, are necessary for the mathematical analysis of the considered model.

- $N_q(t) \equiv$ Queue size waiting in line (queue).
- $N_s(t) \equiv$ Batch size with the server when the server is busy.
- $K(t) \equiv$ Type of vacation, when the server is on vacation.
- $J(t) \equiv$ State of the underlying Markov chain of the MAP.
- $U(t) \equiv$ Remaining service time (if any).
- $V(t) \equiv$ Remaining vacation time (if any).

$N_s(t) = 0$ reflects the server's dormant status at time t. Depending on the considered vacation policy the following Markov process describes the model.

$$\begin{cases} \{(N_q(t), N_s(t)), J(t)\} \cup \{\big(N_q(t), N_s(t), J(t), U(t)\big) \cup \big(N_q(t), K(t), J(t), V(t)\big)\}, & \text{for SV}, \\ \{\big(N_q(t), N_s(t), J(t), U(t)\big) \cup \big(N_q(t), K(t), J(t), V(t)\big)\}, & \text{for MV}, \end{cases}$$

with state space

$$\begin{cases} \{(n, 0, i); 0 \le n \le h-1, 1 \le i \le m\} \bigcup \\ \{(n, r, i, u); n \ge 0, h \le r \le H, 1 \le i \le m, u \ge 0\} \bigcup \\ \{(n, k, i, u); 0 \le k \le h-1, n \ge k, 1 \le i \le m, u \ge 0\}, \quad \text{for SV}, \\ \{(n, r, i, u); n \ge 0, h \le r \le H, 1 \le i \le m, u \ge 0\} \bigcup \\ \{(n, k, i, u); 0 \le k \le h-1, 1 \le i \le m, n \ge k, u \ge 0\}, \quad \text{for MV}. \end{cases}$$

The state probabilities, at time t, are defined as:

- $R_i(n, 0, t) \equiv Pr\{N_q(t) = n, N_s(t) = 0, J(t) = i, u \le U(t) \le u + du\}, 1 \le i \le m, 0 \le n \le h-1$, (for SV only).
- $\xi_i(n, r, u, t)du \equiv Pr\{N_q(t) = n, N_s(t) = r, J(t) = i, u \le U(t) \le u + du\}, 1 \le i \le m, n \ge 0\ ,\ h \le r \le H$.

- $\gamma_i(n,k,u,t)du \equiv Pr\{N_q(t)=n, K(t)=k, J(t)=i, u \le V(t) \le u+du\}, 1 \le i \le m, n \ge k$, $0 \le k \le h-1$.

In steady state, as $t \to \infty$,

$R_i(n,0) = \lim_{t\to\infty} R_i(n,0,t), \quad 0 \le n \le h-1, 1 \le i \le m,$ (exist only for SV),
$\xi_i(n,r,u) = \lim_{t\to\infty} \xi_i(n,r,u,t), \quad n \ge 0, \quad h \le r \le H, \ 1 \le i \le m,$
$\gamma_i(n,k,u) = \lim_{t\to\infty} \gamma_i(n,k,u,t), \quad n \ge k, \quad 0 \le k \le h-1, \ 1 \le i \le m.$
Further, define

- $R(n,0)$=$(R_1(n,0), R_2(n,0), .., R_m(n,0)), \ \ 0 \le n \le h-1.$
- $\xi(n,r,u)$=$(\xi_1(n,r,u), \xi_2(n,r,u), ..., \xi_m(n,r,u)), \ \ n \ge 0, h \le r \le H.$
- $\gamma(n,k,u)$=$(\gamma_1(n,k,u), \gamma_2(n,k,u), ..., \gamma_m(n,k,u)), \ \ n \ge k, 0 \le k \le h-1.$

Following an analysis of the system at time t and $t+dt$, the related steady state equations are obtained as follows:

$$0 = (1-\delta)\Big(R(0,0)C + \gamma(0,0,0)\Big), \tag{4.1}$$

$$0 = (1-\delta)\Big(R(n,0)C + R(n-1,0)D + \sum_{k=0}^{n}\gamma(n,k,0)\Big), 1 \le n \le h-1, \tag{4.2}$$

$$-\frac{d}{du}\xi(0,h,u) = \xi(0,h,u)C + (1-\delta)R(h-1,0)Ds_h(u) + \Big(\sum_{k=0}^{h-1}\gamma(h,k,0) + \sum_{r=h}^{H}\xi(h,r,0)\Big)s_h(u), \tag{4.3}$$

$$-\frac{d}{du}\xi(0,r,u) = \xi(0,r,u)C + \Big(\sum_{k=0}^{h-1}\gamma(r,k,0) + \sum_{r=h}^{H}\xi(r,j,0)\Big)s_r(u), h+1 \le r \le H, \tag{4.4}$$

$$-\frac{d}{du}\xi(n,r,u) = \xi(n,r,u)C + \xi(n-1,r,u)D, \quad h \le r \le H-1, \quad n \ge 1, \tag{4.5}$$

$$-\frac{d}{du}\xi(n,H,u) = \xi(n,H,u)C + \xi(n-1,H,u)D + \Bigg(\sum_{k=0}^{h-1}\gamma(n+H,k,0) + \sum_{r=h}^{H}\xi(n+H,r,0)\Bigg)s_H(u),\ n \geq 1, \quad (4.6)$$

$$-\frac{d}{du}\gamma(k,k,u) = \gamma(k,k,u)C + \Bigg(\sum_{r=h}^{H}\xi(k,r,0) + \delta\sum_{j=0}^{k}\gamma(k,j,0)\Bigg)v_k(u), 0 \leq k \leq h-1, \quad (4.7)$$

$$-\frac{d}{du}\gamma(n,k,u) = \gamma(n,k,u)C + \gamma(n-1,k,u)D, \quad n \geq k+1, \quad 0 \leq k \leq h-1. \quad (4.8)$$

Further, for Re $\theta \geq 0$, define,

$$\tilde{S}_r(\theta) = \int_0^\infty e^{-\theta u} dS_r(u) = \int_0^\infty e^{-\theta u} s_r(u)du, \quad h \leq r \leq H, \quad (4.9)$$

$$\tilde{\xi}(n,r,\theta) = \int_0^\infty e^{-\theta u}\xi(n,r,u)du, \quad h \leq r \leq H,\ n \geq 0, \quad (4.10)$$

$$\xi(n,r) \equiv \tilde{\xi}(n,r,0) = \int_0^\infty \xi(n,r,u)du, \quad h \leq r \leq H,\ n \geq 0, \quad (4.11)$$

$$\tilde{V}_k(\theta) = \int_0^\infty e^{-\theta u} dV_k(u) = \int_0^\infty e^{-\theta u} v_k(u)du, \quad 0 \leq k \leq h-1, \quad (4.12)$$

$$\tilde{\gamma}(n,k,\theta) = \int_0^\infty e^{-\theta u}\gamma(n,k,u)du, \quad 0 \leq k \leq h-1,\ n \geq k, \quad (4.13)$$

$$\gamma(n,k) \equiv \tilde{\gamma}(n,k,0) = \int_0^\infty \gamma(n,k,u)du, \quad 0 \leq k \leq h-1, \quad n \geq k. \quad (4.14)$$

- $R(n,0)$=$(R_1(n,0), R_2(n,0), ..., R_m(n,0)),\ \ 0 \leq n \leq h-1.$
- $\xi(n,r)$=$(\xi_1(n,r), \xi_2(n,r), ..., \xi_m(n,r)),\ \ n \geq 0, h \leq r \leq H.$
- $\gamma(n,k)$=$(\gamma_1(n,k), \gamma_2(n,k), ..., \gamma_m(n,k)),\ \ n \geq k, 0 \leq k \leq h-1.$

Here the probability $(R_i(n,0))$ $\{\xi_i(n,r)\}$ $[\gamma_i(n,k)]$ denotes that (queue size is n and the sever is in dormant state, and the arrival process is in phase i, $0 \leq n \leq h-1$) {queue size is n and r customers are being serviced, and the arrival process is in phase i, $h \leq r \leq H$, $n \geq 0$} [queue size is n and the server is on k^{th} type of vacation, and the arrival process is in phase i, $0 \leq k \leq h-1$, $n \geq k$] at arbitrary epoch.

Multiplying the equations (4.3)-(4.8) by $e^{-\theta u}$ and integrating with respect to u over the limits 0 to ∞, one can get

$$-\theta\tilde{\xi}(0,h,\theta) = \tilde{\xi}(0,h,\theta)C + (1-\delta)R(h-1,0)D\tilde{S}_h(\theta) + \Big(\sum_{k=0}^{h-1}\gamma(h,k,0) + \sum_{r=h}^{H}\xi(h,r,0)\Big)\tilde{S}_h(\theta) - \xi(0,h,0), \quad (4.15)$$

$$-\theta\tilde{\xi}(0,r,\theta) = \tilde{\xi}(0,r,\theta)C + \Big(\sum_{k=0}^{h-1}\gamma(r,k,0) + \sum_{r=h}^{H}\xi(r,j,0)\Big)\tilde{S}_r(\theta) - \xi(0,r,0), \quad h+1 \leq r \leq H, \quad (4.16)$$

$$-\theta\tilde{\xi}(n,r,\theta) = \tilde{\xi}(n,r,\theta)C + \tilde{\xi}(n-1,r,\theta)D - \xi(n,r,0), \quad n \geq 1, h \leq r \leq H-1, \quad (4.17)$$

$$-\theta\tilde{\xi}(n,H,\theta) = \tilde{\xi}(n,H,\theta)C + \tilde{\xi}(n-1,H,\theta)D + \Big(\sum_{k=0}^{h-1}\gamma(n+H,k,0) + \sum_{r=h}^{H}\xi(n+H,r,0)\Big)\tilde{S}_H(\theta) - \xi(n,H,0), \ n \geq 1, \quad (4.18)$$

$$-\theta\tilde{\gamma}(k,k,\theta) = \tilde{\gamma}(k,k,\theta)C + \Big(\sum_{r=h}^{H}\xi(k,r,0) + \delta\sum_{j=0}^{k}\gamma(k,j,0)\Big)\tilde{V}_k(\theta) - \gamma(k,k,0), \quad 0 \leq k \leq h-1, \quad (4.19)$$

$$-\theta\tilde{\gamma}(n,k,\theta) = \tilde{\gamma}(n,k,\theta)C + \tilde{\gamma}(n-1,k,\theta)D - \gamma(n,k,0) \ \ n \geq k+1, \quad 0 \leq k \leq h-1. \quad (4.20)$$

Now the aim is to perceive the probability vector of the joint probabilities of the queue content, server content (queue content, type of vacation), and the phase of the arrival process at any time. However, direct analysis of these is quite challenging. The arbitrary epoch probabilities determine in terms of service (vacation) completion epoch probabilities after characterizing the system's state at the service (vacation) completion epoch. Towards this end, the following probabilities are defined at service (vacation) completion epoch while the arrival process is in phase i $(1 \leq i \leq m)$.

$$\xi_i^+(n,r) = Pr\{\text{At the service completion epoch of a batch of size } r, \text{ queue size is } n.\}, \quad n \geq 0, \quad h \leq r \leq H, \quad (4.21)$$

$$\begin{aligned}
\xi_i^+(n) &= Pr\{\text{At the service completion epoch, queue size is } n\} \\
&= \sum_{r=h}^{H} \xi_i^+(n,r), \quad n \geq 0, \qquad (4.22)\\
\gamma_i^+(n,k) &= Pr\{\text{At } k^{th} \text{ type of vacation termination epoch,} \\
&\quad \text{queue size is } n\}, \quad 0 \leq k \leq h-1, \quad n \geq k, \qquad (4.23)\\
\gamma_i^+(n) &= Pr\{\text{At vacation termination epoch, queue size is } n \\
&= \sum_{k=0}^{min(n,h-1)} \gamma_i^+(n,k), \quad n \geq 0. \qquad (4.24)
\end{aligned}$$

Consequently, the probability vectors are given as follows
$\xi^+(n,r) = (\xi_1^+(n,r), \xi_2^+(n,r), ..., \xi_m^+(n,r)), \ \ n \geq 0, \ \ h \leq r \leq H,$
$\gamma^+(n,k) = (\gamma_1^+(n,k), \gamma_2^+(n,k), ..., \gamma_m^+(n,k)), \ \ n \geq k, \ \ 0 \leq k \leq h-1,$
$\xi^+(n) = (\xi_1^+(n), \xi_2^+(n), ..., \xi_m^+(n)), \ \ n \geq 0,$
$\gamma^+(n) = (\gamma_1^+(n), \gamma_2^+(n), ..., \gamma_m^+(n)), \ \ n \geq 0.$

Lemma 4.1. *The probability vectors* $\xi^+(n,r)$, $\gamma^+(n,k)$, $\xi(n,r,0)$ *and* $\gamma(n,k,0)$ $(h \leq r \leq H, \ 0 \leq k \leq h-1)$ *are given by,*

$$\begin{aligned}
\xi^+(n,r) &= \sigma\xi(n,r,0), \quad n \geq 0, \qquad (4.25)\\
\gamma^+(n,k) &= \sigma\gamma(n,k,0), \quad n \geq k, \qquad (4.26)
\end{aligned}$$

where $\sigma^{-1} = \sum\limits_{m=0}^{\infty} \sum\limits_{r=h}^{H} \xi(m,r,0)\mathbf{e} + \sum\limits_{m=0}^{\infty} \sum\limits_{k=0}^{min(m,h-1)} \gamma(m,k,0)\mathbf{e}.$

Proof. As a result of the fact that $\xi^+(n,r)$ and $\gamma^+(n,k)$ are proportional to $\xi(n,r,0)$ and $\gamma(n,k,0)$, respectively, applying Bayes' theorem and $\sum\limits_{n=0}^{\infty} \sum\limits_{r=h}^{H} \xi^+(n,r)\mathbf{e} + \sum\limits_{n=0}^{\infty} \sum\limits_{k=0}^{min(n,h-1)} \gamma^+(n,k)\mathbf{e}$ =1 the desired outcome is obtained. □

Lemma 4.2. $R(n,0)D\mathbf{e} = \sum\limits_{m=0}^{n} \sum\limits_{k=0}^{m} \gamma(m,k,0)\mathbf{e}$

Proof. using (4.1) and (4.2) after some simplification the intended outcome is accomplished. □

Lemma 4.3. *The expression for* σ^{-1} *is*

$$\sigma^{-1} = \frac{1-(1-\delta)\sum\limits_{n=0}^{h-1} R(n,0)\mathbf{e}}{s_H \sum\limits_{n=H+1}^{\infty} \big(\xi^+(n)+\gamma^+(n)\big)\mathbf{e} + \sum\limits_{n=h}^{H} \big(\xi^+(n)+\gamma^+(n)\big)\mathbf{e}s_n + \sum\limits_{n=0}^{h-1} \big(\xi^+(n)x_n + (1-\delta)\gamma^+(n)s_h + \delta\gamma^+(n)x_n\big)\mathbf{e}}. \tag{4.27}$$

Proof. Post multiplying (4.15)-(4.20) by column vector $\mathbf{e}$ and summing them, using Lemma 4.2, after some simplification one can obtain

$$\begin{aligned}\sum_{m=0}^{\infty}\Big(\sum_{r=h}^{H}\tilde{\xi}(m,r,\theta) + \sum_{k=0}^{min(m,h-1)}\tilde{\gamma}(m,k,\theta)\Big) &= \frac{1-\tilde{S}_H(\theta)}{\theta}\sum_{n=H+1}^{\infty}\Big(\sum_{r=h}^{H}\xi(n,r,0)+\sum_{k=0}^{h-1}\gamma(n,k,0)\Big)\\ &+\sum_{n=h}^{H}\Big(\sum_{r=h}^{H}\xi(n,r,0)+\sum_{k=0}^{h-1}\gamma(n,k,0)\Big)\frac{1-\tilde{S}_n(\theta)}{\theta}\\ &+\sum_{n=0}^{h-1}\Big(\sum_{r=h}^{H}\xi(n,r,0)+\delta\sum_{k=0}^{n}\gamma(n,k,0)\Big)\frac{1-\tilde{V}_n(\theta)}{\theta}\\ &+(1-\delta)\frac{1-\tilde{S}_h(\theta)}{\theta}\sum_{n=0}^{h-1}\sum_{k=0}^{n}\gamma(n,k,0).\end{aligned} \tag{4.28}$$

Applying $\theta \to 0$ in (4.28) and L'Hôspital's rule, and $(1-\delta)\sum\limits_{n=0}^{h-1} R(n,0)\mathbf{e} + \sum\limits_{n=0}^{\infty}\sum\limits_{r=h}^{H}\xi(n,r)\mathbf{e} + \sum\limits_{n=0}^{\infty}\sum\limits_{k=0}^{min(n,h-1)}\gamma(n,k)\mathbf{e}=1$, after few simplification the desired outcome is obtained. □

Further, define a few necessary generating functions, which are as follows:

$$\tilde{\Pi}_i(z,y,\theta) = \sum_{n=0}^{\infty}\sum_{r=h}^{H}\tilde{\xi}(n,r,\theta)z^n y^r, \tag{4.29}$$

$$\Pi_i^+(z,y) = \sum_{n=0}^{\infty}\sum_{r=h}^{H}\xi_i^+(n,r)z^n y^r, \tag{4.30}$$

$$\Psi_i^+(z) = \sum_{n=0}^{\infty}\sum_{r=h}^{H}\xi_i^+(n,r)z^n = \sum_{n=0}^{\infty}\xi_i^+(n)z^n = \Pi_i^+(z,1), \tag{4.31}$$

$$\tilde{O}_i(z,y,\theta) = \sum_{k=0}^{h-1}\sum_{n=k}^{\infty}\tilde{\gamma}_i(n,k,\theta)z^n y^k, \tag{4.32}$$

$$O_i^+(z,y) = \sum_{k=0}^{h-1}\sum_{n=k}^{\infty}\gamma_i^+(n,k)z^n y^k, \tag{4.33}$$

$$O_i^+(z) = \sum_{k=0}^{h-1}\sum_{n=k}^{\infty}\gamma_i^+(n,k)z^n = \sum_{n=0}^{\infty}\sum_{k=0}^{min(n,h-1)}\gamma_i^+(n,k)z^n = \sum_{n=0}^{\infty}\gamma_i^+(n)z^n, \tag{4.34}$$

where, $|z| \leq 1$ and $|y| \leq 1$. Hence,
$\tilde{\Pi}(z,y,\theta) = (\tilde{\Pi}_1(z,y,\theta), \tilde{\Pi}_2(z,y,\theta), ..., \tilde{\Pi}_m(z,y,\theta))$
$\Pi^+(z,y) = (\Pi_1^+(z,y), \Pi_2^+(z,y), ..., \Pi_m^+(z,y))$
$\Psi^+(z) = (\Psi_1^+(z), \Psi_2^+(z), ..., \Psi_m^+(z))$
$\tilde{O}(z,y,\theta) = (\tilde{O}_1(z,y,\theta), \tilde{O}_2(z,y,\theta), ..., \tilde{O}_m(z,y,\theta)$
$O^+(z,y) = (O_1^+(z,y), O_2^+(z,y), ..., O_m^+(z,y))$
$O^+(z) = (O_1^+(z), O_2^+(z), ..., O_m^+(z))$

Lemma 4.4.

$$O^+(z) = \sum_{n=0}^{\infty}\gamma^+(n)z^n = \sum_{k=0}^{h-1}(\xi^+(k) + \delta\gamma^+(k))B^{(k)}(z)z^k \tag{4.35}$$

Proof. Equations (4.19) and (4.20) are multiplied by the appropriate powers of z and y and added throughout the range of n and k, hence, the following expression is obtained

$$\begin{aligned}\tilde{O}(z,y,\theta)(-\theta I - (C+Dz)) &= \sum_{k=0}^{h-1}\left(\sum_{r=h}^{H}\xi(k,r,0) + \delta\sum_{j=0}^{k}\gamma(j,k,0)\right)\tilde{V}_k(\theta)z^k y^k \\ &\quad -\sum_{k=0}^{h-1}\sum_{n=k}^{\infty}\gamma(n,k,0)z^n y^k.\end{aligned} \tag{4.36}$$

If the eigenvalues of $-(C+Dz)$ are $\alpha_1(z), \alpha_2(z), .., \alpha_m(z)$ and $\epsilon_1(z), \epsilon_2(z), ..., \epsilon_m(z)$ be the corresponding eigenvectors, then

$$-(C+Dz)\epsilon_i(z) = \alpha_i(z)\epsilon_i(z), 1 \leq i \leq m. \tag{4.37}$$

Now substituting $\theta = \alpha_i(z)$ in (4.36) and post multiplying by $\epsilon_i(z)$, using Lemma 4.1 and Lemma 4.3 one can get

$$\sum_{k=0}^{h-1}\sum_{n=k}^{\infty}\gamma^+(n,k)z^n y^k\epsilon_i(z) = \sum_{k=0}^{h-1}(\xi^+(k) + \delta\gamma^+(k))\tilde{V}_k(\alpha_i(z))\epsilon_i(z)z^k y^k, \tag{4.38}$$

Equation (4.38) is true for all $\alpha_i(z)$, $1 \leq i \leq m$, hence

$$\sum_{k=0}^{h-1}\sum_{n=k}^{\infty}\gamma^+(n,k)z^ny^k = \sum_{k=0}^{h-1}(\xi^+(k)+\delta\gamma^+(k))\Delta(z)diag\{\tilde{V}_k(\alpha_i(z))\}_{i=1}^m(\Delta(z))^{-1}z^ky^k. \tag{4.39}$$

where $\Delta(z) = (\epsilon_1(z), \epsilon_2(z), ..., \epsilon_m(z))$ and $diag\{\tilde{V}_k(\alpha_i(z))\}_{i=1}^m$ is a diagonal matrix whose (i,i) entry is $\tilde{V}_k(\alpha_i(z))$, $i = 1, 2, ..., m$. Further, define

$(B_l^{(k)}(x))_{i,j}$ = $Pr\{$Given a departure at time 0 which left k costumer in the queue, k^{th} type of vacation begins and the arrival process is in phase i at the end of the k^{th} type of vacation occurs no later than time x, with the arrival process is in phase j, and during the k^{th} vacation type l customers arrive$\}, 0 \leq k \leq h-1$.

Let $B^{(k)}(z)$ be the probability generating function of $B_l^{(k)} = (B_l^{(k)}(x))_{i,j}$, and hence, $B^{(k)}(z) = \sum_{l=0}^{\infty} B_l^{(k)} z^l = \int_0^{\infty} e^{-(C+Dz)t}v_k(t)dt = \Delta(z)diag\{\tilde{V}_k(\alpha_i(z))\}_{i=1}^m(\Delta(z))^{-1}$, $0 \leq k \leq h-1$.

Hence, equation (4.39) expresses as,

$$\sum_{k=0}^{h-1}\sum_{n=k}^{\infty}\gamma^+(n,k)z^ny^k = \sum_{k=0}^{h-1}(\xi^+(k)+\delta\gamma^+(k))B^{(k)}(z)z^ky^k. \tag{4.40}$$

Substituting $y = 1$ in (4.40) the desired result (4.35) is obtained. □

Multiplying (4.15)-(4.18) by the appropriate powers of z and y and adding the results over the range of n and r, the following expression is obtained.

$$\begin{aligned}\tilde{\Pi}(z,y,\theta)(-\theta I-(C+Dz)) &= (1-\delta)\sum_{n=0}^{h-1}\sum_{k=0}^{n}\gamma(n,k,0)\tilde{D}^{h-n}\tilde{S}_h(\theta)y^h \\ &+\sum_{r=h}^{H}\left(\sum_{k=0}^{h-1}\gamma(r,k,0)+\sum_{r=h}^{H}\xi(r,j,0)\right)\tilde{S}_r(\theta)y^r \\ &+\sum_{n=H+1}^{\infty}\left(\sum_{k=0}^{h-1}\gamma(n,k,0)+\sum_{r=h}^{H}\xi(n,r,0)\right)\tilde{S}_H(\theta)z^{n-H}y^H \\ &-\sum_{n=0}^{\infty}\sum_{r=h}^{H}\xi(n,r,0)z^n y^r,\end{aligned} \tag{4.41}$$

where $\tilde{D}=(-C)^{-1}D$. Now substituting $\theta=\alpha_i(z)$ in (4.41) and post multiplying by $\epsilon_i(z)$, using Lemma 4.1 and Lemma 4.3 the following expression is obtained

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{r=h}^{H}\xi^+(n,r)z^n y^r\epsilon_i(z) &= (1-\delta)\sum_{n=0}^{h-1}\gamma^+(n)\tilde{D}^{h-n}\tilde{S}_h(\alpha_i(z))\epsilon_i(z)y^h \\ &+\sum_{r=h}^{H}\left(\gamma^+(r)+\xi^+(r)\right)\tilde{S}_r(\alpha_i(z))\epsilon_i(z)y^r \\ &+\sum_{n=H+1}^{\infty}\left(\gamma^+(n)+\xi^+(n)\right)\tilde{S}_H(\alpha_i(z))\epsilon_i(z)z^{n-H}y^H.\end{aligned} \tag{4.42}$$

Equation (4.42) is true for all $\alpha_i(z)$, $1\le i\le m$, hence

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{r=h}^{H}\xi^+(n,r)z^n y^r &= (1-\delta)\sum_{n=0}^{h-1}\gamma^+(n)\tilde{D}^{h-n}\Delta(z)diag\{\tilde{S}_h(\alpha_i(z))\}_{i=1}^m(\Delta(z))^{-1}y^h \\ &+\sum_{r=h}^{H}\left(\gamma^+(r)+\xi^+(r)\right)\Delta(z)diag\{\tilde{S}_r(\alpha_i(z))\}_{i=1}^m(\Delta(z))^{-1}y^r \\ &+\sum_{n=H+1}^{\infty}\left(\gamma^+(n)+\xi^+(n)\right)\Delta(z)diag\{\tilde{S}_H(\alpha_i(z))\}_{i=1}^m(\Delta(z))^{-1}z^{n-H}y^H.\end{aligned} \tag{4.43}$$

Further, define

$$
\begin{aligned}
(A_l^{(r)}(x))_{i,j} \;=\; & Pr\{\text{Given a departure at time 0 which left } r \\
& (h \le r \le H) \text{ costumer in the queue and the arrival process,} \\
& \text{is in phase } i, \text{ next departure occurs no later than} \\
& \text{time } x \text{ with the arrival process is in phase } j, \text{ and during the} \\
& \text{service time of } r \text{ customers } l \text{ customers arrive}\}.
\end{aligned}
$$

Let $A^{(r)}(z)$ be the probability generating function of $A_l^{(r)} = (A_l^{(r)}(x))_{i,j}$, and hence, $A^{(r)}(z) = \sum_{l=0}^{\infty} A_l^{(r)} z^l = \int_0^{\infty} e^{-(C+Dz)t} s_r(t)dt = \Delta(z) diag\{\tilde{S}_r(\alpha_i(z))\}_{i=1}^{m} (\Delta(z))^{-1}$, $h \le r \le H$.

Substituting $y = 1$ in (4.42) and using Lemma 4.4 and equation (4.31) the following result is obtained

$$
\Psi^+(z) = \frac{\begin{aligned}&\Big\{\sum_{n=0}^{h-1} \big[(\xi^+(n) + \delta\gamma^+(n))\big(B^{(n)}(z) - I\big)A^{(H)}(z)z^n \\ &+(1-\delta)\gamma^+(n)\big(\tilde{D}^{h-n}A^{(h)}(z)z^H - A^{(H)}(z)z^n\big)\big] \\ &+ \sum_{n=h}^{H-1}(\gamma^+(n) + \xi^+(n))\big(A^{(n)}(z)z^H - A^{(H)}(z)z^n\big)\Big\}\end{aligned}}{z^H I - A^{(H)}(z)}. \tag{4.44}
$$

Now using (4.44) in (4.42) after algebraic manipulation the following expression is obtained,

$$
\Pi^+(z,y) = \frac{\begin{aligned}&\sum_{n=0}^{h-1} \big[(1-\delta)\gamma^+(n)\big(\tilde{D}^{h-n}z^H A^{(h)}(z)y^h - A^{(H)}(z)z^n y^H\big) \\ &+(1-\delta)\gamma^+(n)\tilde{D}^{h-n}A^{(h)}(z)A^{(H)}(z)\big(y^H - y^h\big) \\ &+y^H(\xi^+(n) + \delta\gamma^+(n))\big(B^{(n)}(z) - I\big)A^{(H)}(z)z^n\big] \\ &+ \sum_{n=h}^{H-1}\big(\gamma^+(n) + \xi^+(n)\big)\big(z^H y^n A^{(n)}(z) \\ &+\big(y^H - y^n\big)A^{(n)}(z)A^{(H)}(z) - y^H A^{(H)}(z)z^n\big)\end{aligned}}{z^H I - A^{(H)}(z)}. \tag{4.45}
$$

The above bivariate vector generating function given in (4.45) contains H unknown vectors $\{\xi^+(n)\}_{n=0}^{H-1}$, i.e., total mH unknowns $\{\xi_i^+(n)\}_{n=0}^{H-1}$, $1 \le i \le m$ which has to be determined first. From (4.45), the bivariate generating function $\Pi^+(z,y)$ has been represented in compact form, excluding the H unknowns $\{\xi^+(n)\}_{n=0}^{H-1}$. Additionally, if $\xi^+(k)$ $(0 \le k \le$

$h-1$) are known then from Lemma 4.4 the probability vectors $\gamma^+(n,k)$ $(0 \leq k \leq h-1)$ are known. As a result, in order to determine all of the probability vectors at the service (vacation) completion epoch, it is necessary to identify the unknowns $\{\xi^+(n)\}_{n=0}^{H-1}$.

4.3.1 Procedure of obtaining the unknowns $\xi^+(n)$ $(0 \leq n \leq H-1)$

Consider $\tilde{S}_r(\theta)$ and $\tilde{V}_k(\theta)$ both as rational function. Then each element of $A^{(r)}(z)$ $(h \leq r \leq H)$ and $B^{(k)}(z)$ $(0 \leq k \leq h-1)$ are rational functions having same denominator say $d^{(r)}(z)$ $(h \leq r \leq H)$ and $d^{(k)}(z)$ $(0 \leq k \leq h-1)$, respectively. Assign the (i,j)-th element of $A^{(r)}(z)$ say $\frac{f_{i,j}^{(r)}(z)}{d^{(r)}(z)}$, $1 \leq i,j \leq m$, and the (i,j)-th element of $B^{(k)}(z)$ say $\frac{f_{i,j}^{(k)}(z)}{d^{(k)}(z)}$, $1 \leq i,j \leq m$. Consequently, the (i,j)-th element of $z^H I - A^{(H)}(z)$ is

$$(z^H I - A^{(H)}(z))_{i,j} = \frac{\nu_{i,j}(z)}{d^{(H)}(z)}, \tag{4.46}$$

where

$$\nu_{i,j}(z) = \begin{cases} z^H d^{(H)}(z) - f_{i,j}^{(H)}(z), i = j, \\ -f_{i,j}^{(H)}(z), i \neq j. \end{cases}$$

Hence, from (4.44) m system of equations are obtained in the matrix form

$$\Psi^+(z) M(z) = \Omega(z), \tag{4.47}$$

where (i,j)-th entry of the matrix $M(z)$ is $\nu_{i,j}(z)$, and $\Omega(z) = (\Omega_1(z), \Omega_2(z), ..., \Omega_m(z))^T$ is an $m \times 1$ column matrix such that

$$\Omega_j(z) = \frac{\begin{aligned} &(1-\delta)\Big(\prod_{r=0}^{h-1} d^{(r)}(z)\Big)\Big(\prod_{r=h+1}^{H-1} d^{(r)}(z)\Big) \sum_{n=0}^{h-1}\sum_{i=1}^{m} \gamma_i^+(n)\Big(z^H \sum_{l=1}^{m} \tilde{d}_{i,l}^{(h-n)} f_{l,j}^{(h)}(z) d^{(H)}(z) \\ &\quad - z^n d^{(h)}(z) f_{i,j}^{(H)}(z)\Big) + \prod_{r=h}^{H-1} d^{(r)}(z) \sum_{n=0}^{h-1}\sum_{i=1}^{m}(\xi_i^+(n) \\ &\quad + \delta\gamma_i^+(n)) \sum_{l=1}^{m} u_{i,l}^{(n)}(z) f_{l,j}^{(H)}(z) z^n \prod_{r=0, r\neq n}^{h-1} d^{(r)}(z) \\ &\quad + \sum_{r=0}^{h-1} d^{(r)}(z) \sum_{n=h}^{H-1}\sum_{i=1}^{m}(\xi_i^+(n) + \gamma_i^+(n))\Big(z^H d^{(H)}(z) f_{i,j}^{(n)}(z) \\ &\quad - z^n f_{i,j}^{(H)}(z) d^{(n)}(z)\Big) \prod_{r=h, r\neq n}^{H-1} d^{(r)}(z) \end{aligned}}{\prod_{r=0}^{H-1} d^{(r)}(z)}, 1 \leq j \leq m, \tag{4.48}$$

where $u_{i,j}^{(k)}(z) = f_{i,j}^{(k)}(z) - d^{(k)}(z)$, $i \neq j$, $0 \leq k \leq h-1$ and $u_{i,j}^{(k)}(z) = f_{i,j}^{(k)}(z)$, $i = j$, $0 \leq k \leq h-1$. $\tilde{d}_{k,l}^{(h-n)}$ is the (k,l)-th element of $\tilde{D}^{h-n}$. To solve the system of equations given in (4.47), Cramer's rule is applied and the following result is obtained as follows.

$$\Psi_j^+(z) = \frac{|M_j(z)|}{|M(z)|}, \quad 1 \leq j \leq m, \tag{4.49}$$

$$[M_j(z)]_{k,l} = \begin{cases} \Omega_k(z), j = l, \\ \nu_{l,k}(z), j \neq l. \end{cases}$$

Suppose that $|M(z)|$ is a non-zero polynomial in variable z must posses a nonzero coefficient of the power of z. It is clear to observe that $|z^H I - A^{(H)}(z)| = \frac{|M(z)|}{(d^{(H)}(z))^m}$ has precisely mH zeros in $\{z : |z| \leq 1\}$ say $p_1, p_2, ..., p_l$ with multiplicity $q_1, q_2, ..., q_l$, respectively, $\big($where $(l \leq mH-1)$ and $\sum_{i=1}^{l} q_i = (mH-1)\big)$ and $p_H = 1$ is a simple zero. Since, $\Psi_j^+(z)$ is analytic in $|z| \leq 1$, therefore, these zeros are also the zeros of the numerator of $\Psi_j^+(z)$. Hence, taking one component of $\Psi^+(z)$, say $\Psi_j^+(z)$ $(1 \leq j \leq m)$, $mH-1$ equations are obtained as follows

$$\left[\frac{d^{i-1}}{dz^{i-1}}|M_j(z)|\right]_{z=p_x} = 0, \quad 1 \leq x \leq l \;\; \& \;\; 1 \leq i \leq q_j, \tag{4.50}$$

where $\frac{d^0}{dz^0}h(z) = h(z)$.

One more equation is obtained by the normalization condition $\Psi^+(1)\mathbf{e} + O^+(1)\mathbf{e} = 1$, i.e,

$$\sum_{j=1}^{m}\left[\frac{d}{dz}|M_j(z)|\right]_{z=1} + \left[\frac{d}{dz}|M(z)|\right]_{z=1}\sum_{k=0}^{h-1}(\xi^+(k) + \delta\gamma^+(k))e = \left[\frac{d}{dz}|M(z)|\right]_{z=1}. \tag{4.51}$$

Solving (4.50) and (4.51) together mH unknowns $\xi_j^+(n)$ $(1 \leq j \leq m, 0 \leq n \leq H-1)$ are obtained.

Theorem 4.5. *The probability vectors of the joint probability of queue and server content are given by*

$$\xi^+(n,h) = \left((1-\delta)\sum_{m=0}^{h-1}\gamma^+(m)\tilde{D}^{h-m} + \gamma^+(h) + \xi^+(h)\right)A_n^{(h)}, \tag{4.52}$$

$$\xi^+(n,r) = \left(\gamma^+(r) + \xi^+(r)\right)A_n^{(r)}, \quad h+1 \leq r \leq H-1. \tag{4.53}$$

Proof. Using (4.30) in (4.45), then collecting the coefficients of y^r $(h \le r \le H-1)$, one can get

$$\text{coefficient of } y^h : \sum_{n=0}^{\infty} \xi^+(n,h)z^n = \Big((1-\delta)\sum_{m=0}^{h-1}\gamma^+(m)\tilde{D}^{h-m} + \gamma^+(h) + \xi^+(h)\Big)A^{(h)}(z), \tag{4.54}$$

$$\text{coefficient of } y^r : \sum_{n=0}^{\infty} \xi^+(n,r)z^n = \Big(\gamma^+(r) + \xi^+(r)\Big)A^{(r)}(z), h+1 \le r \le H-1. \tag{4.55}$$

Accumulating the coefficients of z^n, from both side of (4.54) and (4.55), the desired results (4.52) and (4.53) are obtained. □

Now the current objective is to collect the remaining probability vectors $\xi^+(n,H)$ $(n \ge 0)$. Towards this end, using (4.30) in (4.45) and then collecting the coefficients of y^H one can get

$$\sum_{n=0}^{\infty}\xi^+(n,H)z^n = \frac{A^{(H)}(z)\Big\{\sum_{n=0}^{h-1}\Big[(\xi^+(n)+\delta\gamma^+(n))(B^{(n)}(z)-I)z^n + (1-\delta)\gamma^+(n)(\tilde{D}^{h-n}A^{(h)}(z)-z^nI)\Big] + \sum_{n=h}^{H-1}(\gamma^+(n)+\xi^+(n))(A^{(n)}(z)-z^nI)\Big\}}{z^HI - A^{(H)}(z)}. \tag{4.56}$$

Assign a symbol $\sum_{n=0}^{\infty}\xi^+(n,H)z^n$ as $\pounds^+(z) = (\pounds_1^+(z), \pounds_2^+(z), ..., \pounds_m^+(z))$, and replacing $\Psi^+(z)$ and $\Omega_j(z)$ by $\pounds^+(z)$ and $\Theta_j(z)$, respectively, where $\Theta_j(z)$, $(1 \le j \le m)$ is given by

$$\Theta_j(z) = \frac{\begin{aligned}&(1-\delta)\Big(\prod_{r=0}^{h-1}d^{(r)}(z)\Big)\Big(\prod_{r=h+1}^{H-1}d^{(r)}(z)\Big)\sum_{n=0}^{h-1}\sum_{i=1}^{m}\gamma_i^+(n)\Big(\sum_{w=1}^{m}\sum_{l=1}^{m}\tilde{d}_{i,l}^{(h-n)}f_{l,w}^{(h)}(z)f_{w,j}^{(H)}(z) \\ &-z^nd^{(h)}(z)f_{i,j}^{(H)}(z)\Big) + \prod_{r=h}^{H-1}d^{(r)}(z)\sum_{n=0}^{h-1}\sum_{i=1}^{m}(\xi_i^+(n) \\ &+\delta\gamma_i^+(n))\sum_{l=1}^{m}u_{i,l}^{(n)}(z)f_{l,j}^{(H)}(z)z^n\prod_{r=0,r\ne n}^{h-1}d^{(r)}(z) \\ &+\prod_{r=0}^{h-1}d^{(r)}(z)\sum_{n=h}^{H-1}\sum_{i=1}^{m}(\xi_i^+(n)+\gamma_i^+(n))\sum_{w=1}^{m}\Big(f_{i,w}^{(n)}(z)f_{w,j}^{(H)}(z) \\ &-z^nf_{i,j}^{(H)}(z)d^{(n)}(z)\Big)\prod_{r=h,r\ne n}^{H-1}d^{(r)}(z)\end{aligned}}{\prod_{r=0}^{H-1}d^{(r)}(z)}, 1 \le j \le m. \tag{4.57}$$

Then $\pounds_j^+(z)$ is expressed as

$$\pounds_j^+(z) = \frac{|N_j(z)|}{|M(z)|}, \quad 1 \le j \le m, \tag{4.58}$$

where,

$$[N_j(z)]_{k,l} = \begin{cases} \Theta_k(z), j = l, \\ \nu_{l,k}(z), j \ne l. \end{cases}$$

It is assumed that degree of $|N_j(z)|$ is $\hat{d}_j$ and degree of $|M(z)|$ is $\hat{d}$.

The zeros of $|N_j(z)|$ of modules with more than one must be known in order to derive the probability vectors $\xi^+(n, H)$ $(n \ge 0)$.

Since (4.56) is analytic, in $|z| \le 1$, the roots of $|M(z)|$ lying in $|z| \le 1$ are also the roots of $|N_j(z)|$, hence, the roots lying in $|z| \le 1$ can not be used to calculate $\xi^+(n, H)$ $(n \ge 0)$. Assume that $\beta_1, \beta_2, ..., \beta_l$ are the zeros of $|M(z)|$ of modules greater than one having multiplicity $\eta_1, \eta_2, ..., \eta_l$, respectively, and $\sum\limits_{j=1}^{l} \eta_j < \hat{d}$. Here, two cases arise

Case A: $\hat{d} \le \hat{d}_j$

Now applying the partial fraction method on (4.58), $\pounds_j^+(z)$ can be written as,

$$\pounds_j^+(z) = \sum_{i=0}^{\hat{d}_j - \hat{d}} \varrho_{i,j} z^i + \sum_{w=1}^{l} \sum_{i=1}^{\eta_w} \frac{B_{i,w,j}}{(z - \beta_w)^{\eta_w - i + 1}}, \tag{4.59}$$

where

$$B_{i,w,j} = \frac{1}{(i-1)!} \left[\frac{d^{i-1}}{dz^{i-1}} \left(\frac{|N_j(z)| \frac{d^{\eta_w}}{dz^{\eta_w}} (z - \beta_w)^{\eta_w}}{\frac{d^{\eta_w}}{dz^{\eta_w}} |M(z)|} \right) \right]_{z=\beta_w}, w = 1, 2, ..., l, \; i = 1, 2, ..., \eta_w,$$
$$j = 1, 2, ..., m.$$

Collecting the coefficients of z^n $(n \ge 0)$ from both side of (4.59) for $(1 \le j \le m)$ one can obtain

$$\xi_j^+(n, H) = \begin{cases} \varrho_{n,j} + \sum\limits_{w=1}^{l} \sum\limits_{i=1}^{\eta_w} \frac{B_{i,w,j}}{(-1)^{\eta_w - i + 1} \beta_w^{\eta_w + n - i + 1}} \binom{\eta_w - i + n}{\eta_w - i}, & 0 \le n \le \hat{d}_j - \hat{d}, \\ \sum\limits_{w=1}^{l} \sum\limits_{i=1}^{\eta_w} \frac{B_{i,w,j}}{(-1)^{\eta_w - i + 1} \beta_w^{\eta_w + n - i + 1}} \binom{\eta_w - i + n}{\eta_w - i}, & n > \hat{d}_j - \hat{d}. \end{cases}$$

Case B: $\hat{d} > \hat{d}_j$

Removing the first summation term from the right hand side of (4.59) one can have

$$\xi_j^+(n,H) = \sum_{w=1}^{l}\sum_{i=1}^{\eta_w} \frac{B_{i,w,j}}{(-1)^{\eta_w-i+1}\beta_w^{\eta_w+n-i+1}}\binom{\eta_w-i+n}{\eta_w-i}, \quad n \geq 0. \tag{4.60}$$

Theorem 4.6. *Arbitrary epoch probability vectors are given by,*

$$R(n,0) = \frac{\sum_{m=0}^{n}\gamma^+(m)\tilde{D}^{n-m}(-C)^{-1}}{E}, \quad 0 \leq n \leq h-1 \ \ (\text{exist only for } SV) \tag{4.61}$$

$$\xi(0,h) = (1-\delta)R(h-1,0)D + \left(\frac{\xi^+(h)+\gamma^+(h)-\xi^+(0,h)}{E}\right)(-C)^{-1}, \quad n \geq 0, \tag{4.62}$$

$$\xi(0,r) = \left(\frac{\xi^+(r)+\gamma^+(r)-\xi^+(0,r)}{E}\right)(-C)^{-1}, \quad n \geq 0, \ h+1 \leq r \leq H-1, \tag{4.63}$$

$$\xi(n,r) = \left(\xi(n-1,r)D - \frac{\xi^+(n,r)}{E}\right)(-C)^{-1}, \quad n \geq 1, \tag{4.64}$$

$$\xi(n,H) = \left(\xi(n-1,H)D + \frac{\xi^+(n+H)+\gamma^+(n+H)-\xi^+(n,H)}{E}\right)(-C)^{-1}, \ n \geq 0, \tag{4.65}$$

$$\gamma(k,k) = \left(\frac{\xi^+(k)+\delta\gamma^+(k)-\gamma^+(k,k)}{E}\right)(-C)^{-1}, \ 0 \leq k \leq h-1, \tag{4.66}$$

$$\gamma(n,k) = \left(\gamma(n-1,k) - \frac{\gamma^+(n,k)}{E}\right)(-C)^{-1}, \quad n \geq k+1, \ 0 \leq k \leq h-1, \tag{4.67}$$

where $E = \hat{w} + (1-\delta)\sum_{n=0}^{h-1}\sum_{m=0}^{n}\gamma^+(m)\tilde{D}^{(n-m)}(-C)^{-1}\mathbf{e}$,
$\hat{w} = s_H\sum_{n=H+1}^{\infty}\big(\xi^+(n)+\gamma^+(n)\big)\mathbf{e}+\sum_{n=h}^{H}\big(\xi^+(n)+\gamma^+(n)\big)\mathbf{e}s_n+\sum_{n=0}^{h-1}\big(\xi^+(n)\mathbf{e}x_n+(1-\delta)\gamma^+(n)\mathbf{e}s_h+\delta\gamma^+(n)\mathbf{e}x_n\big)$.

Proof. Dividing equation (4.1) and (4.2) by σ^{-1} after simple algebraic manipulation, equation (4.61) is obtained. Further, taking $\theta \to 0$ in (4.15)-(4.20) and then diving by σ^{-1} after simple algebraic manipulation desired outcome (4.62)-(4.67) is obtained. □

4.4 Marginal Probabilities

Some marginal probabilities are given as follows:

1. Queue length distribution is given by

$$P_n^{queue} = \begin{cases} (1-\delta)R(n,0)\mathbf{e} + \sum\limits_{r=h}^{H} \xi(n,r)\mathbf{e} + \sum\limits_{k=0}^{min(n,h-1)} \gamma(n,k)\mathbf{e}, \ 0 \le n \le h-1, \\ \sum\limits_{r=h}^{H} \xi(n,r)\mathbf{e} + \sum\limits_{k=0}^{min(n,h-1)} \gamma(n,k)\mathbf{e}, \ n \ge h. \end{cases}$$

2. The probability that the server is in dormant state (P^{dor})=$\sum\limits_{n=0}^{h-1} R(n,0)\mathbf{e}$.

3. Probability that r customers are with the server (P_r^{ser})=$\sum\limits_{n=0}^{\infty} \xi(n,r)\mathbf{e}$, $h \le r \le H$.

4. Probability that server is in k^{th} type of vacation ($Q_{vac}^{[k]}$)=$\sum\limits_{n=k}^{\infty} \gamma(n,k)\mathbf{e}$, $0 \le k \le h-1$.

5. The probability that the server is busy (P_{busy})=$\sum\limits_{r=h}^{H} \sum\limits_{n=0}^{\infty} \xi(n,r)\mathbf{e}$.

6. The probability that the server is on vacation (Q_{vac})=$\sum\limits_{k=0}^{h-1} \sum\limits_{n=k}^{\infty} \gamma(n,k)\mathbf{e}$.

4.5 Performance measure

Performance measures are presented for observing the system performance. It helps the system manager for observe the system behavior so that he can modify the system for more efficient result. In this section, a significant performance measures are obtained which are as follows.

1. The expected number in the queue (L_q) $= (1-\delta)\sum\limits_{n=0}^{h-1} nR(n,0)\mathbf{e} + \sum\limits_{n=0}^{\infty}\sum\limits_{r=h}^{H} n\xi(n,r)\mathbf{e} + \sum\limits_{k=0}^{h-1}\sum\limits_{n=k}^{\infty} n\gamma(n,k)\mathbf{e}$=$(1-\delta)\sum\limits_{n=0}^{h-1} nP_n^{queue} + \sum\limits_{n=h-\delta h}^{\infty} nP_n^{queue}$.

2. The expected number in the system (L_s) $= (1-\delta)\sum\limits_{n=0}^{h-1} nR(n,0)\mathbf{e} + \sum\limits_{n=0}^{\infty}\sum\limits_{r=h}^{H}(n+r)\xi(n,r)\mathbf{e} + \sum\limits_{k=0}^{h-1}\sum\limits_{n=k}^{\infty} n\gamma(n,k)\mathbf{e}$.

3. The expected waiting time of a customer in the queue (W_q) $=\frac{L_q}{\lambda}$.

4. The expected waiting time of a customer in the system (W_s) $=\frac{L_s}{\lambda}$.

5. Expected number with the server when server is busy (L^{ser}) $=\sum\limits_{r=h}^{H}(rP_r^{ser}/P_{busy})$.

6. Expected type of vacation when server is on vacation $(L^{vac}) = \sum_{k=0}^{h-1}(kQ_{vac}^{[k]}/Q_{vac})$.

4.6 Numerical results

The main objective for presenting this section is to validate the mathematical results, derived in the previous section, with some numerical results. These results are displayed in the tabular and graphical form, considering the service (vacation) time distribution as phase (PH) - type, which is usually represented as (α, T), where α is a row vector of order $1 \times n$, and T is a square matrix of order n. The joint probabilities with predefined notations are presented for $MAP/G_r^{(5,9)}/1$ queue with queue size dependent SV (MV) in the tabular form in Table 4.1 - Table 4.8. The input parameters are given below.

The MAP is represented by the matrices $C = \begin{pmatrix} -91.8125 & 14.1250 \\ 49.4375 & -77.6875 \end{pmatrix}$ and $D = \begin{pmatrix} 49.4375 & 28.2500 \\ 7.0625 & 21.1875 \end{pmatrix}$. The service time of each batch under service follow the Erlang (E_3) distribution having PH-type representation (α_r, T_r), where $T_r = \begin{pmatrix} -\mu_r & \mu_r & 0.0 \\ 0.0 & -\mu_r & \mu_r \\ 0.0 & 0.0 & -\mu_r \end{pmatrix}$, $\mu_r = \frac{r(5.2)}{2}$, $\alpha_r = \begin{pmatrix} 1.0 & 0.0 & 0.0, \end{pmatrix}$, $5 \leq r \leq 9$. The vacation time of the server follows E_2 distribution, having PH-type representation (α_k, T_k), $T_k = \begin{pmatrix} -\nu_k & \nu_k \\ 0.0 & -\nu_k \end{pmatrix}$, $\nu_k = (k + 1.0)^2 0.5$, $0 \leq k \leq 4$. $\alpha_k = \begin{pmatrix} 1.0 & 0.0 \end{pmatrix}$, $0 \leq k \leq 4$. ξ=[0.57143, 0.42857], $\lambda = 56.50$.

Table 4.1: Joint probabilities (queue size and server content) at service completion epoch for SV

n	$\xi_1^+(n,5)$	$\xi_2^+(n,5)$	$\xi_1^+(n,6)$	$\xi_2^+(n,6)$	$\xi_1^+(n,7)$	$\xi_2^+(n,7)$	$\xi_1^+(n,8)$	$\xi_2^+(n,8)$	$\xi_1^+(n,9)$	$\xi_2^+(n,9)$
0	0.00243	0.00173	0.00227	0.00162	0.00256	0.00183	0.00274	0.00196	0.00283	0.00203
1	0.00430	0.00317	0.00371	0.00275	0.00390	0.00289	0.00389	0.00289	0.00626	0.00459
2	0.00508	0.00379	0.00405	0.00304	0.00395	0.00297	0.00368	0.00278	0.00888	0.00659
3	0.00500	0.00376	0.00369	0.00278	0.00334	0.00253	0.00291	0.00221	0.01035	0.00773
4	0.00443	0.00334	0.00302	0.00229	0.00254	0.00193	0.00207	0.00158	0.01086	0.00814
5	0.00366	0.00277	0.00231	0.00175	0.00181	0.00138	0.00137	0.00105	0.01074	0.00807
6	0.00288	0.00219	0.00168	0.00128	0.00122	0.00093	0.00087	0.00066	0.01027	0.00772
7	0.00219	0.00166	0.00118	0.00090	0.00080	0.00061	0.00053	0.00040	0.00964	0.00725
8	0.00162	0.00123	0.00080	0.00061	0.00050	0.00039	0.00031	0.00024	0.00897	0.00675
9	0.00117	0.00089	0.00054	0.00041	0.00031	0.00024	0.00018	0.00014	0.00832	0.00626
10	0.00083	0.00063	0.00035	0.00027	0.00019	0.00015	0.00010	0.00008	0.00773	0.00581
31	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00294	0.00220
32	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00285	0.00214
152	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00046	0.00035
153	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00046	0.00034
302	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00006	0.00004
303	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00006	0.00004
372	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00002	0.00001
373	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00002	0.00001
442	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00001
443	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00000
444	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00000
459	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00000
460	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00000
461	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00000
≥ 462	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
Total	0.0353	0.0265	0.0242	0.0181	0.021	0.0160	0.0188	0.0141	0.363	0.272

After tabular representation the behavior of the considered model is presented in graphical form which represents the comparison of the queue size dependent vacation (QSDV) and queue size independent vacation (QSIV) policy. The following two cases are considered for the comparison scenario.
Case 1. The QSDV rates are considered as $\nu_k = (k+1)^2(1.1),\ \ 0 \le k \le 2$.
Case 2. The QSIV rates are considered as $\nu_k = \nu_0,\ \ 0 \le k \le 2$.

The vacation time decreases with increasing queue size at vacation initiation epoch for case 1, however, for Case 2, the vacation time remains constant irrespective of queue size, at vacation initiation epoch. The other input parameters for Figure 4.1 to Figure 4.2 are taken as follows.

- The vacation time of the server follows E_2 distribution, having PH-type representation (α_k, T_k), T_k= $\begin{pmatrix} -\nu_k & \nu_k \\ 0.0 & -\nu_k \end{pmatrix}$, α_k= $\begin{pmatrix} 1.0 & 0.0 \end{pmatrix}$, $0 \le k \le 2$.
- The service time of each batch under service follow the Erlang (E_3) distribution having PH-type representation (α_r, T_r), where T_r= $\begin{pmatrix} -\mu_r & \mu_r & 0.0 \\ 0.0 & -\mu_r & \mu_r \\ 0.0 & 0.0 & -\mu_r \end{pmatrix}$, $\mu_r = 0.3r$, α_r= $\begin{pmatrix} 1.0 & 0.0 & 0.0, \end{pmatrix}$, $3 \le r \le 5$.
- The MAP is represented by the matrices $C_l = \begin{pmatrix} -4.657l & 1.761l \\ 1.128l & -3.941l \end{pmatrix}$ and $D_l = \begin{pmatrix} 1.657l & 1.239l \\ 0.872l & 1.941l \end{pmatrix}$. $\xi(C_l + D_l)$=$\mathbf{0}$. l=1.0, 1.1,..., 2.0. ξ=[0.4, 0.6].

It is observed from Figure 4.1 to Figure 4.2 that as the effective arrival rate λ increases, the expected queue length L_q increases in both the cases, this is because increasing the effective arrival rate increases the traffic intensity, and this behavior reflects the increase in L_q. Also, it can be marked here, L_q is lower in Case 1 than Case 2 for a fixed λ. Hence, the consideration of QSDV policy is more versed, because consideration of QSDV minimizes L_q in comparison to the QSIV.

TABLE 4.2: Joint probabilities (queue size and type of vacation) at vacation termination epoch for SV

n	$\gamma_1^+(n,0)$	$\gamma_2^+(n,0)$	$\gamma_1^+(n,1)$	$\gamma_2^+(n,1)$	$\gamma_1^+(n,2)$	$\gamma_2^+(n,2)$	$\gamma_1^+(n,3)$	$\gamma_2^+(n,3)$	$\gamma_1^+(n,4)$	$\gamma_2^+(n,4)$	P_n^{queue+}
0	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.02200
1	0.00001	0.00001	0.00010	0.00007	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.03853
2	0.00001	0.00001	0.00018	0.00013	0.00049	0.00035	0.00000	0.00000	0.00000	0.00000	0.04598
3	0.00001	0.00001	0.00025	0.00019	0.00084	0.00062	0.00124	0.00089	0.00000	0.00000	0.04834
4	0.00002	0.00001	0.00031	0.00023	0.00108	0.00080	0.00193	0.00143	0.00218	0.00156	0.04975
5	0.00002	0.00002	0.00036	0.00027	0.00124	0.00093	0.00225	0.00168	0.00301	0.00224	0.04692
6	0.00002	0.00002	0.00041	0.00031	0.00134	0.00100	0.00234	0.00175	0.00312	0.00234	0.04236
7	0.00003	0.00002	0.00045	0.00033	0.00139	0.00104	0.00228	0.00171	0.00288	0.00217	0.03745
8	0.00003	0.00002	0.00048	0.00036	0.00140	0.00105	0.00213	0.00160	0.00250	0.00188	0.03286
9	0.00003	0.00002	0.00050	0.00037	0.00138	0.00103	0.00193	0.00146	0.00207	0.00157	0.02883
10	0.00004	0.00003	0.00052	0.00039	0.00133	0.00100	0.00172	0.00130	0.00168	0.00127	0.02540
31	0.00007	0.00005	0.00038	0.00029	0.00020	0.00015	0.00003	0.00002	0.00000	0.00000	0.00634
32	0.00007	0.00005	0.00037	0.00028	0.00018	0.00013	0.00003	0.00002	0.00000	0.00000	0.00612
33	0.00007	0.00005	0.00035	0.00027	0.00016	0.00012	0.00002	0.00002	0.00000	0.00000	0.00592
151	0.00004	0.00003	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00089
152	0.00004	0.00003	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00088
153	0.00004	0.00003	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00087
301	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00011
302	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00011
303	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00011
501	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001
502	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001
503	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001
≥504	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
Total	0.0123	0.0094	0.02192	0.0164	0.0256	0.0192	0.0253	0.0190	0.02297	0.0172	0.9999

$$P_n^{queue+} = \sum_{r=h}^{H} \xi^+(n,r)\mathbf{e} + \sum_{k=0}^{min(n,h-1)} \gamma^+(n,k)\mathbf{e}$$

TABLE 4.3: Joint probabilities (queue size and server content) at arbitrary epoch for SV

n	$R_1(n,0)$	$R_2(n,0)$	$\xi_1(n,5)$	$\xi_2(n,5)$	$\xi_1(n,6)$	$\xi_2(n,6)$	$\xi_1(n,7)$	$\xi_2(n,7)$	$\xi_1(n,8)$	$\xi_2(n,8)$	$\xi_1(n,9)$	$\xi_2(n,9)$
0	0.00000	0.00000	0.00421	0.00317	0.00268	0.00202	0.00220	0.00166	0.00179	0.00135	0.00324	0.00240
1	0.00002	0.00001	0.00346	0.00262	0.00209	0.00158	0.00162	0.00123	0.00125	0.00095	0.00383	0.00286
2	0.00011	0.00008	0.00273	0.00207	0.00154	0.00117	0.00113	0.00086	0.00082	0.00063	0.00400	0.00300
3	0.00046	0.00033	0.00208	0.00158	0.00110	0.00084	0.00075	0.00057	0.00051	0.00039	0.00392	0.00295
4	0.00126	0.00092	0.00154	0.00117	0.00076	0.00058	0.00049	0.00037	0.00031	0.00024	0.00373	0.00280
5			0.00112	0.00085	0.00051	0.00039	0.00031	0.00023	0.00018	0.00014	0.00348	0.00262
6			0.00079	0.00060	0.00034	0.00026	0.00019	0.00014	0.00011	0.00008	0.00323	0.00243
7			0.00056	0.00042	0.00022	0.00017	0.00011	0.00009	0.00006	0.00005	0.00299	0.00225
8			0.00038	0.00029	0.00014	0.00011	0.00007	0.00005	0.00003	0.00003	0.00277	0.00208
9			0.00026	0.00020	0.00009	0.00007	0.00004	0.00003	0.00002	0.00001	0.00257	0.00194
10			0.00018	0.00014	0.00006	0.00004	0.00002	0.00002	0.00001	0.00001	0.00240	0.00180
31			0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00099	0.00074
32			0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00096	0.00072
33			0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00093	0.00070
34			0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00091	0.00068
35			0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00089	0.00067
301			0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00002	0.00001
302			0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00002	0.00001
303			0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00002	0.00001
373			0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00001
374			0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00000
392			0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00000
≥ 393			0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
Total	0.00185	0.00134	0.01767	0.01338	0.00962	0.00730	0.00696	0.00529	0.00511	0.00389	0.12674	0.09514

TABLE 4.4: Joint probabilities (queue size and type of vacation) at arbitrary epoch for SV

n	$\gamma_1(n,0)$	$\gamma_2(n,0)$	$\gamma_1(n,1)$	$\gamma_2(n,1)$	$\gamma_1(n,2)$	$\gamma_2(n,2)$	$\gamma_1(n,3)$	$\gamma_2(n,3)$	$\gamma_1(n,4)$	$\gamma_2(n,4)$	P_n^{queue}
0	0.00187	0.00132	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.02966
1	0.00185	0.00137	0.00323	0.00232	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.03354
2	0.00184	0.00138	0.00318	0.00237	0.00372	0.00269	0.00000	0.00000	0.00000	0.00000	0.03687
3	0.00184	0.00138	0.00314	0.00236	0.00357	0.00267	0.00356	0.00258	0.00000	0.00000	0.03983
4	0.00184	0.00138	0.00310	0.00232	0.00341	0.00256	0.00326	0.00244	0.00308	0.00224	0.04250
5	0.00183	0.00138	0.00304	0.00228	0.00323	0.00243	0.00292	0.00220	0.00262	0.00197	0.03589
6	0.00183	0.00137	0.00298	0.00224	0.00303	0.00228	0.00258	0.00195	0.00216	0.00163	0.03188
7	0.00183	0.00137	0.00292	0.00219	0.00283	0.00213	0.00225	0.00169	0.00173	0.00131	0.02841
8	0.00182	0.00137	0.00285	0.00214	0.00262	0.00197	0.00193	0.00146	0.00137	0.00103	0.02547
9	0.00182	0.00136	0.00277	0.00208	0.00242	0.00182	0.00165	0.00125	0.00106	0.00080	0.02299
10	0.00181	0.00136	0.00270	0.00203	0.00222	0.00167	0.00140	0.00105	0.00081	0.00062	0.02091
31	0.00164	0.00123	0.00118	0.00088	0.00023	0.00017	0.00002	0.00001	0.00000	0.00000	0.00714
32	0.00163	0.00122	0.00112	0.00084	0.00020	0.00015	0.00002	0.00001	0.00000	0.00000	0.00693
33	0.00162	0.00121	0.00107	0.00080	0.00018	0.00013	0.00001	0.00001	0.00000	0.00000	0.00672
34	0.00161	0.00121	0.00102	0.00077	0.00016	0.00012	0.00001	0.00001	0.00000	0.00000	0.00653
35	0.00160	0.00120	0.00097	0.00073	0.00014	0.00011	0.00001	0.00001	0.00000	0.00000	0.00635
301	0.00006	0.00004	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00014
302	0.00006	0.00004	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00013
303	0.00006	0.00004	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00013
373	0.00002	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00005
374	0.00002	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00005
392	0.00001	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00003
393	0.00001	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00003
394	0.00001	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00003
446	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001
447	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001
514	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001
515	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001
≥ 516	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
Total	0.20851	0.15635	0.09092	0.06817	0.04722	0.03541	0.02624	0.01969	0.01524	0.01144	0.99994

L_q=48.547, L_s=51.133, W_q=0.859, W_s= 0.905, L^{ser}=8.141, L^{vac}=0.838
P_{dor}=0.0032, P_{idle}=0.682, P_{busy}=0.318

Table 4.5: Joint probabilities (queue size and server content) at service completion epoch for MV

n	$\xi_1^+(n,5)$	$\xi_2^+(n,5)$	$\xi_1^+(n,6)$	$\xi_2^+(n,6)$	$\xi_1^+(n,7)$	$\xi_2^+(n,7)$	$\xi_1^+(n,8)$	$\xi_2^+(n,8)$	$\xi_1^+(n,9)$	$\xi_2^+(n,9)$
0	0.00188	0.00133	0.00233	0.00166	0.00264	0.00189	0.00283	0.00202	0.00293	0.00209
1	0.00332	0.00245	0.00381	0.00282	0.00402	0.00298	0.00402	0.00299	0.00647	0.00475
2	0.00393	0.00293	0.00416	0.00312	0.00408	0.00307	0.00381	0.00287	0.00916	0.00680
3	0.00387	0.00291	0.00379	0.00286	0.00345	0.00261	0.00301	0.00228	0.01066	0.00797
4	0.00343	0.00259	0.00310	0.00235	0.00262	0.00199	0.00214	0.00163	0.01118	0.00838
5	0.00283	0.00214	0.00237	0.00180	0.00186	0.00142	0.00142	0.00108	0.01103	0.00829
6	0.00223	0.00169	0.00172	0.00131	0.00126	0.00096	0.00089	0.00068	0.01052	0.00791
7	0.00169	0.00129	0.00121	0.00092	0.00082	0.00063	0.00054	0.00042	0.00985	0.00741
8	0.00125	0.00095	0.00083	0.00063	0.00052	0.00040	0.00032	0.00025	0.00914	0.00687
9	0.00090	0.00069	0.00055	0.00042	0.00032	0.00025	0.00019	0.00014	0.00845	0.00636
10	0.00064	0.00049	0.00036	0.00028	0.00020	0.00015	0.00011	0.00008	0.00782	0.00588
20	0.00001	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00431	0.00324
21	0.00001	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00413	0.00310
251	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00012	0.00009
457	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00000
458	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00000
459	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00000
$\geq$ 460	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
Total	0.02734	0.02051	0.02486	0.01865	0.02207	0.01656	0.01940	0.01455	0.36214	0.27164

TABLE 4.6: Joint probabilities (queue size and type of vacation) at vacation termination epoch for MV

n	$\gamma_1^+(n,0)$	$\gamma_2^+(n,0)$	$\gamma_1^+(n,1)$	$\gamma_2^+(n,1)$	$\gamma_1^+(n,2)$	$\gamma_2^+(n,2)$	$\gamma_1^+(n,3)$	$\gamma_2^+(n,3)$	$\gamma_1^+(n,4)$	$\gamma_2^+(n,4)$	P_n^{queue+}
0	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.02161
1	0.00001	0.00001	0.00009	0.00007	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.03780
2	0.00001	0.00001	0.00018	0.00013	0.00049	0.00035	0.00000	0.00000	0.00000	0.00000	0.04509
3	0.00001	0.00001	0.00025	0.00018	0.00084	0.00062	0.00133	0.00095	0.00000	0.00000	0.04760
4	0.00002	0.00001	0.00031	0.00023	0.00109	0.00081	0.00207	0.00153	0.00272	0.00194	0.05013
5	0.00002	0.00002	0.00036	0.00027	0.00125	0.00093	0.00242	0.00181	0.00375	0.00279	0.04785
6	0.00002	0.00002	0.00040	0.00030	0.00135	0.00101	0.00251	0.00188	0.00389	0.00292	0.04351
7	0.00003	0.00002	0.00044	0.00033	0.00140	0.00105	0.00245	0.00184	0.00360	0.00271	0.03862
8	0.00003	0.00002	0.00047	0.00035	0.00140	0.00105	0.00229	0.00172	0.00311	0.00235	0.03395
9	0.00003	0.00002	0.00049	0.00037	0.00138	0.00104	0.00208	0.00157	0.00259	0.00196	0.02979
10	0.00003	0.00003	0.00051	0.00038	0.00134	0.00101	0.00185	0.00139	0.00209	0.00158	0.02622
20	0.00006	0.00004	0.00052	0.00039	0.00065	0.00049	0.00034	0.00026	0.00013	0.00010	0.01054
21	0.00006	0.00004	0.00051	0.00038	0.00059	0.00044	0.00028	0.00021	0.00009	0.00007	0.00992
251	0.00001	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00023
457	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001
458	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001
459	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001
460	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001
500	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001
501	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001
502	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001
≥ 503	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
Total	0.01235	0.00926	0.02160	0.01620	0.02577	0.01932	0.02720	0.02040	0.02864	0.02148	0.99993

TABLE 4.7: Joint probabilities (queue size and server content) at arbitrary epoch for MV

n	$\xi_1(n,5)$	$\xi_2(n,5)$	$\xi_1(n,6)$	$\xi_2(n,6)$	$\xi_1(n,7)$	$\xi_2(n,7)$	$\xi_1(n,8)$	$\xi_2(n,8)$	$\xi_1(n,9)$	$\xi_2(n,9)$
0	0.00329	0.00247	0.00278	0.00209	0.00229	0.00173	0.00187	0.00141	0.00338	0.00250
1	0.00271	0.00205	0.00216	0.00164	0.00169	0.00128	0.00130	0.00099	0.00399	0.00298
2	0.00213	0.00162	0.00160	0.00122	0.00117	0.00089	0.00085	0.00065	0.00416	0.00312
3	0.00162	0.00123	0.00114	0.00087	0.00078	0.00060	0.00054	0.00041	0.00407	0.00306
4	0.00120	0.00091	0.00079	0.00060	0.00051	0.00039	0.00033	0.00025	0.00386	0.00290
5	0.00087	0.00066	0.00053	0.00041	0.00032	0.00024	0.00019	0.00015	0.00359	0.00270
6	0.00062	0.00047	0.00035	0.00027	0.00020	0.00015	0.00011	0.00009	0.00332	0.00250
7	0.00043	0.00033	0.00023	0.00017	0.00012	0.00009	0.00006	0.00005	0.00306	0.00230
8	0.00030	0.00023	0.00015	0.00011	0.00007	0.00005	0.00004	0.00003	0.00283	0.00213
9	0.00021	0.00016	0.00009	0.00007	0.00004	0.00003	0.00002	0.00001	0.00262	0.00197
10	0.00014	0.00011	0.00006	0.00004	0.00002	0.00002	0.00001	0.00001	0.00244	0.00183
101	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00030	0.00023
102	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00030	0.00022
389	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00000
390	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00000
391	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00000
≥ 392	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
Total	0.01380	0.01045	0.00997	0.00756	0.00725	0.00551	0.00533	0.00406	0.12747	0.09570

TABLE 4.8: Joint probabilities (queue size and type of vacation) at arbitrary epoch for MV

n	$\gamma_1(n,0)$	$\gamma_2(n,0)$	$\gamma_1(n,1)$	$\gamma_2(n,1)$	$\gamma_1(n,2)$	$\gamma_2(n,2)$	$\gamma_1(n,3)$	$\gamma_2(n,3)$	$\gamma_1(n,4)$	$\gamma_2(n,4)$	P_n^{queue}
0	0.00185	0.00131	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.02851
1	0.00183	0.00136	0.00322	0.00231	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.03254
2	0.00183	0.00137	0.00317	0.00236	0.00378	0.00273	0.00000	0.00000	0.00000	0.00000	0.03601
3	0.00183	0.00137	0.00313	0.00234	0.00363	0.00271	0.00387	0.00280	0.00000	0.00000	0.03908
4	0.00182	0.00137	0.00308	0.00231	0.00346	0.00260	0.00353	0.00265	0.00388	0.00281	0.04183
5	0.00182	0.00136	0.00303	0.00227	0.00328	0.00246	0.00317	0.00239	0.00330	0.00248	0.03728
6	0.00182	0.00136	0.00297	0.00223	0.00308	0.00231	0.00280	0.00211	0.00272	0.00205	0.03311
7	0.00181	0.00136	0.00290	0.00218	0.00287	0.00216	0.00244	0.00184	0.00218	0.00165	0.02946
8	0.00181	0.00136	0.00283	0.00213	0.00266	0.00200	0.00210	0.00158	0.00172	0.00130	0.02635
9	0.00180	0.00135	0.00276	0.00207	0.00246	0.00185	0.00179	0.00135	0.00134	0.00101	0.02371
10	0.00180	0.00135	0.00268	0.00202	0.00226	0.00170	0.00151	0.00114	0.00102	0.00078	0.02149
101	0.00085	0.00064	0.00002	0.00002	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00206
102	0.00084	0.00063	0.00002	0.00002	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00204
389	0.00002	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00004
390	0.00001	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00004
391	0.00001	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00003
392	0.00001	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00003
458	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001
459	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001
460	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001
514	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001
515	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001
≥ 516	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
Total	0.20682	0.15507	0.09046	0.06782	0.04796	0.03596	0.02847	0.02135	0.01919	0.01439	0.99994

L_q=48.268, L_s=50.837, W_q=0.854, W_s= 0.899, L^{ser}=8.224, L^{vac}=0.887

P_{idle}=0.687, P_{busy}=0.312

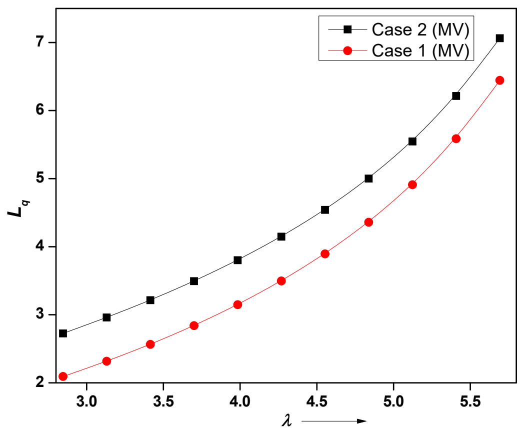

FIGURE 4.1: Effect of λ on L_q

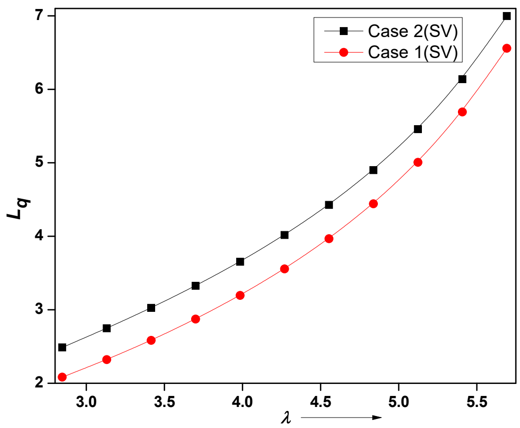

FIGURE 4.2: Effect of λ on L_q

4.6.1 Deduction of the results for $M/M/1$ queue

The model considered in this chapter reduces to $M/M/1$ model if $a = 1$, $b = 1$, $C = -\lambda$, $D = \lambda$, service time follows exponential distribution and the vacation rate is taken to be considerably large (i.e., vacation time almost tends to zero). Table 4.9 and Table 4.10 are presented to show the values of L_q, W_s, L^{ser} and P_{idle} which are obtained for $M/M/1$ model for the following two cases.

Case I: Results for $M/M/1$ model deduced from the analytical results presented in this chapter by considering $C = -\lambda$, $D = \lambda$, $a = b = 1$, exponential service time distribution and $\nu_0 \longrightarrow \infty$ ($\nu_0 = 200000$).
Case II: Results for classical $M/M/1$ model, for which performance measures L_q, W_s and probability P_{idle} are calculated using standard formula $L_q = \frac{\lambda^2}{\mu_1(\mu_1-\lambda)}$, $W_s = \frac{1}{\mu_1-\lambda}$ and $P_{idle} = 1 - \rho$.

Table 4.9 and Table 4.10 are described as follows.

- 1st and 2nd column present the values of input parameters λ and μ_1, respectively, for which ρ varies from 0.4166 to 0.833.
- 3rd, 4th, 5th and 6th column present the values of L_q, W_s, L^{ser} and P_{idle}, respectively, for Case I.
- 7th, 8th and 9th column present the values of L_q, W_s and P_{idle}, respectively, for Case II.

It is clearly observed from Table 4.9 and Table 4.10 that the results deduced from current study as a special case matches exactly with the results obtained from $M/M/1$ model. Also, the value of L^{ser} calculated from the current study as a special case always gives the value 1 which is obvious and shows the correctness of present study.

TABLE 4.9: Table for Case I and Case II, for SV

		Case I				Case II		
λ	μ_1	L_q	W_s	L^{ser}	P_{idle}	L_q	W_s	P_{idle}
5	6	4.1666658	0.9999998	1.0000000	0.1666667	4.1666667	1.0000000	0.1666667
5	9	0.6944444	0.2500000	1.0000000	0.4444444	0.6944444	0.2500000	0.4444444
5	12	0.2976190	0.1428571	1.0000000	0.5833333	0.2976190	0.1428571	0.5833333

TABLE 4.10: Table for Case I and Case II, for MV

		Case I				Case II		
λ	μ_1	L_q	W_s	L^{ser}	P_{idle}	L_q	W_s	P_{idle}
5	6	4.1666691	1.0000005	1.0000000	0.1666667	4.1666667	1.0000000	0.1666667
5	9	0.6944469	0.2500005	1.0000000	0.4444444	0.6944444	0.2500000	0.4444444
5	12	0.2976215	0.1428576	1.0000000	0.5833333	0.2976190	0.1428571	0.5833333

4.7 Cost model

A cost model is also presented in this section which helps the manager to determine the optimal value of desired input parameters . The following cost parameters are taken for this purpose.

$C_{st}\equiv$ Startup cost per customer per unit time.

$C_b\equiv$ Holding cost per customer per unit time when the server is busy.

$C_v\equiv$ Holding cost per customer per unit time when the server is on vacation.

$C_d\equiv$ Holding cost per customer per unit time when the server is dormant (exists only for SV).

$C_o\equiv$ Operating cost per customer per unit time. Thus in long run,

$$\text{total system cost (TSC)} = \lambda C_{st} + C_b \sum_{n=0}^{\infty}\sum_{r=a}^{b} n\frac{\xi(n,r)\mathbf{e}}{P_{busy}} + C_v \sum_{n=0}^{\infty}\sum_{k=0}^{min(n,a-1)} n\frac{\gamma(n,k)\mathbf{e}}{Q_{vac}} + (1-\delta)C_d \sum_{n=0}^{a-1} n\frac{R(n,0)\mathbf{e}}{P_{dor}} + C_o L^{ser}.$$

Figure 4.3 reflects the behavior of TSC for different values of a $(1 \le a \le 10)$ for SV and for λ= 0.5, 0.9. The maximum capacity of the server is fixed at $b = 10$. Service time follows E_4 distribution with service rate $\mu_r = \frac{r}{75}$, $(a \le r \le b)$. Vacation time follows E_2

distribution with vacation rate $\nu_k = (2k+1)0.3$, $(0 \le k \le a-1)$. The MAP representation is taken as follows

$$\begin{cases} C = \begin{pmatrix} -0.8125 & 0.1250 \\ 0.4375 & -0.6875 \end{pmatrix} \text{ and } D = \begin{pmatrix} 0.4375 & 0.2500 \\ 0.0625 & 0.1875 \end{pmatrix}, & \text{for } \lambda = 0.5, \\ C = \begin{pmatrix} -1.4625 & 0.2250 \\ 0.7875 & -1.2375 \end{pmatrix} \text{ and } D = \begin{pmatrix} 0.7875 & 0.4500 \\ 0.1125 & 0.3375 \end{pmatrix}, & \text{for } \lambda = 0.9. \end{cases}$$

TSC are calculated with the following cost parameters: $C_{st} = 0.4$, $C_b = 1.2$, $C_v = 1.5$, $C_d = (1-\delta)1.5$ and $C_o = 4.2$. Here our objective is to identify the optimum value of a at which TSC is minimum. From Figure 4.3, it is clear that for $\lambda = 0.5$ (0.9) the optimum value for a is 2 (3) and the corresponding minimum value of TSC is 35.409 (52.838). Similarely, Figure 4.4 depicts the behavior of TSC for different values of a $(1 \le a \le 10)$ for MV and for λ= 0.5 (0.9). The input parameters, cost parameters and the service (vacation) time distribution are taken same as taken for Figure 4.3. For $\lambda = 0.5$ (0.9), the optimum value for a is 3 (4) and the corresponding minimum value of TSC is 33.913 (49.825). The minimum values of TSC, in each figure, are indicated by arrow sign.

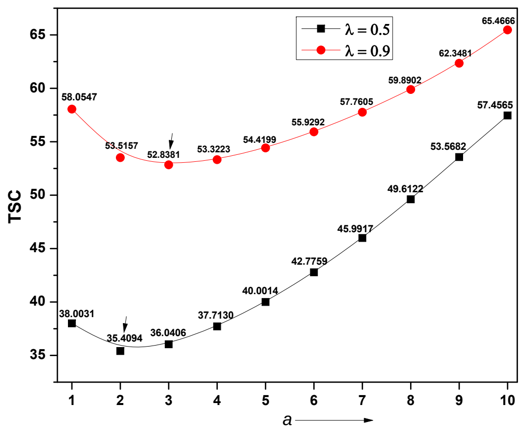

FIGURE 4.3: Effect of a on TSC for SV

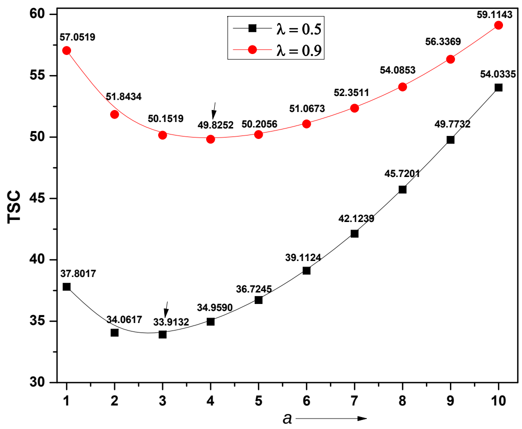

FIGURE 4.4: Effect of a on TSC for MV

4.8 Conclusion

An infinite capacity $MAP/G_r^{(a,b)}/1$ queue with queue size dependent SV (MV) is discussed in this chapter. Bivariate vector generating function method and the supplementary variable approach have been used to extract steady state joint probabilities of queue content, server content (type of vacation) and the phase of arrival process. The present model can be extended for analyzing different queueing models with batch Markovian arrival process (BMAP) and different vacation policies (*viz.*, $BMAP/G_r^{(a,b)}/1$ queue with queue size dependent single and multiple working vacation), which is left for the future study.