Chapter 3

Analysis of $M/G_r^{(a,b)}/1$ queue with queue length dependent single and multiple vacation

3.1 Introduction

Theory of the batch service queue with vacation has huge application in various congestion situations, *viz.*, telecommunication, manufacturing, transportation, etc. In telecommunication system messages, data, digital signals are first broken into cells (packets) and are transmitted over the common transmission line in batches with a minimum threshold and maximum capacity and whenever the minimum number of packets is not available for the transmission, the multiplexer enters into the predefined single vacation or multiple vacation.

Most of the literature on batch service queue with vacations deals with the derivation of the distribution of the queue size at various epochs only, in which the service time is considered to be independent of the batch size under service. Though, Gupta et al. [101] considered $M/G_r^{(a,b)}/1$ finite buffer queue with SV (MV). They obtained the required joint distributions at various epoch. By using the embedded Markov chain technique, they first derived transition probability matrix (TPM) to obtain the joint distribution of the queue size and batch size with the server at the epoch of service completion and the queue size

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and type of vacation at vacation termination epoch. However, it seems difficult to handle TPM with a considerably large buffer size or an infinite buffer queue (see, e.g., Bar-Lev et al. [13]). To address this difficulty, in this chapter an analytical study of the batch size dependent bulk service queue with SV (MV) and infinite buffer is presented, where the service time and vacation time depend on the batch size with the server and queue size at the vacation initiation epoch, respectively. Using SVT, the bivariate generating function approach for the joint probabilities of the queue size and batch size with the server at service completion epoch and the queue size and type of vacation at vacation termination epoch are obtained. Then all the required service (vacation) completion epoch joint probabilities are successfully extracted which are eventually used to obtain corresponding arbitrary epoch joint probabilities. The analytical study presented in this chapter is completely different than that is presented in Gupta et al. [101] and result obtained in the considered model cannot be obtained as a special case of their result. The novelty of the present work is that where the queueing practitioners stuck with the results provided by Gupta et al. [101] while handling an infinite buffer system, results obtained in the considered model will definitely help them to find out the proper solution.

The practical motivation for the considered model comes up with the way of the sample testing procedure in the pandemic situation (*viz.*, COVID-19). An effective way of fighting some dangerous virus (pandemic) such as the corona virus is to test all suspected people in whom the virus is likely to be found. However, at the beginning of an epidemic, the shortage of test kits causes major problems which aggravate the pandemic. To deal with such a situation group testing play a key role instead of an individual test. In a group test, samples of multiple swabs are mixed to form a 'mixed sample', which is then tested. If the test result is found to be negative, then all samples in the mixture are negative for the virus, however, a positive test shows one or more samples infected with the virus in the mixture. Then that particular sample will be further tested to identify the infected samples. Such group testing process is already justified during the COVID-19 pandemic, see the references [124, 125, 126].

The model presented in this chapter may be helpful in policy-making for group testing methods to deal with a pandemic situation for a particular country. For better understanding an example of group testing is considered in which the samples are coming according to the Poisson stream and the health worker test the samples in bulk (i.e., mixed sample) with a lower threshold 'a' to upper threshold 'b' (following GBS rule). On completion of a test if the number of waiting samples to be tested is r ($\geq a$) then a mixed sample of size $\min(r, b)$ is taken for the test. The mixing time depends on the number of samples which is going to be tested, therefore, batch size dependent service is justified. On the other hand, if at the completion of the test the number of waiting samples to be tested is k ($< a$) then the health worker stops testing and goes on vacation. During vacation, the health worker will be engaged in some additional works (stocking of health care inventory, increase people's awareness, visiting the quarantine room, etc.). Before going for the vacation, the health worker always checks the queue size, and depending on the queue size he set his vacation time which will increase the expected number of samples tested per unit time. Hence, queue size dependent vacation is also justified.

The rest of the chapter is devoted as follows: Description of the considered model presented in Section 3.2. The steady state joint probabilities is investigated at various epoch in Section 3.3. Significant marginal probabilities are presented in Section 3.4. Some necessary performance measures are reported in Section 3.5. Numerical results and their discussion are presented in Section 3.6. In Section 3.7, a cost model is developed in order to obtain the optimum value of the lower threshold a. The conclusion section ends the chapter.

3.2 Model description

The present chapter investigates infinite capacity single server batch service queue with SV (MV) where the service time (T_r) ($1 \leq a \leq r \leq b$) and vacation time (V_k) ($0 \leq k \leq a-1$), respectively, depend on batch size under service and queue size at vacation initiation epoch, respectively. The customers reach, one by one, to the system following the Poison distribution with a rate of λ, and are received the service in batches with a lower threshold a (≥ 1) and upper threshold b ($b \geq a$) as per the GBS rule. After service, if the queue size is $\geq a$ then the server renders the service as per the GBS rule, otherwise, the server goes for vacation. The vacation time of the server depends on the size of the queue k ($0 \leq k \leq a-1$) at the vacation initiation epoch. Such a vacation that depends on the queue size k is known as k^{th} type of vacation. The customers are not allowed to join any running service even if the server is serving less number of customers than its maximum capacity. The service time (T_r) ($1 \leq a \leq r \leq b$) and the vacation time (V_k) ($0 \leq k \leq a-1$) are distributed generally. Some other assumptions are as follows:

- $s_r(t)$ = probability density function (pdf) of T_r,
- $S_r(t)$ = distribution function (DF) of T_r,
- $\tilde{S}_r(\theta)$ = Laplace-Stieltjes transform (LST) of T_r,

- mean service time $= \frac{1}{\mu_r} = s_r = -\tilde{S}_r^{(1)}(0) =$ derivative of $\tilde{S}_r(\theta)$ evaluated at θ=0,
- $v_k(t)$= pdf of V_k,
- $V_k(t)$ = DF of V_k,
- $\tilde{V}_k(\theta)$= LST of V_k,
- mean vacation time $= \frac{1}{\nu_k} = x_k = -\tilde{V}_k^{(1)}(0) =$ derivative of $\tilde{V}_k(\theta)$ evaluated at θ=0.

On completion of k^{th} $(0 \leq k \leq a-1)$ type of vacation, if the queue size $\geq a$, then the server renders the service as per the GBS rule, otherwise, following SV rule, the server remains in the dormant state until the queue size reaches the lower threshold a, or under MV rule, takes repeated vacation until it finds queue length $\geq a$ at the end of the vacation. The condition that ensures the system stability is $\frac{\lambda s_b}{b}(< 1)$. In this chapter, both SV and MV queues are studied simultaneously. By substituting $\delta = 1$ in the steady state analysis, results for MV can be obtained and by substituting $\delta = 0$ results for SV can be obtained.

3.3 System analysis

This section covers the analysis of the joint probabilities of the queue size and batch size with the server at the service completion epoch, and the joint probabilities of queue size and type of vacation taken by the server at the vacation termination epoch. Then arbitrary epoch joint probabilities are obtained by establishing the relation between the joint probabilities of service (vacation) completion epoch and random epoch. For the mathematical analysis, the steady state equations are obtained by defining additional variable for remaining service (vacation) time. To this end, the system state at time t is introduced by random variables as follows:

- $N_q(t)$ represents the queue size.
- $N_s(t)$ represents the batch size with the server when the server is busy.
- $K(t)$ represents the type of vacation taken by the server, when the server is on vacation.
- $U(t)$ represents remaining service time of the batch in service, if any.
- $V(t)$ represents remaining vacation time of a vacation period, if any.

Note that the dormant state of the server at time t will be represented by $N_s(t) = 0$.

For SV, $\{(N_q(t), N_s(t))\} \cup \{(N_q(t), N_s(t), U(t)) \cup (N_q(t), K(t), V(t))\}$ forms a Markov process with state space $\{(n, 0); 0 \leq n \leq a-1\} \bigcup \{(n, r, u); n \geq 0, a \leq r \leq b, u \geq 0\} \bigcup \{(n, k, u); 0 \leq k \leq a-1, n \geq k, u \geq 0\}$.

For MV, $\{(N_q(t), N_s(t), U(t)) \cup (N_q(t), K(t), V(t))\}$ forms a Markov process with state space $\{(n, r, u); n \geq 0, a \leq r \leq b, u \geq 0\} \bigcup \{(n, k, u); 0 \leq k \leq a-1, n \geq k, u \geq 0\}$.

Further, the state probabilities, at time t, are defined as follows:

- $R_n(t) \equiv Pr\{N_q(t) = n, N_s(t) = 0\}, \quad 0 \leq n \leq a-1$ (exist only for SV).
- $P_{n,r}(u, t)du \equiv Pr\{N_q(t) = n, N_s(t) = r, u \leq U(t) \leq u + du\}, \quad n \geq 0\ ,\ a \leq r \leq b.$
- $Q_n^{[k]}(u, t)du \equiv\ Pr\{N_q(t) = n, K(t) = k, u \leq V(t) \leq u + du\}, \ n \geq k\ ,\ 0 \leq k \leq a-1.$

In steady state, as $t \to \infty$,

$R_n = \lim_{t \to \infty} R_n(t)$ $(0 \leq n \leq a-1)$ (exist only for SV),
$P_{n,r}(u) = \lim_{t \to \infty} P_{n,r}(u, t), \quad n \geq 0, \quad a \leq r \leq b,$
$Q_n^{[k]}(u) = \lim_{t \to \infty} Q_n^{[k]}(u, t), \quad n \geq k, \quad 0 \leq k \leq a-1.$
Now the system equation that governs the system behavior is obtained. Analyzing the model at time t and $t + dt$, the governing equations in steady state are obtained as follows:

$$0 = (1-\delta)\Big(-\lambda R_0 + Q_0^{[0]}(0)\Big), \tag{3.1}$$

$$0 = (1-\delta)\Big(-\lambda R_n + \lambda R_{n-1} + \sum_{k=0}^{n} Q_n^{[k]}(0)\Big), 1 \leq n \leq a-1, \tag{3.2}$$

$$-\frac{d}{du}P_{0,a}(u) = -\lambda P_{0,a}(u) + (1-\delta)\lambda R_{a-1} s_a(u) + \Big(\sum_{k=0}^{a-1} Q_a^{[k]}(0) + \sum_{r=a}^{b} P_{a,r}(0)\Big) s_a(u), \tag{3.3}$$

$$-\frac{d}{du}P_{0,r}(u) = -\lambda P_{0,r}(u) + \Big(\sum_{k=0}^{a-1} Q_r^{[k]}(0) + \sum_{j=a}^{b} P_{r,j}(0)\Big) s_r(u), \tag{3.4}$$

$$a+1 \leq r \leq b,$$

$$-\frac{d}{du}P_{n,r}(u) = -\lambda P_{n,r}(u)+\lambda P_{n-1,r}(u), \quad a\le r\le b-1, \quad n\ge 1, \tag{3.5}$$

$$-\frac{d}{du}P_{n,b}(u) = -\lambda P_{n,b}(u)+\lambda P_{n-1,b}(u) + \left(\sum_{k=0}^{a-1}Q_{n+b}^{[k]}(0)+\sum_{r=a}^{b}P_{n+b,r}(0)\right)s_b(u), \ n\ge 1, \tag{3.6}$$

$$-\frac{d}{du}Q_k^{[k]}(u) = -\lambda Q_k^{[k]}(u)+\left(\sum_{r=a}^{b}P_{k,r}(0)+\delta\sum_{j=0}^{k}Q_k^{[j]}(0)\right)\nu_k(u), \quad 0\le k\le a-1, \tag{3.7}$$

$$-\frac{d}{du}Q_n^{[k]}(u) = -\lambda Q_n^{[k]}(u)+\lambda Q_{n-1}^{[k]}(u), \quad n\ge k+1, \quad 0\le k\le a-1. \tag{3.8}$$

Further, define for Re $\theta\ge 0$,

$$\tilde{S}_r(\theta)=\int_0^\infty e^{-\theta u}dS_r(u)=\int_0^\infty e^{-\theta u}s_r(u)du, \quad a\le r\le b, \tag{3.9}$$

$$\tilde{P}_{n,r}(\theta)=\int_0^\infty e^{-\theta u}P_{n,r}(u)du, \quad a\le r\le b, \ n\ge 0, \tag{3.10}$$

$$P_{n,r}\equiv\tilde{P}_{n,r}(0)=\int_0^\infty P_{n,r}(u)du, \quad a\le r\le b, \ n\ge 0, \tag{3.11}$$

$$\tilde{V}_k(\theta)=\int_0^\infty e^{-\theta u}dV_k(u)=\int_0^\infty e^{-\theta u}v_k(u)du, \quad 0\le k\le a-1, \tag{3.12}$$

$$\tilde{Q}_n^{[k]}(\theta)=\int_0^\infty e^{-\theta u}Q_n^{[k]}(u)du, \quad 0\le k\le a-1, \ n\ge k, \tag{3.13}$$

$$Q_n^{[k]}\equiv\tilde{Q}_n^{[k]}(0)=\int_0^\infty Q_n^{[k]}(u)du, \quad 0\le k\le a-1, \quad n\ge k. \tag{3.14}$$

One may note here that (R_n) $\{P_{n,r}\}$ $[Q_n^{[k]}]$ denotes the probability of (queue size is n and the sever is dormant, $0\le n\le a-1$) {queue size is n and batch size with server is r, $a\le r\le b$, $n\ge 0$} [queue size is n and the server is on k^{th} type of vacation, $0\le k\le a-1$, $n\ge k$] at arbitrary epoch.

Now main objective is to obtain $R_n, P_{n,r}$ and $Q_n^{[k]}$ using (3.1)-(3.8). Keeping this in mind multiplying (3.3)-(3.8) by $e^{-\theta u}$ and integrate with respect to u over the limits 0 to ∞, one can obtain,

$$(\lambda-\theta)\tilde{P}_{0,a}(\theta) = (1-\delta)\lambda R_{a-1}\tilde{S}_a(\theta)+\left(\sum_{k=0}^{a-1}Q_a^{[k]}(0)+\sum_{r=a}^{b}P_{a,r}(0)\right)\tilde{S}_a(\theta) - P_{0,a}(0), \tag{3.15}$$

$$
\begin{aligned}
(\lambda-\theta)\tilde{P}_{0,r}(\theta) &= \left(\sum_{k=0}^{a-1} Q_r^{[k]}(0) + \sum_{j=a}^{b} P_{r,j}(0)\right)\tilde{S}_r(\theta) \\
&\quad -P_{0,r}(0), \quad a+1 \le r \le b, && (3.16)\\
(\lambda-\theta)\tilde{P}_{n,r}(\theta) &= \lambda\tilde{P}_{n-1,r}(\theta) - P_{n,r}(0), \quad n \ge 1, a \le r \le b-1, && (3.17)\\
(\lambda-\theta)\tilde{P}_{n,b}(\theta) &= \lambda\tilde{P}_{n-1,b}(\theta) + \left(\sum_{k=0}^{a-1} Q_{n+b}^{[k]}(0) + \sum_{r=a}^{b} P_{n+b,r}(0)\right)\tilde{S}_b(\theta) \\
&\quad -P_{n,b}(0), \; n \ge 1, && (3.18)\\
(\lambda-\theta)\tilde{Q}_k^{[k]}(\theta) &= \left(\sum_{r=a}^{b} P_{k,r}(0) + \delta\sum_{j=0}^{k} Q_k^{[j]}(0)\right)\tilde{V}_k(\theta) \\
&\quad -Q_k^{[k]}(0), \quad 0 \le k \le a-1, && (3.19)\\
(\lambda-\theta)\tilde{Q}_n^{[k]}(\theta) &= \lambda\tilde{Q}_{n-1}^{[k]}(\theta) - Q_n^{[k]}(0) \;\; n \ge k+1, \quad 0 \le k \le a-1. && (3.20)
\end{aligned}
$$

As the primary aim is to find the joint probabilities of the queue size as well as the batch size with the server under service (queue size and the type of vacation of server) at an arbitrary epoch, which seems to be difficult to obtain directly from (3.15)-(3.20). Hence, characterizing the system state at service (vacation) completion epoch which reduces the continuous time Markov process into an embedded Markov chain where embedded Markov points are defined as service completion epoch and vacation termination epoch. The approach of finding an embedded Markov chain reduces the complexity of the system for mathematical evaluation. Towards this end, the probabilities at service (vacation) completion epoch are defined as follows:

$$
\begin{aligned}
P_{n,r}^{+} &= Pr\{\text{queue size is } n \text{ at service completion epoch of a batch size } r\}, \\
&\qquad n \ge 0, \quad a \le r \le b, && (3.21)\\
P_n^{+} &= Pr\{\text{queue size is } n \text{ at service completion epoch of a batch}\} \\
&= \sum_{r=a}^{b} P_{n,r}^{+}, \quad n \ge 0, && (3.22)\\
Q_n^{[k]+} &= Pr\{\text{queue size is } n \text{ at } k^{th} \text{ type of vacation termination epoch}\}, \\
&\qquad 0 \le k \le a-1, \quad n \ge k, && (3.23)
\end{aligned}
$$

$$
\begin{aligned}
Q_n^{+} &= Pr\{\text{queue size is } n \text{ at the vacation termination epoch}\} \\
&= \sum_{k=0}^{min(n,a-1)} Q_n^{[k]+}, \quad n \ge 0. && (3.24)
\end{aligned}
$$

3.3.1 Joint probabilities at service (vacation) completion epoch

In this section, the main objective is to find $P^+_{n,r}$ $(a \le r \le b, n \ge 0,)$ and $Q^{[k]+}_n$ $(0 \le k \le a-1, n \ge k)$. In this connection few generating functions are defined as follows:

$$P(z,y,\theta) = \sum_{n=0}^{\infty}\sum_{r=a}^{b} \tilde{P}_{n,r}(\theta) z^n y^r, |z| \le 1, \ |y| \le 1, \tag{3.25}$$

$$P^+(z,y) = \sum_{n=0}^{\infty}\sum_{r=a}^{b} P^+_{n,r} z^n y^r, |z| \le 1, \ |y| \le 1, \tag{3.26}$$

$$P^+(z) = \sum_{n=0}^{\infty}\sum_{r=a}^{b} P^+_{n,r} z^n = \sum_{n=0}^{\infty} P^+_n z^n, \quad |z| \le 1, \tag{3.27}$$

$$Q(z,y,\theta) = \sum_{k=0}^{a-1}\sum_{n=k}^{\infty} \tilde{Q}^{[k]}_n(\theta) z^n y^k, |z| \le 1, \ |y| \le 1, \tag{3.28}$$

$$Q^+(z,y) = \sum_{k=0}^{a-1}\sum_{n=k}^{\infty} Q^{[k]+}_n z^n y^k, |z| \le 1, \ |y| \le 1, \tag{3.29}$$

$$\begin{aligned} Q^+(z) &= \sum_{k=0}^{a-1}\sum_{n=k}^{\infty} Q^{[k]+}_n z^n = \sum_{n=0}^{\infty}\sum_{k=0}^{min(n,a-1)} Q^{[k]+}_n z^n \\ &= \sum_{n=0}^{\infty} Q^+_n z^n, \quad |z| \le 1. \end{aligned} \tag{3.30}$$

Further, define the following probabilities as follows:

$$\begin{aligned} m_j^{(r)} &= Pr\{j \text{ customers arrive during the service of the batch size } r\}, \\ &= \int_0^{\infty} \frac{e^{-\lambda t}(\lambda t)^j}{j!} s_r(t)dt, \ j \ge 0, \quad a \le r \le b, \end{aligned} \tag{3.31}$$

$$\begin{aligned} w_j^{(k)} &= Pr\{j \text{ customers arrive during the } k^{th} \text{ type of vacation }\}, \\ &= \int_0^{\infty} \frac{e^{-\lambda t}(\lambda t)^j}{j!} v_k(t)dt, \quad j \ge 0, \ 0 \le k \le a-1, \end{aligned} \tag{3.32}$$

such that $\sum\limits_{j=0}^{\infty} m_j^{(r)}$=1, $\sum\limits_{j=0}^{\infty} w_j^{(k)}$=1.

The PGF (probability generating function) of $m_j^{(r)}$ and $w_j^{(k)}$ are defined as follows:

$$M^{(r)}(z) = \sum_{j=0}^{\infty} m_j^{(r)} z^j = \tilde{S}_r(\lambda - \lambda z), \quad a \le r \le b, \quad |z| \le 1, \tag{3.33}$$

$$N^{(k)}(z) = \sum_{j=0}^{\infty} w_j^{(k)} z^j = \tilde{V}_k(\lambda - \lambda z), \quad 0 \le k \le a-1, \quad |z| \le 1. \tag{3.34}$$

Lemma 3.1. *The joint probabilities* $P^+_{n,r}$, $Q_n^{[k]+}$, $P_{n,r}(0)$ *and* $Q_n^{[k]}(0)$ $(a \le r \le b, 0 \le k \le a-1)$ *are associated with the following relation*

$$P^+_{n,r} = \sigma P_{n,r}(0), \quad n \ge 0, \tag{3.35}$$

$$Q_n^{[k]+} = \sigma Q_n^{[k]}(0), \quad n \ge k, \tag{3.36}$$

where $\sigma^{-1} = \sum\limits_{m=0}^{\infty}\sum\limits_{r=a}^{b} P_{m,r}(0) + \sum\limits_{m=0}^{\infty}\sum\limits_{k=0}^{min(m,a-1)} Q_m^{[k]}(0).$

Proof. Since $P^+_{n,r}$ and $Q_n^{[k]+}$ are proportional to $P_{n,r}(0)$ and $Q_n^{[k]}(0)$, respectively, using the concept of Bayes' theorem and $\sum\limits_{n=0}^{\infty}\sum\limits_{r=a}^{b} P^+_{n,r} + \sum\limits_{n=0}^{\infty}\sum\limits_{k=0}^{min(n,a-1)} Q_n^{[k]+}=1$ the desired outcome is obtained. ☐

Lemma 3.2. *The value* σ^{-1} *is given by*

$$\sigma^{-1} = \frac{1-(1-\delta)\sum\limits_{n=0}^{a-1} R_n}{s_b \sum\limits_{n=b+1}^{\infty}\left(P_n^+ + Q_n^+\right) + \sum\limits_{n=a}^{b}\left(P_n^+ + Q_n^+\right)s_n + \sum\limits_{n=0}^{a-1}\left(P_n^+ x_n + (1-\delta)Q_n^+ s_a + \delta Q_n^+ x_n\right)}. \tag{3.37}$$

Proof. Using (3.1) and (3.2), one can get

$$\lambda R_n = \sum_{m=0}^{n}\sum_{k=0}^{m} Q_m^{[k]}(0), \quad 0 \le n \le a-1. \tag{3.38}$$

Using (3.38) in (3.15), one can get

$$(\lambda-\theta)\tilde{P}_{0,a}(\theta) = \sum_{n=0}^{a-1}\sum_{k=0}^{n} Q_n^{[k]}(0)\tilde{S}_a(\theta) + \left(\sum_{k=0}^{a-1} Q_a^{[k]}(0) + \sum_{r=a}^{b} P_{a,r}(0)\right)\tilde{S}_a(\theta) - P_{0,a}(0). \tag{3.39}$$

Summing (3.39) and (3.16)-(3.20), one can obtain

$$\begin{aligned}\sum_{m=0}^{\infty}\sum_{r=a}^{b}P_{m,r}(\theta)+\sum_{m=0}^{\infty}\sum_{k=0}^{min(m,a-1)}Q_m^{[k]}(\theta) &= \frac{1-\tilde{S}_b(\theta)}{\theta}\sum_{n=b+1}^{\infty}\left(\sum_{r=a}^{b}P_{n,r}(0)+\sum_{k=0}^{a-1}Q_n^{[k]}(0)\right)\\ &+\sum_{n=a}^{b}\left(\sum_{r=a}^{b}P_{n,r}(0)+\sum_{k=0}^{a-1}Q_n^{[k]}(0)\right)\frac{1-\tilde{S}_n(\theta)}{\theta}\\ &+\sum_{n=0}^{a-1}\left(\sum_{r=a}^{b}P_{n,r}(0)+\delta\sum_{k=0}^{n}Q_n^{[k]}(0)\right)\frac{1-\tilde{V}_n(\theta)}{\theta}\\ &+(1-\delta)\frac{1-\tilde{S}_a(\theta)}{\theta}\sum_{n=0}^{a-1}\sum_{k=0}^{n}Q_n^{[k]}(0). \qquad (3.40)\end{aligned}$$

Taking $\theta \to 0$ in (3.40) and using L'Hôspital's rule, the normalization condition $(1-\delta)\sum_{n=0}^{a-1}R_n+\sum_{n=0}^{\infty}\sum_{r=a}^{b}P_{n,r}+\sum_{n=0}^{\infty}\sum_{k=0}^{min(n,a-1)}Q_n^{[k]}=1$, after few algebraic calculation the desired outcome is obtained. □

Lemma 3.3.

$$Q^{+}(z)=\sum_{k=0}^{a-1}\sum_{n=k}^{\infty}Q_n^{[k]+}z^n=\sum_{k=0}^{a-1}(P_k^{+}+\delta Q_k^{+})N^{(k)}(z)z^k \qquad (3.41)$$

Proof. Multiplying (3.19) and (3.20) by the appropriate power of z and y, and adding them over the range of n and k, the following expression is obtained

$$\begin{aligned}(\lambda-\theta-\lambda z)Q(z,y,\theta) &= \sum_{k=0}^{a-1}\left(\sum_{r=a}^{b}P_{k,r}(0)+\delta\sum_{j=0}^{k}Q_j^{[k]}(0)\right)\tilde{V}_k(\theta)z^ky^k\\ &-\sum_{k=0}^{a-1}\sum_{n=k}^{\infty}Q_n^{[k]}(0)z^ny^k. \qquad (3.42)\end{aligned}$$

Now substituting $\theta{=}\lambda-\lambda z$ in (3.42) and using Lemma 3.1, (3.22) and (3.24) the following expression is obtained

$$\sum_{k=0}^{a-1}\sum_{n=k}^{\infty}Q_n^{[k]+}z^ny^k=\sum_{k=0}^{a-1}(P_k^{+}+\delta Q_k^{+})N^{(k)}(z)z^ky^k. \qquad (3.43)$$

Substituting $y=1$ in (3.43) the desired outcome is acquired. □

Lemma 3.4.

$$Q_n^{[k]+} = \left(P_k^+ + \delta\sum_{j=0}^{k} Q_k^{[j]+}\right) w_{n-k}^{(k)}, \quad 0 \le k \le a-1, n \ge k. \tag{3.44}$$

Proof. From (3.43) collecting the coefficients of y^k $(0 \le k \le a-1)$ the following expression is obtained,

$$\sum_{n=k}^{\infty} Q_n^{[k]+} z^n = (P_k^+ + \delta Q_k^+) N^{(k)}(z) z^k. \tag{3.45}$$

Now using (3.34) and (3.24), in (3.45) and collecting the coefficients of z^n $(n \ge k)$ desired result (3.44) is obtained. □

Hence from Lemma 3.4 it is clear that once P_k^+ $(0 \le k \le a-1)$ are known, the joint probabilities $Q_n^{[k]+}$ $(0 \le k \le a-1, n \ge k)$ are all known.

Now, the main focus is to find the bivariate generating function for the queue size and batch size with the server at service completion epoch. Towards this end, multiplying (3.15)-(3.18) by appropriate power of z and y and adding them over the range of n and r one can get,

$$\begin{aligned}(\lambda-\theta-\lambda z)P(z,y,\theta) \;=\;& (1-\delta)\sum_{n=0}^{a-1}\sum_{k=0}^{n} Q_n^{[k]}(0)\tilde{S}_a(\theta)y^a \\ &+\sum_{r=a}^{b}\left(\sum_{k=0}^{a-1} Q_r^{[k]}(0) + \sum_{j=a}^{b} P_{r,j}(0)\right)\tilde{S}_r(\theta)y^r \\ &+\sum_{n=b+1}^{\infty}\left(\sum_{k=0}^{a-1} Q_n^{[k]}(0) + \sum_{r=a}^{b} P_{n,r}(0)\right)\tilde{S}_b(\theta)z^{n-b}y^b \\ &-\sum_{n=0}^{\infty}\sum_{r=a}^{b} P_{n,r}(0)z^n y^r.\end{aligned} \tag{3.46}$$

Substituting $\theta = (\lambda - \lambda z)$ in the above expression and using Lemma 3.1, (3.22), (3.24) and (3.26), one have

$$\begin{aligned}P^+(z,y) \;=\;& (1-\delta)y^a M^{(a)}(z)\sum_{n=0}^{a-1} Q_n^+ + \sum_{r=a}^{b}\left(Q_r^+ + P_r^+\right)M^{(r)}(z)y^r \\ &+\sum_{n=b+1}^{\infty}\left(Q_n^+ + P_n^+\right)M^{(b)}(z)z^{n-b}y^b.\end{aligned} \tag{3.47}$$

Substituting $y = 1$ in (3.47) and using Lemma 3.3 and (3.27), the following result is obtained,

$$P^+(z) = \frac{\begin{array}{r}\Big\{\sum_{n=0}^{a-1}\big[M^{(b)}(z)(P_n^+ + \delta Q_n^+)\big(N^{(n)}(z)-1\big)z^n \\ +(1-\delta)Q_n^+\big(M^{(a)}(z)z^b - M^{(b)}(z)z^n\big)\big] \\ +\sum_{n=a}^{b-1}(Q_n^+ + P_n^+)\big(M^{(n)}(z)z^b - M^{(b)}(z)z^n\big)\Big\}\end{array}}{z^b - M^{(b)}(z)}. \quad (3.48)$$

Finally, using (3.48) in (3.47) after some algebraic manipulation the expression $P^+(z,y)$ is given by,

$$P^+(z,y) = \frac{\begin{array}{r}\sum_{n=0}^{a-1}\big[(1-\delta)Q_n^+\big(z^b y^a M^{(a)}(z) - y^b M^{(b)}(z)z^n\big) \\ +(1-\delta)M^{(a)}(z)M^{(b)}(z)\big(y^b - y^a\big)Q_n^+ \\ +y^b(P_n^+ + \delta Q_n^+)\big(N^{(n)}(z)-1\big)M^{(b)}(z)z^n\big] \\ +\sum_{n=a}^{b-1}\big(Q_n^+ + P_n^+\big)\big(z^b y^n M^{(n)}(z) \\ +\big(y^b - y^n\big)M^{(n)}(z)M^{(b)}(z) - y^b M^{(b)}(z)z^n\big)\end{array}}{z^b - M^{(b)}(z)}. \quad (3.49)$$

Remark 1: The bivariate generation function (3.45) of the queue size and the type of vacation at vacation termination epoch, and the bivariate generating function of the queue size and batch size with the server at service completion epoch, i.e., (3.49) have not analyzed in the literature so far.

It may be observed from (3.49) that the generating function $P^+(z,y)$ is in the compact form with b unknowns $\{P_n^+\}_{n=0}^{b-1}$. One may further note from Lemma 3.4 that once P_k^+ $(0 \le k \le a-1)$ are known then the joint probabilities $Q_n^{[k]+}$ $(0 \le k \le a-1)$ are completely known. Hence, in order to find $P_{n,r}^+$ $(a \le r \le b, n \ge 0)$ and $Q_n^{[k]+}$ $(0 \le k \le a-1, n \ge k)$, one should first find the unknowns $\{P_n^+\}_{n=0}^{b-1}$. Next section is dedicated in getting these unknowns $\{P_n^+\}_{n=0}^{b-1}$.

3.3.2 Procedure of getting the unknowns P_n^+ $(0 \le n \le b-1)$

Note that the unknowns P_n^+ $(0 \le n \le b-1)$, as appeared in (3.49), are the same as the unknowns which are appeared in (3.48). Using the Rouche's theorem one may conclude that, for $\rho < 1$, $z^b - M^{(b)}(z)$ has $(b-1)$ zeros, say $\hat{e}_1, \hat{e}_2, ..., \hat{e}_l$, with multiplicity $r_1, r_2, ..., r_l$, respectively, inside the unit circle $|z| = 1$ $\big($where $(l \le b-1)$ and $\sum_{i=1}^{l} r_i = (b-1)\big)$ and one simple zero, say, $z_b = 1$, on the unit circle $|z|$=1. Due to analyticity of (3.48) in $|z| \le 1$ these zeros are also the zeros of numerator of (3.48). Hence, from (3.48) $(b-1)$ linearly independent equations are given by,

$$\begin{aligned}\Bigg[\frac{d^{i-1}}{dz^{i-1}}\Bigg(\sum_{n=0}^{a-1}\big\{M^{(b)}(z)(P_n^+ + \delta Q_n^+)\big(N^{(n)}(z)-1\big)z^n \\ +(1-\delta)Q_n^+\big(M^{(a)}(z)z^b - M^{(b)}(z)z^n\big)\big\} \\ +\sum_{n=a}^{b-1}(Q_n^+ + P_n^+)\big(M^{(n)}(z)z^b - M^{(b)}(z)z^n\big)\Bigg)\Bigg]_{z=\hat{e}_j} = 0, \\ 1 \le j \le l \ \&\ 1 \le i \le r_j,\end{aligned} \tag{3.50}$$

where $\frac{d^0}{dz^0}h(z) \equiv h(z)$.

Now using (3.48), Lemma 3.3 and the normalization condition $P^+(1)+Q^+(1) = 1$, applying L'Hôspital's rule, the following expression is obtained

$$\begin{aligned}\sum_{n=0}^{a-1}\Big[(P_n^+ + \delta Q_n^+)(\lambda x_n + b - b\rho) + (1-\delta)Q_n^+(b-n)\Big] \\ +\sum_{n=a}^{b-1}\Bigg[\Bigg(Q_n^+ + P_n^+\Bigg)\Bigg(b - n + \lambda(s_n - s_b)\Bigg)\Bigg] = b - b\rho.\end{aligned} \tag{3.51}$$

Hence, (3.50) and (3.51) together forms non-homogenous system of b linearly independent equations in b unknowns P_n^+ $(0 \le n \le b-1)$, solving them P_n^+ $(0 \le n \le b-1)$ are uniquely determined.

Theorem 3.5. *The joint probabilities* $P_{n,r}^+$ $(1 \le a \le r \le b-1, n \ge 0)$ *are given by*

$$P_{n,a}^+ = \Bigg((1-\delta)\sum_{m=0}^{a-1} Q_m^+ + Q_a^+ + P_a^+\Bigg)m_n^{(a)}, \tag{3.52}$$

$$P_{n,r}^+ = \Bigg(Q_r^+ + P_r^+\Bigg)m_n^{(r)}, \quad a+1 \le r \le b-1. \tag{3.53}$$

Proof. Using (3.26) in (3.49) and then accumulating the coefficients of y^r $(1 \leq a \leq r \leq b-1)$, the following expression is obtained

$$\text{coefficient of } y^a : \sum_{n=0}^{\infty} P_{n,a}^+ z^n = \left((1-\delta)\sum_{m=0}^{a-1} Q_m^+ + Q_a^+ + P_a^+\right) M^{(a)}(z). \tag{3.54}$$

$$\text{coefficient of } y^r : \sum_{n=0}^{\infty} P_{n,r}^+ z^n = \left(Q_r^+ + P_r^+\right) M^{(r)}(z), a+1 \leq r \leq b-1. \tag{3.55}$$

Using (3.33) in (3.54) and (3.55), and then accumulating the coefficients of z^n, the desired result is obtained. □

Now the next objective is to find the remaining joint probabilities $P_{n,b}^+$ $(n \geq 0)$. To get these, using (3.26) in (3.49), and then accumulating the coefficient of y^b, one can obtain

$$\sum_{n=0}^{\infty} P_{n,b}^+ z^n = \frac{\begin{aligned} M^{(b)}(z)\Bigg\{ \sum_{n=0}^{a-1} \Big[&(P_n^+ + \delta Q_n^+)(N^{(n)}(z)-1)z^n \\ &+(1-\delta)Q_n^+(M^{(a)}(z)-z^n)\Big] \\ &+\sum_{n=a}^{b-1}(Q_n^+ + P_n^+)(M^{(n)}(z)-z^n)\Bigg\} \end{aligned}}{z^b - M^{(b)}(z)}. \tag{3.56}$$

To derive $P_{n,b}^+$ $(n \geq 0)$ completely it is necessary to invert the right hand side of (3.56) and towards this direction assume that LST of service time and the vacation time distribution as $\tilde{S}_r(\theta) = \frac{P_r(\theta)}{Q_r(\theta)}$, $a \leq r \leq b$, and $\tilde{V}_k(\theta) = \frac{P_k(\theta)}{Q_k(\theta)}$, $0 \leq k \leq a-1$, respectively. Here one should note that the logic behind the consideration of $\tilde{S}_r(\theta)$ and $\tilde{V}_k(\theta)$ in rational form is that, in most of real life queueing model service (vacation) time distribution can be expressed as rational function. However, the transcendental LST (for example LST of deterministic distribution) can be handle using Padé approximation.

Now substituting $M^{(r)}(z) = \tilde{S}_r(\lambda-\lambda z) = \frac{P_r(\lambda-\lambda z)}{Q_r(\lambda-\lambda z)}$, $a \leq r \leq b$ and $N^{(k)}(z) = \tilde{V}_k(\lambda-\lambda z) = \frac{P_k(\lambda-\lambda z)}{Q_k(\lambda-\lambda z)}$, $0 \leq k \leq a-1$, in the right hand side of (3.56), after some simplification (3.56) can be converted as,

$$\sum_{n=0}^{\infty} P_{n,b}^+ z^n = \frac{U(z)}{D(z)}, \tag{3.57}$$

where $U(z)$ and $D(z)$ are polynomials of degree $\bar{u}$ and d, respectively, and $D(z)$ is a monic polynomial (i.e., the coefficient of z^d in $D(z)$ is 1). To extract the joint probabilities $P_{n,b}^+$

$(n \geq 0)$ the zeros of $D(z)$ of modules greater than one must be known. Due to analyticity of (3.57), in $|z| \leq 1$, the zeros of $D(z)$ which lie inside and on the unit circle are also the zeros of $U(z)$, therefore, they can not play any role in extracting $P^+_{n,b}$ $(n \geq 0)$. Towards this end, let $\gamma_1, \gamma_2, ..., \gamma_l$ be the zeros of $D(z)$ of modules greater than one with multiplicity $\tau_1, \tau_2, ..., \tau_l$, respectively, such that $\sum\limits_{j=1}^{l} \tau_j \leq d$. Now two cases may arise depending on d and $\bar{u}$ which are discussed here below

Case A: $d \leq \bar{u}$
Applying the method of partial fraction on (3.57), $\sum\limits_{n=0}^{\infty} P^+_{n,b} z^n$ is given by,

$$\sum_{n=0}^{\infty} P^+_{n,b} z^n = \sum_{i=0}^{\bar{u}-d} \varrho_i z^i + \sum_{j=1}^{l} \sum_{i=1}^{\tau_j} \frac{B_{i,j}}{(z-\gamma_j)^{\tau_j - i+1}}, \tag{3.58}$$

where

$$B_{i,j} = \frac{1}{(i-1)!}\left[\frac{d^{i-1}}{dz^{i-1}}\left(\frac{U(z)\frac{d^{\tau_j}}{dz^{\tau_j}}(z-\gamma_j)^{\tau_j}}{\frac{d^{\tau_j}}{dz^{\tau_j}}(D(z))}\right)\right]_{z=\gamma_j}, j = 1, 2, ..., l, \; i = 1, 2, ..., \tau_j. \tag{3.59}$$

Accumulating the coefficients of z^n $(n \geq 0)$ from (3.58) one can obtain

$$P^+_{n,b} = \begin{cases} \varrho_n + \sum\limits_{j=1}^{l}\sum\limits_{i=1}^{\tau_j} \frac{B_{i,j}}{(-1)^{\tau_j-i+1}\gamma_j^{\tau_j+n-i+1}}\binom{\tau_j-i+n}{\tau_j-i}, & 0 \leq n \leq \bar{u}-d, \\ \sum\limits_{j=1}^{l}\sum\limits_{i=1}^{\tau_j} \frac{B_{i,j}}{(-1)^{\tau_j-i+1}\gamma_j^{\tau_j+n-i+1}}\binom{\tau_j-i+n}{\tau_j-i}, & n > \bar{u}-d. \end{cases}$$

Case B: $d > \bar{u}$
Eliminating first summation of the right-hand side of (3.58) and hence $P^+_{n,b}$ is given by,

$$P^+_{n,b} = \sum_{j=1}^{l}\sum_{i=1}^{\tau_j} \frac{B_{i,j}}{(-1)^{\tau_j-i+1}\gamma_j^{\tau_j+n-i+1}}\binom{\tau_j - i + n}{\tau_j - i}, \quad n \geq 0. \tag{3.60}$$

Thus, the analysis of the joint probabilities $P^+_{n,r}$ $(a \leq r \leq b, n \geq 0)$ at service completion epoch and $Q_n^{[k]+}$ $(0 \leq k \leq a-1, n \geq k)$ at the vacation termination epoch have been completed. Now the main objective is centered for getting these probabilities at arbitrary epoch.

Remark 2. By inverting the PGF $M^{(r)}(z)$ and $N^{(k)}(z)$ one can easily compute $m_j^{(r)}$, $a \leq r \leq b, \; j \geq 0$ and $w_j^{(k)}$, $0 \leq k \leq a-1, \; j \geq 0$.

Theorem 3.6. *The probabilities* R_n $(0 \le n \le a-1)$*,* $P_{n,r}$ $(n \ge 0, a \le r \le b)$ *and* $Q_n^{[k]}$ $(n \ge k, 0 \le k \le a-1)$ *are given by,*

$$R_n = \frac{\sum_{m=0}^{n} Q_m^+}{E}, \quad 0 \le n \le a-1 \quad (for\ SV), \tag{3.61}$$

$$P_{n,a} = \frac{(1-\delta)\sum_{m=0}^{a-1} Q_m^+ + Q_a^+ - \sum_{j=0}^{n} P_{j,a}^+ + P_a^+}{E}, \quad n \ge 0, \tag{3.62}$$

$$P_{n,r} = \frac{Q_r^+ + P_r^+ - \sum_{j=0}^{n} P_{j,r}^+}{E}, \quad n \ge 0, \ a+1 \le r \le b-1, \tag{3.63}$$

$$P_{n,b} = \frac{\sum_{j=0}^{n} (Q_{b+j}^+ + P_{b+j}^+ - P_{j,b}^+)}{E}, \quad n \ge 0, \tag{3.64}$$

$$Q_n^{[k]} = \frac{P_k^+ + \delta Q_k^+ - \sum_{j=k}^{n} Q_j^{[k]+}}{E}, \quad n \ge k, \ 0 \le k \le a-1, \tag{3.65}$$

where $E = \lambda g + (1-\delta)\sum_{i=0}^{a-1}(a-i)Q_i^+$,

$g = s_b \sum_{n=b+1}^{\infty} \big(P_n^+ + Q_n^+\big) + \sum_{n=a}^{b} \big(P_n^+ + Q_n^+\big)s_n + \sum_{n=0}^{a-1} \big(P_n^+ x_n + (1-\delta)Q_n^+ s_a + \delta Q_n^+ x_n\big).$

Proof. Dividing (3.1) by σ^{-1} and using Lemma 3.1, Lemma 3.2 and (3.24), one can get

$$R_0 = \frac{(1 - \sum_{n=0}^{a-1} R_n)Q_0^+}{\lambda g}. \tag{3.66}$$

Similarly, from (3.38), one can obtain

$$R_n = \frac{(1 - \sum_{i=0}^{a-1} R_i) \sum_{m=0}^{n} Q_m^+}{\lambda g}, \quad 0 \le n \le a-1. \tag{3.67}$$

Using (3.66) in (3.67), one have

$$R_n = \frac{R_0}{Q_0^+} \sum_{m=0}^{n} Q_m^+, \quad 0 \le n \le a-1. \tag{3.68}$$

Using (3.68) in (3.66) after some algebraic manipulation, one can get

$$R_0 = \frac{Q_0^+}{\lambda g + \sum_{i=0}^{a-1}(a-i)Q_i^+}. \tag{3.69}$$

Using (3.69) in (3.68), one can obtain

$$R_n = \frac{\sum_{m=0}^{n} Q_m^+}{\lambda g + \sum_{i=0}^{a-1}(a-i)Q_i^+}, \quad 0 \le n \le a-1. \tag{3.70}$$

Setting θ=0 in (3.15)-(3.20), one have

$$\lambda P_{0,a} = (1-\delta)\sum_{m=0}^{a-1}\sum_{k=0}^{m} Q_m^{[k]}(0) + \sum_{k=0}^{a-1} Q_a^{[k]}(0) + \sum_{r=a}^{b} P_{a,r}(0) - P_{0,a}(0), \tag{3.71}$$

$$\lambda P_{0,r} = \sum_{k=0}^{a-1} Q_r^{[k]}(0) + \sum_{j=a}^{b} P_{r,j}(0) - P_{0,r}(0), \quad a+1 \le r \le b, \tag{3.72}$$

$$\lambda P_{n,r} = \lambda P_{n-1,r} - P_{n,r}(0), \quad n \ge 1, \ a \le r \le b-1, \tag{3.73}$$

$$\lambda P_{n,b} = \lambda P_{n-1,b} + \sum_{k=0}^{a-1} Q_{n+b}^{[k]}(0) + \sum_{r=a}^{b} P_{n+b,r}(0) - P_{n,b}(0), \quad n \ge 1, \tag{3.74}$$

$$\lambda Q_k^{[k]} = \sum_{r=a}^{b} P_{k,r}(0) + \delta\sum_{j=0}^{k} Q_k^{[j]}(0) - Q_k^{[k]}(0), \quad 0 \le k \le a-1, \tag{3.75}$$

$$\lambda Q_n^{[k]} = \lambda Q_{n-1}^{[k]} - Q_n^{[k]}(0), \quad n \ge k+1, \ 0 \le k \le a-1. \tag{3.76}$$

Dividing (3.71) and (3.72) by σ^{-1}, respectively, and then using Lemma 3.1, Lemma 3.2, (3.22) and (3.24), one can get

$$P_{0,a} = \frac{\left(1-(1-\delta)\sum_{i=0}^{a-1} R_i\right)\left((1-\delta)\sum_{m=0}^{a-1} Q_m^+ + Q_a^+ + P_a^+ - P_{0,a}^+\right)}{\lambda g}, \tag{3.77}$$

$$P_{0,r} = \frac{\left(1-(1-\delta)\sum_{i=0}^{a-1} R_i\right)\left(Q_r^+ + P_r^+ - P_{0,r}^+\right)}{\lambda g}. \tag{3.78}$$

Using (3.66) and (3.69) in (3.77)-(3.78), respectively, one can get

$$P_{0,a} = \frac{\left((1-\delta)\sum_{m=0}^{a-1} Q_m^+ + Q_a^+ + P_a^+ - P_{0,a}^+\right)}{\lambda g + (1-\delta)\sum_{i=0}^{a-1}(a-i)Q_i^+}, \tag{3.79}$$

$$P_{0,r} = \frac{\left(Q_r^+ + P_r^+ - P_{0,r}^+\right)}{\lambda g + (1-\delta)\sum_{i=0}^{a-1}(a-i)Q_i^+}, \quad a+1 \le r \le b. \tag{3.80}$$

Applying similar process for (3.73)-(3.74) and using (3.79) and (3.80), one can obtain

$$P_{n,a} = \frac{\left((1-\delta)\sum_{m=0}^{a-1} Q_m^+ + Q_a^+ + P_a^+ - \sum_{j=0}^{n} P_{j,a}^+\right)}{\lambda g + (1-\delta)\sum_{i=0}^{a-1}(a-i)Q_i^+}, \quad n \ge 1, \tag{3.81}$$

$$P_{n,r} = \frac{\left(Q_r^+ + P_r^+ - \sum_{j=0}^{n} P_{j,r}^+\right)}{\lambda g + (1-\delta)\sum_{i=0}^{a-1}(a-i)Q_i^+}, \quad n \ge 1, \ a+1 \le r \le b-1, \tag{3.82}$$

$$P_{n,b} = \frac{\sum_{j=0}^{n}(Q_{b+j}^+ + P_{b+j}^+ - P_{j,b}^+)}{\lambda g + (1-\delta)\sum_{i=0}^{a-1}(a-i)Q_i^+}, \quad n \ge 1. \tag{3.83}$$

Combining (3.79) and (3.81) equation (3.62) is obtained. Combining (3.80) and (3.82) over the range r, (3.63) and (3.64) are obtained.
Dividing (3.75) by σ^{-1} and using Lemma 3.1, Lemma 3.2 and (3.22), one can get

$$Q_k^{[k]} = \frac{\left(1-(1-\delta)\sum_{i=0}^{a-1} R_i\right)(P_k^+ + \delta Q_k^+ - Q_k^{[k]+})}{\lambda g}, \quad 0 \le k \le a-1. \tag{3.84}$$

Using (3.67) and (3.69) in (3.84), one have

$$Q_k^{[k]} = \frac{(P_k^+ + \delta Q_k^+ - Q_k^{[k]+})}{\lambda g + (1-\delta)\sum_{i=0}^{a-1}(a-i)Q_i^+}, \quad 0 \le k \le a-1. \tag{3.85}$$

Applying similar process for (3.76) after some algebraic manipulation, the following expression is obtained

$$Q_n^{[k]} = \frac{(P_k^+ + \delta Q_k^+ - \sum_{j=k}^{n} Q_j^{[k]+})}{\lambda g + (1-\delta)\sum_{i=0}^{a-1}(a-i)Q_i^+}, \quad n \geq k+1, \ 0 \leq k \leq a-1. \tag{3.86}$$

Combining (3.85)-(3.86) equation (3.65) is obtained. Now by back substitution method the joint probabilities $Q_n^{[k]}$ $(0 \leq k \leq a-1, \ n \geq k)$ are obtained from (3.65). □

3.4 Marginal Probabilities

Marginal probabilities that can be found from the earlier results are as follows:

1. Queue size distribution is

$$P_n^{queue} = \begin{cases} (1-\delta)R_n + \sum_{r=a}^{b} P_{n,r} + \sum_{k=0}^{min(n,a-1)} Q_n^{[k]}, \ 0 \leq n \leq a-1, \\ \sum_{r=a}^{b} P_{n,r} + \sum_{k=0}^{min(n,a-1)} Q_n^{[k]}, \ n \geq a. \end{cases}$$

2. Probability that the server is in a dormant state $(= P^{dor}) = (1-\delta)\sum_{n=0}^{a-1} R_n$.

3. Probability of the batch size with the server is r $(= P_r^{ser}) = \sum_{n=0}^{\infty} P_{n,r}, \ a \leq r \leq b$.

4. Probability that the server is in k^{th} type of vacation $(= Q_{vac}^{[k]}) = \sum_{n=k}^{\infty} Q_n^{[k]}, \ 0 \leq k \leq a-1$.

5. Probability that the server is busy $(= P_{busy}) = \sum_{r=a}^{b}\sum_{n=0}^{\infty} P_{n,r}$.

6. Probability that the server is on vacation $(= Q_{vac}) = \sum_{k=0}^{a-1}\sum_{n=k}^{\infty} Q_n^{[k]}$.

7. Probability that the server is idle $(= P_{idle}) = (1-\delta)P^{dor} + Q_{vac}$.

3.4.1 Some results as particular case

In this section, some useful results are presented which seem to be new in the literature. These results derive from the results obtained for the considered queuing model as particular cases.

- The present chapter analyzes the batch size dependent service $M/G_r^{(a,b)}/1$ queue and queue size dependent SV (MV). Hence, If $\mu_i = \mu$, $a \le i \le b$; $\nu_i = \nu$, $0 \le i \le a-1$ then the considered model is reduces to batch size independent bulk service $M/G^{(a,b)}/1$ queue with SV and MV where the vacation time is also not dependent to queue size at vacation initiation epoch. This reduced model is analyzed by Sikdar and Gupta [87] for SV only, and they obtained the queue size distributions at various epoch. For verification, substituting $\delta = 0$, $M^{(r)}(z) = \tilde{M}(z)$, $(a \le r \le b)$ and $N^{(k)}(z) = \tilde{N}(z)$ $(0 \le k \le a-1)$ in (3.48) and (3.41) the service completion epoch (vacation completion epoch) generating function for queue size distribution are obtained as follows,

$$P^+(z) = \frac{\tilde{M}(z)\Big[\sum_{n=0}^{a-1}\big(P_n^+\big(\tilde{N}(z)-1\big)z^n + Q_n^+(z^b - z^n)\big) + \sum_{n=a}^{b-1}(Q_n^+ + P_n^+)\big(z^b - z^n\big)\Big]}{z^b - \tilde{M}(z)}, \tag{3.87}$$

$$Q^+(z) = \tilde{N}(z)\sum_{k=0}^{a-1} P_k^+ z^k, \tag{3.88}$$

which matches exactly with the results obtained in Sikdar and Gupta [87, eq(43), eq(44), page 953]. Further, from (3.43) and (3.49) the following expressions are obtained

$$Q^+(z,y) = \tilde{N}(z)\sum_{k=0}^{a-1}(P_k^+ + \delta Q_k^+)z^k y^k, \tag{3.89}$$

$$P^{+}(z,y)=\frac{\begin{aligned}&\tilde{M}(z)\sum_{n=0}^{a-1}\Big[(1-\delta)Q_n^{+}\big(z^by^a-y^bz^n\big)+(1-\delta)\tilde{M}(z)\big(y^b-y^a\big)Q_n^{+}\\&\qquad+y^b(P_n^{+}+\delta Q_n^{+})\big(\tilde{N}(z)-1\big)z^n\Big]\\&+\tilde{M}(z)\sum_{n=a}^{b-1}\big(Q_n^{+}+P_n^{+}\big)\big(z^by^n+\big(y^b-y^n\big)\tilde{M}(z)-y^bz^n\big)\end{aligned}}{z^b-\tilde{M}(z)}, \quad (3.90)$$

which are bivariate generating functions at vacation termination epoch, and service completion epoch, respectively, and is not available so far in the literature. From these bivariate generating functions ((3.89) and (3.90)), applying similar procedure as presented in this chapter, one can obtain the complete joint probabilities of the queue size and batch size with the server, also, the joint probabilities of the queue size and type of vacation at any time point.

- If $a=1$ then the presented model converts to $M/G_r^{(1,b)}/1$ queue with SV (MV). According to the analysis done in this chapter, one can extract joint probabilities at different epochs. Such results are also not available directly in the literature.
- If $a=b$ then the present model reduces to $M/G^b/1$ queue with queue size dependent SV and MV.

3.5 Performance measure

The performance measure is the values that collects the information of the system and helps the manager to run the system smoothly. The present section covers significant performance measures of the considered model.

1. Expected queue size (L_q) is given by
   $L_q=(1-\delta)\sum_{n=0}^{a-1}nR_n+\sum_{n=0}^{\infty}\sum_{r=a}^{b}nP_{n,r}+\sum_{k=0}^{a-1}\sum_{n=k}^{\infty}nQ_n^{[k]}$
   $=(1-\delta)\sum_{n=0}^{a-1}nP_n^{queue}+\sum_{n=a-\delta a}^{\infty}nP_n^{queue}.$
2. Expected system size (L_s) is given by
   $L_s=(1-\delta)\sum_{n=0}^{a-1}nR_n+\sum_{n=0}^{\infty}\sum_{r=a}^{b}(n+r)P_{n,r}+\sum_{k=0}^{a-1}\sum_{n=k}^{\infty}nQ_n^{[k]}.$
3. Expected waiting time of a customer in the queue (W_q) is given by
   $W_q=\frac{L_q}{\lambda}.$

4. Expected waiting time of a customer in the system (W_s) is given by
   $W_s = \frac{L_s}{\lambda}$.

5. Expected batch size with the server when server is busy (L^{ser}) is given by
   $L^{ser} = \sum_{r=a}^{b} (rP_r^{ser}/P_{busy})$.

6. Expected type of vacation taken by server when server is in vacation (L^{vac}) is given by
   $L^{vac} = \sum_{k=0}^{a-1} (kQ_{vac}^{[k]}/Q_{vac})$.

3.6 Numerical results

In this section, a variety of numerical results are presented to show the behavior of the performance measures of the model under study, using graphs and tables. In this connection, first consider the example of a sugarcane mill, which is presented in Chapter 2 (Section 2.6), however, with proper modifications as per considered model. This example will reflect the more original scenario of the sugarcane mill example and also the real life applicability of the considered model. Consider that the sugarcane machine can takes 3 to 6 packets of sugarcane for producing juice. After a production if the machine finds 3 or more packets in the queue then it produces juice as per GBS rule, i.e., machine takes l=min$(r, 6)$ packets for producing the juice with service rate μ_l, (μ_3= 5.16666, μ_4=3.87500, μ_5=3.100000, and μ_6=2.583333) otherwise, the machine performs either 0^{th} type of vacation or 1^{th} type of vacation or 2^{th} type of vacation. The service (juice producing) time and the vacation time follow E_3 distribution and exponential distribution, respectively. In the 0^{th} type of vacation, the machine removes waste, checks the machinery parts, and purifies the extracted juice assembled in the containers, however, in 1^{th} or 2^{th} type of vacation, it checks the machinery parts. Assume that the packets are arriving with rate λ= 4.5 following the Poisson manner. Then the following results are observed, where ν_0=1.3, ν_1=1.7, and ν_2=1.9. (i.e., for the case of queue size dependent vacation (QSDV)).

Average packet (customer) size in the queue	2.817 (for the case of SV)	3.373 (for the case of MV)
Average waiting time of a packet in the queue	0.626 (for the case of SV)	0.749 (for the case of MV)

The following results are observed for the case of queue size independent vacation (QSIV) with $\nu_0 = \nu_1 = \nu_2 = 1.3$.

Average packet (customer) size in the queue	3.334 (for the case of SV)	4.036 (for the case of MV)
Average waiting time of a packet in the queue	0.740 (for the case of SV)	0.897 (for the case of MV)

From the above findings one can conclude that, for this particular example, the average queue size (waiting time) for QSDV is less than the average queue size (waiting time) for QSIV. Hence, the consideration of QSDV makes the model more efficient than QSIV.

For further justification of the considered model graphically, a comparison between QSDV and QSIV are presented (see, Figure 3.1 to Figure 3.7).

For the comparison purpose the following two cases are considered.
Case 1. The QSDV rates are taken as $\nu_k = (k+1)^2 0.85, \ \ 0 \leq k \leq a-1$.
Case 2. The QSIV rates are taken as $\nu_k = \nu_0, \ \ 0 \leq k \leq a-1$.

In Case 1, the vacation rates are chosen in such a way that as queue size at vacation initiation epoch increases the vacation time decreases accordingly. However, for Case 2, the vacation time remains constant irrespective of the queue size at vacation initiation epoch. In Figure 3.1 to Figure 3.7 the comparison between QSDV and QSIV for the $M/G_r^{(4,9)}/1$ queue with SV (MV) are prsented. Figure 3.1 to Figure 3.4 represent the influence of the arrival rate λ on L_q and W_q. The service time follows Erlang (E_2) distribution with batch size dependent service rate $\mu_r = \frac{\mu}{r}, \quad 4 \leq r \leq 9$ where $\mu = 12.5$, and the vacation time distributed exponentially. The above consideration holds for both the cases, i.e., Case 1 and Case 2. It is observed from Figure 3.1 to Figure 3.4 that as λ increases from 5.5 to 11.5 (i.e., ρ varies from 0.32 to 0.92) the performance measures L_q and W_q increases in both the cases SV (MV) and Case 1 and Case 2. Also, it is noted that for a fixed λ, L_q (W_q) is lower for Case 1 as compared to Case 2. Hence, the above studies reveals that assumption of QSDV policy in the batch size dependent service model is more efficient, as QSDV is minimizing L_q and W_q, in comparison to the QSIV policy.

In Figure 3.5, P^{dor} is depicted versus λ for SV. It is observed that as λ increases from 5.5 to 11.5, i.e., ρ varies from 0.32 to 0.92, P^{dor} decreases for both the cases. This is because the mean vacation time of the server is longer for Case 2 than in Case 1, which means that the returning time (from vacation) of the server in the system from the vacation is shorter in Case 1 as compared to Case 2. Hence, the reflexion in Figure 3.5 is on the expected direction in the sense that for a fixed value of λ, P^{dor} is greater for Case 1 in comparison to Case 2.

In Figure 3.6 and Figure 3.7 the effect of λ on Q_{vac} for SV and MV are presented, respectively. It is observed that the increase in λ from 5.5 to 11.5, results in a decrease in Q_{vac} for both the cases, which is on the expected line.

In Figure 3.8 the effect of λ versus Q_{vac} for SV (Case 1) and MV (Case 1) has plotted. The input parameters and the service (vacation) time distributions are taken exactly as taken for Figure 3.1 to Figure 3.7. From Figure 3.8 it is observed that as λ increases from 5.5 to 11.5, Q_{vac} decreases. Since increase in value of ρ from 0.32 to 0.92 results in increase in L_q significantly, i.e., the probability that the server is busy should also increase. The influence of λ on Q_{vac} presented in Figure 3.8 is on the expected direction, as for the case of SV, $Q_{vac} = P_{idle} - P^{dor}$ and for the case of MV $Q_{vac} = P_{idle}$.

In Figure 3.9 and Figure 3.10 the effect of λ on L_q and Q_{vac} are presented, respectively, for $M/G_r^{(3,7)}/1$ queue with SV for different vacation time distribution, e.g., (Exponential (M), Erlang (E_2) and Deterministic (D)) with rate $\nu_k = (k+1)^2 0.95$ $(0 \leq k \leq 2)$. Service time of each batch distributed exponentially with rate $\mu_r = \frac{\mu}{r}$, $3 \leq r \leq 7$ where $\mu = 7.5$, irrespective of vacation time distribution. From Figure 3.9 and Figure 3.10, as λ increases from 3 to 7, i.e., ρ increases from 0.4 to 0.93, L_q increases and Q_{vac} decreases. All the numerical experiments that are presented here in the form of graphs (Figure 3.1 to Figure 3.10) helps us to understand that whether the main objective of studying the proposed model is actually achieved or not. One can conclude from the explanation presented above that the considered model helps reducing congestion in the real life queues because of QSDV.

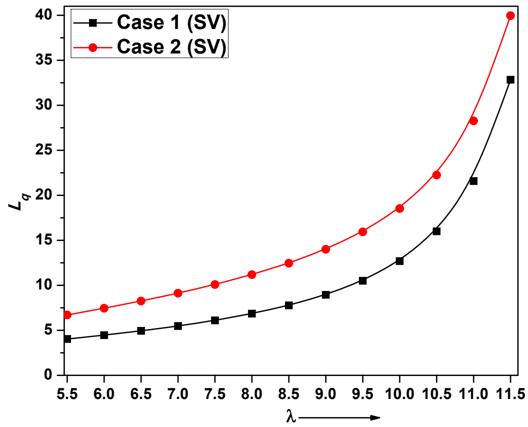

FIGURE 3.1: Effect of λ on L_q

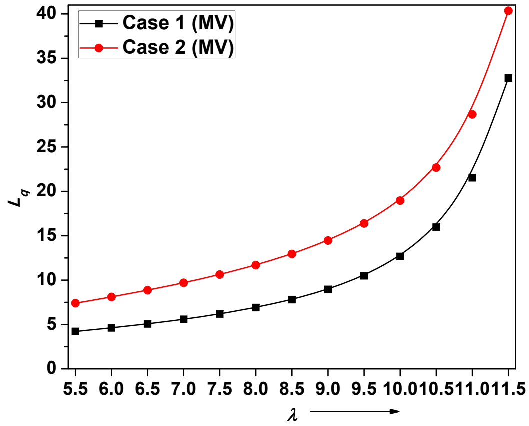

FIGURE 3.2: Effect of λ on L_q

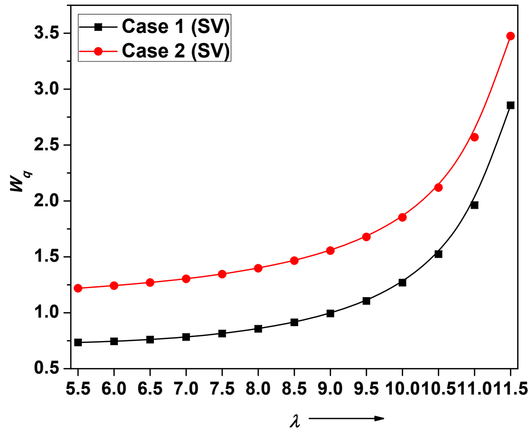

FIGURE 3.3: Effect of λ on W_q

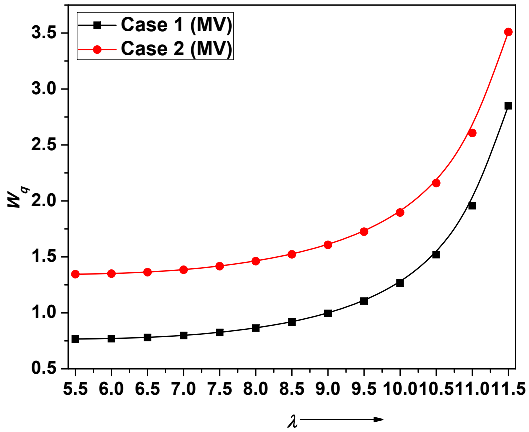

FIGURE 3.4: Effect of λ on W_q

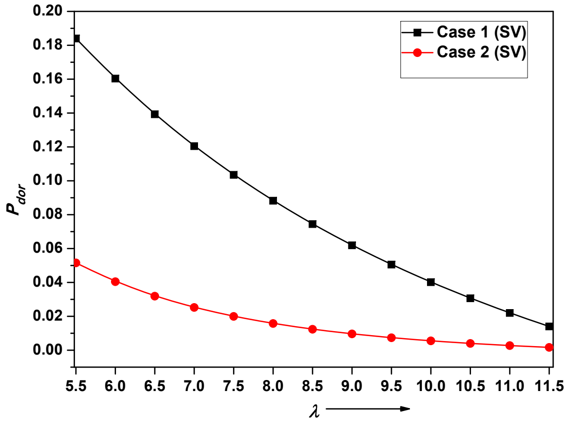

FIGURE 3.5: Effect of λ on P^{dor}

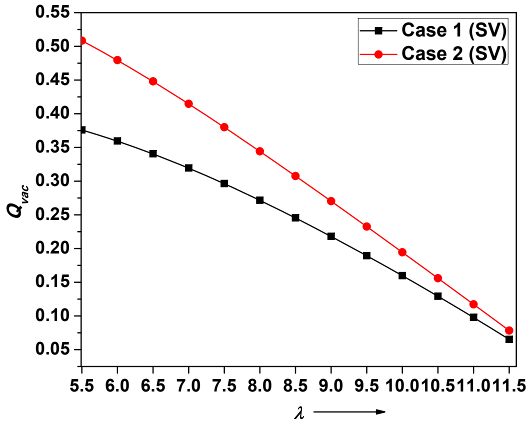

FIGURE 3.6: Effect of λ on Q_{vac}

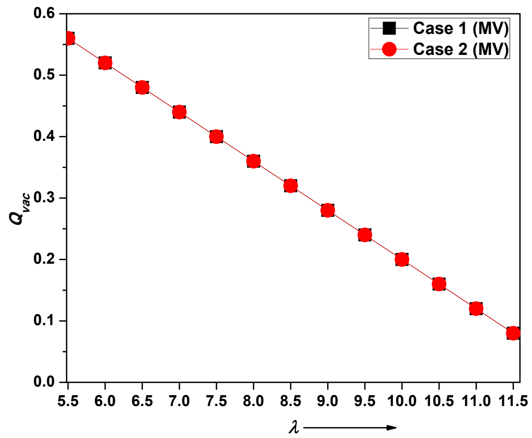

FIGURE 3.7: Effect of λ on Q_{vac}

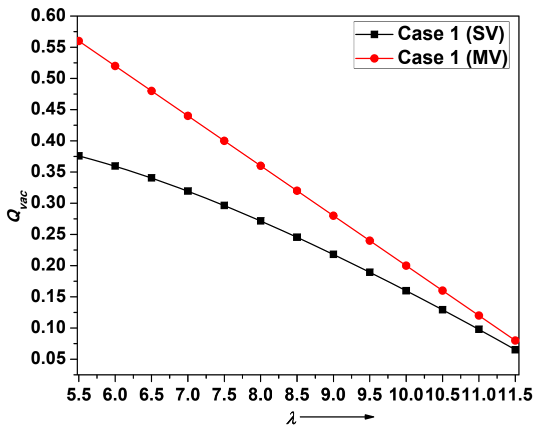

FIGURE 3.8: Effect of λ on Q_{vac}

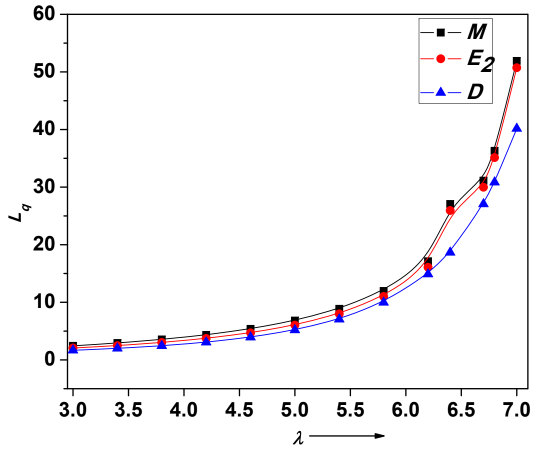

FIGURE 3.9: Effect of λ on L_q

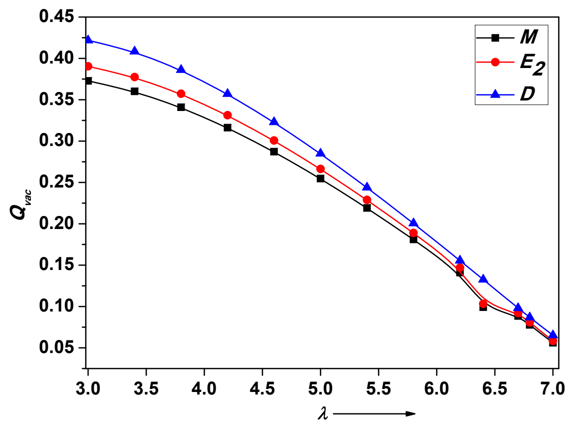

FIGURE 3.10: Effect of λ on Q_{vac}

3.6.1 Deduction of the results for $M/M/1$ queue

The numerical results for $M/M/1$ queue can be easily derived from the analytical results presented in this chapter by considereing $a = 1$, $b = 1$, exponential service time distribution and considerably large vacation rate (so that vacation time almost tends to zero). Table 3.1 and Table 3.2 provide the values of the important performance measures (L_q, W_s, L^{ser}) and probability (P_{idle}) which are obtained for $M/M/1$ model for following two cases.

Case I: Results for $M/M/1$ model deduced from the analytical results presented in this chapter by considering $a = b = 1$, exponential service time distribution and $\nu_0 \longrightarrow \infty$ ($\nu_0 = 200000$).

Case II: Results for classical $M/M/1$ model, for which performance measures L_q, W_s and probability P_{idle} are calculated using standard formula $L_q = \frac{\lambda^2}{\mu_1(\mu_1-\lambda)}$, $W_s = \frac{1}{\mu_1-\lambda}$ and $P_{idle} = 1 - \rho$.

The detail descriptions of Table 3.1 and Table 3.2 are as follows.

- 1st and 2nd column present the values of the input parameters λ and μ_1, respectively, for which ρ varies from 0.42857 to 0.75000.
- 3rd, 4th, 5th and 6th column present the values of L_q, W_s, L^{ser} and P_{idle}, respectively, for Case I.
- 7th, 8th and 9th column present the values of L_q, W_s and P_{idle}, respectively, for Case II.

It is clearly observed from Table 3.1 and Table 3.2 that the results deduced from current study as a special case matches exactly with the results obtained from classical $M/M/1$ model. Also, the value of L^{ser} calculated from the current study as a special case always gives the value 1 (approximately) which is obvious and shows the correctness of present study.

TABLE 3.1: Table for Case I and Case II, for SV

		Case I				Case II		
λ	μ_1	L_q	W_s	L^{ser}	P_{idle}	L_q	W_s	P_{idle}
3	4	2.2500000	1.0000000	1.0000000	0.2500000	2.2500000	1.0000000	0.2500000
3	5	0.9000000	0.5000000	1.0000000	0.4000000	0.9000000	0.5000000	0.4000000
3	6	0.5000000	0.3333333	1.0000000	0.5000000	0.5000000	0.3333333	0.5000000
3	7	0.3214286	0.2500000	0.9999999	0.5714286	0.3214286	0.2500000	0.5714286

TABLE 3.2: Table for Case I and Case II, for MV

		Case I				Case II		
λ	μ_1	L_q	W_s	L^{ser}	P_{idle}	L_q	W_s	P_{idle}
3	4	2.2500015	1.0000005	1.0000000	0.2500000	2.2500000	1.0000000	0.2500000
3	5	0.9000015	0.5000005	1.0000000	0.4000000	0.9000000	0.5000000	0.4000000
3	6	0.5000015	0.3333338	1.0000000	0.5000000	0.5000000	0.3333333	0.5000000
3	7	0.3214301	0.2500005	0.9999999	0.5714286	0.3214286	0.2500000	0.5714286

3.7 Cost model

In this section, a cost model is presented which may help in deciding the optimal values of the system parameters to minimize the total system cost. This type of cost model may be applicable in the example of group testing (*viz.*, for sample of testing COVID-19) that has been discussed in introduction section. For this purpose, consider the following cost parameters:

$C_{st}\equiv$ Startup cost (i.e., cost that bring the sample to the system) per sample per unit time.

$C_b\equiv$ Holding cost (i.e., cost to preserve the sample waiting in the queue for test, during health worker's busy period) per sample per unit time.

$C_v\equiv$ Holding cost (i.e., cost to preserve the sample waiting in the queue for test, during health worker's vacation period) per sample per unit time.

$C_d\equiv$ Holding cost (i.e., cost to preserve the sample waiting in the queue for test, during health worker's dormant period) per sample per unit time (exists only for SV).

$C_o\equiv$ Testing cost (i.e., cost when sample is taken for testing by health worker) per sample per unit time. Thus in long run,

$$\begin{aligned}\text{Total System Cost (TSC)} &= \lambda C_{st} + C_b\sum_{n=0}^{\infty}\sum_{r=a}^{b} n\frac{P_{n,r}}{P_{busy}} + C_v\sum_{n=0}^{\infty}\sum_{k=0}^{min(n,a-1)} n\frac{Q_n^{[k]}}{Q_{vac}} \\ &\quad +(1-\delta)C_d\sum_{n=0}^{a-1} n\frac{R_n}{P^{dor}} + C_oL^{ser}.\end{aligned}$$

Here, a numerical result is presented by considering particular values of the system parameters. The values of a varies from 1 to 12 and the value of b is fixed at 12. In Table 3.3 the values of TSC are presented for MV (see, column 2 to 5 of Table 3.3) and for SV (see, column 6 to 9 of Table 3.3) and for different values of λ=7, 10, 13, 18. The service time distribution is considered to be E_2 with batch size dependent service rate μ_r=$r\mu$ ($\mu = 0.15$) and the vacation time distribution is considered to be exponential distribution with queue size dependent vacation rate $\nu_k = \nu_{k-1}+1$ ($1 \le k \le a-1$, $\nu_0 = 1.5$). The TSC are obtained under the consideration C_{st}=0.1, C_b=1.0, C_v=1.5, C_d=1.7, and C_o=3.5.

Table 3.3: TSC for the different values of a and λ.

	MV				SV			
a	λ=7 ρ=0.324	λ=10 ρ=0.461	λ=13 ρ=0.602	λ=18 ρ=0.833	λ=7 ρ=0.324	λ=10 ρ=0.461	λ=13 ρ=0.602	λ=18 ρ=0.833
1	45.6097	59.8795	74.5118	105.1803	45.9564	60.2416	73.7764	105.5137
2	38.8260	50.4910	62.5858	90.3496	40.4422	52.2853	64.4433	92.3205
3	**37.5457**	47.6842	58.3782	84.0541	**40.2614**	**50.7451**	61.6620	87.5598
4	37.8280	**46.8892**	56.5833	80.7099	41.4895	51.0672	**61.1300**	85.6435
5	38.9235	47.0889	**55.9508**	78.7737	43.3893	52.2396	61.6259	**85.0612**
6	40.5979	47.9446	56.0471	77.6705	45.7371	53.9310	62.7181	85.2009
7	42.7503	49.3144	56.6847	77.1392	48.4473	56.0102	64.2233	85.8075
8	45.3111	51.1227	57.7699	**77.0478**	51.4753	58.4199	66.0572	86.7268
9	48.2072	53.3082	59.2421	77.3285	54.7816	61.1311	68.1802	87.8919
10	51.3455	55.7913	61.0382	77.9483	58.3193	64.1203	70.5733	89.2624
11	54.5852	58.4396	63.0583	78.8905	62.0354	67.3569	73.2220	90.8208
12	57.6421	61.0017	65.1240	80.1430	65.8783	70.7994	76.1048	92.5641

By considering this type of numerical experiment with desired values of system parameters and service time distribution (vacation time distribution) system analysts may easily achieve the minimum TSC by considering an optimal value of a. For fixed λ the minimum values of TSC are indicated in bold letters and the corresponding values of a are the desired optimum values of the lower threshold. For example if $\lambda = 10$ (MV), then the minimum TSC is 46.8892 which is achieved at $a = 4$, hence, $a = 4$ is the corresponding optimum value. A similar conclusion may be drawn for all other values of λ and for SV and MV. The graphical representation of Table 3.3 is presented in Figure 3.11 (for MV) and Figure 3.12 (for SV) and the corresponding minimum value of TSC is indicated by arrow sign in the figures.

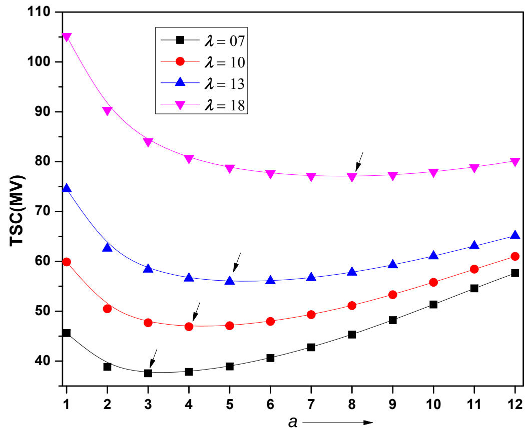

FIGURE 3.11: Effect of a on TSC for MV

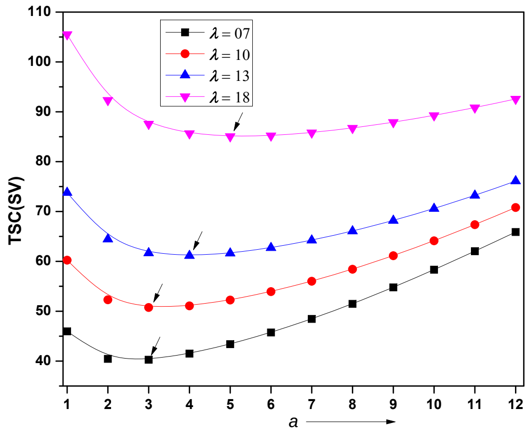

FIGURE 3.12: Effect of a on TSC for SV

3.8 Conclusion

In this chapter, an infinite capacity batch service queue with single and multiple vacations have been analyzed where the service time of the batches depend on the size of the batch under service, and the vacation time of the server depends on the queue size at vacation initiation epoch. Steady state joint probabilities have been achieved at various epochs by using the supplementary variable approach and the bivariate generating function method. Finally, various performance measures have been discussed to appraise the applicability of the considered model in the numerical section. In the present model interarrival time has been considered to be exponentially distributed, however, in computer communication and telecommunication system the arrivals of the data are bursty in nature and hence cannot be model using exponential interarrival time. To model these type of real life system, the analysis of the present model may motivate researchers to analyze a more complex model for the steady state joint probabilities for different service rules (different vacation policies) with more general arrival (service) process, *viz.*, MAP (MSP) or $BMAP$ ($BMSP$).