Chapter 2

Analysis of $M/M^{(a,b)}/1$ queue with single and multiple vacation for steady state joint distributions

2.1 Introduction

The bulk service queue with vacation has a remarkable influence in real life problems. For example, in a transport system, when a ship undergoes maintenance then it may be designed as the vacation model by considering the maintenance time as the vacation time. The bulk service queues together with SV and MV have been studied by Lee et al. [24], Sikdar and Gupta [99], Sikdar and Samanta [100], etc. In [24] authors presented the analysis of $M/G/1$ queue in which customers are served in batches of fixed size. Sikdar and Gupta [99] analyzed $M^X/G^Y/1/N$ queue and obtained stationary queue length distribution. Sikdar and Samanta [100] considered $BMAP/G^Y/1/N$ queue, they obtained queue length distribution at service (vacation) completion as well as arbitrary epoch. Recently, Gupta et al. [101] analyzed $M/G_r^{(a,b)}/1/N$ queue with SV and MV and obtained the joint distributions at various epoch.

In this chapter, bulk service Poisson queue has been considered in which the arrivals and departure both follow Poisson distribution, i.e., the inter-arrival time and the service time both are exponentially distributed. A single server serves the customers in batches with a minimum threshold value a and a maximum capacity b following general bulk service

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(GBS) rule. At the end of a service if the server finds that the queue length k is less than a, then the server will go for a k^{th} type of vacation of random length following exponential distribution, otherwise, it will keep on servicing customers as per the GBS rule. Using the probability generating function (PGF) technique, the joint distribution of queue and server content (when the server is busy) as well as the joint distribution of queue content and the type of vacation taken by the server (when the server is on vacation) have been obtained.

There are few articles available in the literature which motivated the author to analyze the queueing model under study. It may be remarked here that almost all the literature on bulk service vacation queues have been discussed which provide only either queue length distribution or PGF of the queue length, except Choi and Han [91] and Gupta et al. [101]. In [91], authors obtained the joint distribution of queue and server content when the server is busy and queue length distribution when the server is on vacation at pre-arrival and arbitrary epoch for $G/M^{(a,b)}/1$ queue with MV. However, the model under consideration differ from the model considered by Choi and Han [91] in such a way that firstly, in [91] authors considered $G/M^{(a,b)}/1$ queue with MV only, whereas this chapter consider $M/M^{(a,b)}/1$ queue with both SV and MV, secondly, in [91], authors obtained the joint distribution of queue and server content when the server is busy and queue length distribution when the server is on vacation at pre-arrival and arbitrary epoch, whereas, the joint distribution of queue and server content (when the server is busy) also the joint distribution of queue content and type of vacation (when the server is on vacation) have been obtained here. Gupta et al. [101] considered $M/G_r^{(a,b)}/1/N$ queue with SV and MV and derived the joint probabilities of the queue and server content, also the joint probabilities of the queue content and type of vacation at various epoch. Infinite buffer $M/M^{(a,b)}/1$ queue with SV and MV has been considered and the required joint probabilities are obtained, however, the consideration of the infinite buffer queue makes the model mathematically complex and the mathematical analysis is completely different from that of Gupta et al. [101]. To the best of the author's knowledge, the considered model has not been analyzed so far in the literature for obtaining steady state joint probabilities of the queue and server content, as well as the joint probabilities of queue content and the type of vacation for infinite buffer queue.

The rest of the chapter is organized as follows, In Section 2.2 model description is presented. Using bivariate probability generating function method the steady state joint probabilities have been obtained in Section 2.3. Section 2.4 presents some marginal probabilities. Section 2.5 presents some important performance measures. For better system efficiency the numerical results are discussed in Section 2.6, and the chapter ends with Section 2.7 where conclusion and future scope is discussed.

2.2 Model description

In this chapter, an infinite buffer, single server, bulk service queue with vacation is considered. The customers are arriving to the system according to the Poisson process with rate λ. The customers are served in batches according to the general bulk service (GBS) rule. In GBS rule server serves the customers in batches with minimum threshold limit $a(\geq 1)$ and maximum capacity b $(b > a)$. That is, after one service if there are at least l $(l \geq a)$ customers waiting in the queue then the server serves a batch of size $\min(l, b)$, otherwise remain in the idle state (which includes dormant state and/or vacation state) until the queue length reaches minimum threshold limit a for starting another busy period. The service time distribution of the batches are exponentially distributed with rate μ. Newly arriving customers are not allowed to join the ongoing service even if there is a free capacity. At the end of a busy period when server finds that the queue length is less than a then it takes k^{th} type of vacation (where k $(0 \leq k \leq a-1)$ is the number of customers waiting in the queue at vacation initiation epoch) following exponential vacation time distribution with rate ν. At the end of the vacation if server finds that at least a customers are waiting in the queue then the server provides service according to the GBS rule, otherwise, it will remain in dormant state until required number of customers to accumulate in the queue for providing service, or takes another vacation depending on the vacation policy under consideration, i.e., either single vacation (SV) or multiple vacation (MV). In this chapter, SV and MV queues have been studied in an unified way by defining a variable δ as follows.

$$\delta = \begin{cases} 1, & \text{for MV}, \\ 0, & \text{for SV}. \end{cases}$$

The traffic intensity of the system is defined by $\rho = \frac{\lambda}{b\mu}(< 1)$. The considered model is presented schematically in Figure 2.1 for the case of SV, and in Figure 2.2 for the case of MV.

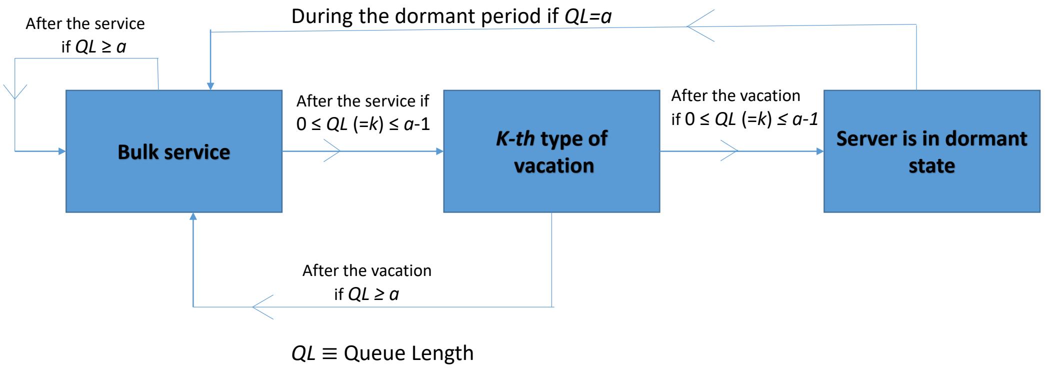

FIGURE 2.1: Schematic representation of the considered model for SV

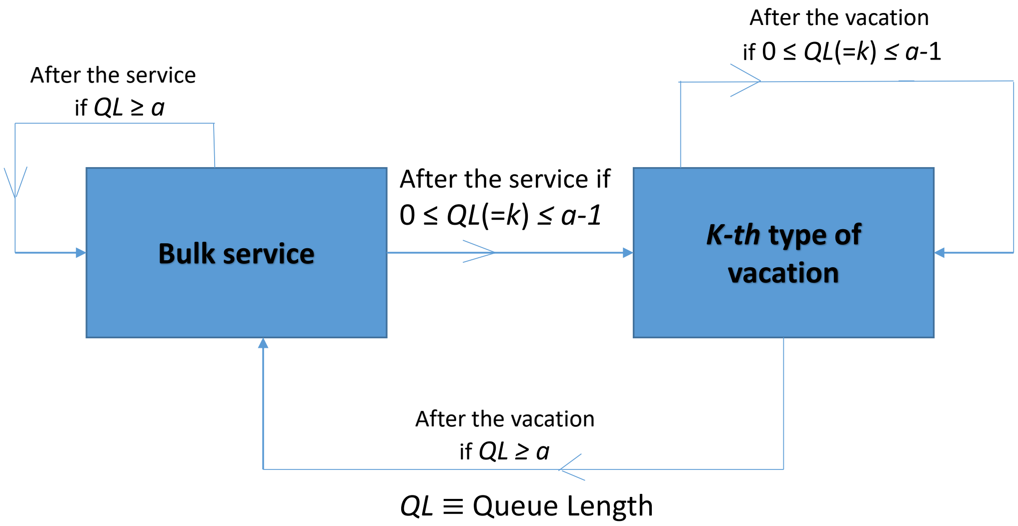

FIGURE 2.2: Schematic representation of the considered model for MV

2.3 Steady state analysis

In this section, author obtains the steady state busy period joint distribution of the queue and server content, and vacation period joint distribution of queue content and type of vacation taken by the server. To this end, the following notations are defined, at time t, for use in sequel.

- $N_q(t)$: be the number in the queue,

- $N_s(t)$: be the number with the server, when server is busy,
- $K(t)$: be the type of vacation taken by the server, when server is in vacation.

Remark:- It is to be noted here that, $N_s(t) = 0$ represents that the server is in dormant state for the case of SV model, and the type of vacation taken by the server denotes that the queue content (i.e., number present in the queue) at vacation initiation epoch.

Now $\{(N_q(t), N_s(t)) \cup (N_q(t), K(t))\}$ constitute two dimensional continuous time Markov chain with state space $\{(n,0); 0 \le n \le a-1\} \bigcup \{(n,r); n \ge 0, a \le r \le b\} \bigcup \{(n,k); 0 \le k \le a-1, n \ge k\}$, $\{(n,r); n \ge 0, a \le r \le b\} \bigcup \{(n,k); 0 \le k \le a-1, n \ge k\}$ for SV and MV, respectively. Further, define the state probabilities, at time t, as follows.

- $R_n(t) \equiv Pr\{N_q(t) = n, N_s(t) = 0\},\ 0 \le n \le a-1$, (exist only for SV),
- $P_{n,r}(t) \equiv Pr\{N_q(t) = n, N_s(t) = r\},\ n \ge 0,\ a \le r \le b$,
- $Q_n^{[k]}(t) \equiv\ Pr\{N_q(t) = n, K(t) = k\},\ n \ge k,\ 0 \le k \le a-1$.

In steady state, as $t \to \infty$, define,
$R_n = \lim_{t\to\infty} R_n(t),\ 0 \le n \le a-1$ (for SV),
$P_{n,r} = \lim_{t\to\infty} P_{n,r}(t),\ n \ge 0,\ a \le r \le b$,
$Q_n^{[k]} = \lim_{t\to\infty} Q_n^{[k]}(t),\ 0 \le k \le a-1,\ n \ge k$.
More preciously, R_n represents the steady state probability that the server is in dormant state (which exist only for SV) and queue length is n $(0 \le n \le a-1)$; $P_{n,r}$ represents the steady state joint probability that the server is busy in serving a batch of r $(a \le r \le b)$ customers and queue length is n (≥ 0) and $Q_n^{[k]}$ represents the steady state joint probability that the server is in k^{th} type of vacation and queue length is $n \ge k$. Now observing the system at time t and $t + dt$, the Kolmogorov equations of the model under consideration are obtained as follows.

$$\frac{dR_0(t)}{dt} = (1-\delta)\Big(-\lambda R_0(t) + \nu Q_0^{[0]}(t)\Big), \tag{2.1}$$

$$\frac{dR_n(t)}{dt} = (1-\delta)\Big(-\lambda R_n(t) + \lambda R_{n-1}(t) + \nu \sum_{k=0}^{n} Q_n^{[k]}(t)\Big),\quad 1 \le n \le a-1, \tag{2.2}$$

$$\frac{dP_{0,a}(t)}{dt} = -(\lambda+\mu)P_{0,a}(t) + (1-\delta)\lambda R_{a-1}(t) + \nu\sum_{k=0}^{a-1} Q_a^{[k]}(t) + \mu\sum_{r=a}^{b} P_{a,r}(t), \quad (2.3)$$

$$\frac{dP_{0,r}(t)}{dt} = -(\lambda+\mu)P_{0,r}(t) + \nu\sum_{k=0}^{a-1} Q_r^{[k]}(t) + \mu\sum_{j=a}^{b} P_{r,j}(t), \quad a+1 \le r \le b, \quad (2.4)$$

$$\frac{dP_{n,r}(t)}{dt} = -(\lambda+\mu)P_{n,r}(t) + \lambda P_{n-1,r}(t), \quad a \le r \le b-1, \ n \ge 1, \quad (2.5)$$

$$\frac{dP_{n,b}(t)}{dt} = -(\lambda+\mu)P_{n,b}(t) + \lambda P_{n-1,b}(t) + \nu\sum_{k=0}^{a-1} Q_{n+b}^{[k]}(t) + \mu\sum_{r=a}^{b} P_{n+b,r}(t), \ n \ge 1, \quad (2.6)$$

$$\frac{dQ_k^{[k]}(t)}{dt} = -(\lambda+\nu)Q_k^{[k]}(t) + \mu\sum_{r=a}^{b} P_{k,r}(t) + \delta\Big(\nu\sum_{j=0}^{k} Q_k^{[j]}(t)\Big), \quad 0 \le k \le a-1, \quad (2.7)$$

$$\frac{dQ_n^{[k]}(t)}{dt} = -(\lambda+\nu)Q_n^{[k]}(t) + \lambda Q_{n-1}^{[k]}(t), \quad 0 \le k \le a-1, \ n \ge k+1. \quad (2.8)$$

Now letting $t \longrightarrow \infty$ in (2.1) to (2.8), the corresponding steady state equations are obtained as follows.

$$0 = (1-\delta)\Big(-\lambda R_0 + \nu Q_0^{[0]}\Big), \quad (2.9)$$

$$0 = (1-\delta)\Big(-\lambda R_n + \lambda R_{n-1} + \nu\sum_{k=0}^{n} Q_n^{[k]}\Big), \quad 1 \le n \le a-1, \quad (2.10)$$

$$0 = -(\lambda+\mu)P_{0,a} + (1-\delta)\lambda R_{a-1} + \nu\sum_{k=0}^{a-1} Q_a^{[k]} + \mu\sum_{r=a}^{b} P_{a,r}, \quad (2.11)$$

$$0 = -(\lambda+\mu)P_{0,r} + \nu\sum_{k=0}^{a-1} Q_r^{[k]} + \mu\sum_{j=a}^{b} P_{r,j}, \quad a+1 \le r \le b, \quad (2.12)$$

$$0 = -(\lambda+\mu)P_{n,r} + \lambda P_{n-1,r}, \quad a \le r \le b-1, \quad n \ge 1, \quad (2.13)$$

$$0 = -(\lambda+\mu)P_{n,b} + \lambda P_{n-1,b} + \nu\sum_{k=0}^{a-1} Q_{n+b}^{[k]} + \mu\sum_{r=a}^{b} P_{n+b,r}, \quad n \ge 1, \quad (2.14)$$

$$0 = -(\lambda+\nu)Q_k^{[k]} + \mu\sum_{r=a}^{b} P_{k,r} + \delta\Big(\nu\sum_{j=0}^{k} Q_k^{[j]}\Big), \quad 0 \le k \le a-1, \quad (2.15)$$

$$0 = -(\lambda+\nu)Q_n^{[k]} + \lambda Q_{n-1}^{[k]}, \quad 0 \le k \le a-1, \ n \ge k+1. \quad (2.16)$$

The normalizing condition is given by

$$\sum_{n=0}^{\infty}\sum_{r=a}^{b} P_{n,r} + \sum_{k=0}^{a-1}\sum_{n=k}^{\infty} Q_n^{[k]} + (1-\delta)\sum_{n=0}^{a-1} R_n = 1. \quad (2.17)$$

The primary objective of the author is to solve (2.9) - (2.16) along with (2.17) for obtaining the required joint probabilities. Towards this end, the bivariate probability generating function is defined as follows:
$R(z) + P(z, y) + Q(z, y)$, $|z| \le 1$, $|y| \le 1$, where

$$R(z) = \sum_{n=0}^{a-1} R_n z^n, \ |z| \le 1, \tag{2.18}$$

$$P(z, y) = \sum_{n=0}^{\infty} \sum_{r=a}^{b} P_{n,r} z^n y^r, \ |z| \le 1, \ |y| \le 1, \tag{2.19}$$

$$Q(z, y) = \sum_{k=0}^{a-1} \sum_{n=k}^{\infty} Q_n^{[k]} z^n y^k, \ |z| \le 1, \ |y| \le 1. \tag{2.20}$$

Substituting $y = 1$ in (2.19) and (2.20), respectively, the following expressions are obtained.

$$P(z, 1) = \sum_{n=0}^{\infty} \sum_{r=a}^{b} P_{n,r} z^n = \sum_{n=0}^{\infty} P_n^* z^n = P^*(z), \quad |z| \le 1, \tag{2.21}$$

$$\textit{and} \ \ Q(z, 1) = \sum_{k=0}^{a-1} \sum_{n=k}^{\infty} Q_n^{[k]} z^n = \sum_{n=0}^{\infty} \sum_{k=0}^{min(n,a-1)} Q_n^{[k]} z^n = \sum_{n=0}^{\infty} Q_n^* z^n = Q^*(z), \quad |z| \le 1, \tag{2.22}$$

where

$$P_n^* = \sum_{r=a}^{b} P_{n,r}, \ n \ge 0, \tag{2.23}$$

and

$$Q_n^* = \sum_{k=0}^{min(n,a-1)} Q_n^{[k]}, \ n \ge 0. \tag{2.24}$$

Lemma 2.1. *For the case of SV the following result is obtained*

$$\lambda R_n = \nu \sum_{r=0}^{n} Q_r^*, \quad 0 \le n \le a - 1. \tag{2.25}$$

Proof. From equation (2.9) and (2.10), recursively the desired result (2.25) is obtained. □

In order to obtain steady state joint probabilities, multiplying (2.11)-(2.16) by proper power of z and y and summing over the range of n, r and k, and using (2.21), (2.22) and

Lemma 2.1 (for SV), the following expressions are obtained.

$$P(z,y) = \frac{\begin{array}{r} z^b \sum_{r=a}^{b-1} \big(\nu Q_r^* + \mu P_r^*\big) y^r + (1-\delta)\big(z^b \nu \sum_{r=0}^{a-1} Q_r^* y^a\big) \\ + y^b \nu \big(Q^*(z) - \sum_{n=0}^{b-1} Q_n^* z^n\big) \\ + \mu y^b \big(P^*(z) - \sum_{n=0}^{b-1} P_n^* z^n\big) \end{array}}{z^b(\lambda + \mu - \lambda z)}, |z| \leq 1, \ |y| \leq 1, \quad (2.26)$$

$$Q(z,y) = \frac{\sum_{k=0}^{a-1} \big(\mu P_k^* + \delta \nu Q_k^*\big) z^k y^k}{(\lambda + \nu - \lambda z)}, \quad |z| \leq 1, \ |y| \leq 1. \quad (2.27)$$

Lemma 2.2. *For the case of SV the probabilities* $Q_n^{[k]}$ $(0 \leq k \leq a-1)$ *and* P_k^* $(0 \leq k \leq a-1)$ *are connected by the following relation*

$$Q_n^{[k]} = \frac{\mu P_k^*}{\lambda \alpha^{n-k+1}}, \quad 0 \leq k \leq a-1 \text{ and } n \geq k. \quad (2.28)$$

Proof. From (2.27), after simplification, the following expression is obtained,

$$Q(z,y) = \frac{\mu}{\lambda \alpha} \sum_{k=0}^{a-1} P_k^* z^k y^k \sum_{n=0}^{\infty} \frac{z^n}{\alpha^n}, \quad \text{where } \alpha = 1 + \frac{\nu}{\lambda}. \quad (2.29)$$

Now equating (2.20) and (2.29) and then collecting the coefficients of y^k $(0 \leq k \leq a-1)$ the following expression is obtained,

$$\sum_{n=k}^{\infty} Q_n^{[k]} z^n = \sum_{n=k}^{\infty} \frac{\mu P_k^*}{\lambda \alpha^{n-k+1}} z^n, \quad 0 \leq k \leq a-1. \quad (2.30)$$

Finally, collecting the coefficients of z^n from both the sides of (2.30), desired result (2.28) is obtained. $\square$

Lemma 2.3. *For the case of MV the probabilities* $Q_n^{[k]}$ $(0 \leq k \leq a-1)$ *and* P_k^* $(0 \leq k \leq a-1)$ *are connected by the following relation*

$$Q_n^{[k]} = \frac{\left(\mu P_k^* + \nu \sum_{j=0}^{k} \alpha^{j-k} Q_j^{[j]}\right)}{\lambda \alpha^{n-k+1}}, \quad 0 \leq k \leq a-1, \ n \geq k. \quad (2.31)$$

Proof. From the equation (2.15) and (2.16), and using (2.23), after certain manipulation desired result (2.31) is obtained. □

Hence, from Lemma 2.2 (or Lemma 2.3) the joint probabilities $Q_n^{[k]}$ $(0 \leq k \leq a-1, n \geq k)$ are known if P_n^* $(0 \leq n \leq a-1)$ are known.
Now substituting $y = 1$ in (2.27) the following expression is obtained.

$$Q^*(z) \;=\; \frac{\sum\limits_{k=0}^{a-1}\Big(\mu P_k^* + \delta\nu Q_k^*\Big)z^k}{(\lambda+\nu-\lambda z)}, \quad |z| \leq 1. \tag{2.32}$$

Substituting $y = 1$ in (2.26) and using (2.24), (2.32), Lemma 2.2 (or Lemma 2.3) after simplification, one can get,

$$P^*(z) = \frac{H(z)}{L(z)}, \quad |z| \leq 1, \tag{2.33}$$

where

$$H(z) = \begin{cases} (\lambda+\nu-\lambda z)\mu\Big(z^b\sum\limits_{r=a}^{b-1}P_r^* + \nu\sum\limits_{n=0}^{b-1}\sum\limits_{k=0}^{min(n,a-1)}\frac{P_k^*}{\lambda\alpha^{n-k+1}}(z^b - z^n) - \sum\limits_{n=0}^{b-1}P_n^* z^n\Big) \\ +\mu\nu\sum\limits_{k=0}^{a-1}P_k^* z^k, \quad \text{for } SV, \\ (\lambda+\nu-\lambda z)\Bigg[\sum\limits_{j=a}^{b-1}\Big(\nu z^b\sum\limits_{k=0}^{a-1}\alpha^{k-j}Q_k^{[k]} + \mu z^b P_j^*\Big) \\ -\sum\limits_{n=0}^{b-1}\Big(\mu P_n^* z^n + \nu\sum\limits_{k=0}^{min(n,a-1)}\alpha^{k-n}Q_k^{[k]}z^n\Big)\Bigg] \\ +\nu\sum\limits_{k=0}^{a-1}\Big(\mu P_k^* z^k + \nu\sum\limits_{j=0}^{k}\alpha^{j-k}Q_j^{[j]}z^k\Big), \quad \text{for } MV, \end{cases}$$

and $\quad L(z) = (\lambda+\nu-\lambda z)(z^b(\lambda+\mu-\lambda z)-\mu)$.
It is to be noted here that the only unknown terms in $Q^*(z)$ and $P^*(z)$, as appeared in (2.32) and (2.33), respectively, are P_n^* $(0 \leq n \leq b-1)$. Therefore, one can conclude here that once P_n^* $(0 \leq n \leq b-1)$ is known completely one can derive the steady state joint probabilities $P_{n,r}$ $(n \geq 0, a \leq r \leq b)$, $Q_n^{[k]}$ $(0 \leq k \leq a-1, n \geq k)$ from (2.26), Lemma 2.2 (or Lemma 2.3). Hence forth, next section is dedicated for derivation of P_n^* $(0 \leq n \leq b-1)$.

2.3.1 Determination of unknown terms P_n^* $(0 \leq n \leq b-1)$ as appeared in $P^*(z)$

To determine the unknowns P_n^* $(0 \leq n \leq b-1)$ the knowledge of the zeros of $L(z)$ inside and on the unit circle $\{z \,\epsilon\, \mathbb{C} : |z| = 1\}$ is required. In $L(z)$ the first factor, i.e., $(\lambda+\nu-\lambda z)$ is a linear polynomial having zero α $(= 1 + \frac{\nu}{\lambda} > 1)$, Therefore, it does not play any role in getting unknowns P_n^* $(0 \leq n \leq b-1)$. To know the nature of the zeros of the second factor of $L(z)$, i.e., $z^b(\lambda+\mu-\lambda z)-\mu$, consider $f(z) = (\lambda+\mu)z^b$ and $g(z) = -\lambda z^{b+1} - \mu$ and assume that C be a closed contour defined by $|z| = 1+\delta$, where δ is small positive real number. It can be easily verified that $|f(z)|>|g(z)|$ on C if and only if $\frac{\lambda}{b\mu}<1$. Henceforth, Rouche's theorem states that, $f(z)+g(z)\big(=z^b(\lambda+\mu-\lambda z)-\mu\big)$ has exactly b zeros inside and on the unit circle. Assume these zeros as $z_1, z_2, ..., z_{b-1}, z_b = 1$. Note that $z_b = 1$ is the only zero of unit modulus of $f(z)+g(z)$. Therefore, $f(z)+g(z)$ has only one zero say z_0, out side the unit circle. Hence, $L(z)$ has b zeros inside and on the unit circle and they are $z_1, z_2, ..., z_{b-1}, z_b = 1$, and has two zeros out side the unit circle, and they are z_0 and α. Due to analyticity of $P^*(z)$ in $|z| \leq 1$ the zeros z_i $(1 \leq i \leq b)$ of $L(z)$ must be the zeros of the numerator $H(z)$. Hence,

$$H(z_i) = 0, \quad i = 1, 2, ..., b. \tag{2.34}$$

Remark: Note that i=b gives the trivial equation, therefore, ultimately from (2.34) $(b-1)$ equations in b unknowns P_n^* $(0 \leq n \leq b-1)$ are obtained.

Now the next objective is to solve equation (2.34) for obtaining the unknowns P_n^* $(0 \leq n \leq b-1)$. It should be noted here that the zeros of $L(z)$, lying inside the unit circle, may be all distinct or some of them are repeated. Therefore, depending on the nature of the zeros following two cases are discussed.

Case 1: when all the zeros of $L(z)$ in $|z| < 1$ are distinct

Assume that all the zeros of $L(z)$ lying inside the unit circle are distinct, i.e., $z_i \neq z_j$ for all $i \neq j$ and $1 \leq i, j \leq b-1$. Hence, from (2.34) one can derive $(b-1)$ linearly independent homogeneous equations in b unknowns P_n^* $(0 \leq n \leq b-1)$, which may results in $P_n^* = \xi_n P_0^*$, $0 \leq n \leq b-1$, where each ξ_n is known constants. (It is to be noted here that ξ_0=1).

Case 2: when some of the zeros of $L(z)$ in $|z|<1$ are repeated

Assume that some of the zeros of $L(z)$ lying inside the unit circle are repeated, and these repeated zeros are denoted by $x_1, x_2, ..., x_l$ with multiplicity $r_1, r_2, ..., r_l$, respectively, so that $m = \sum\limits_{i=1}^{l} r_i$. The remaining distinct zeros lying inside the unit circle are denoted by

$x_{m+1}, x_{m+2}, ..., x_{b-1}$. Using the property of analyticity of $P^*(z)$ in $|z| < 1$, the following system of equations are obtained.

$$H^{i-1}(x_j) = 0, \quad j = 1, 2, ..., l, i = 1, 2, ..., r_j, \tag{2.35}$$

$$H(x_i) = 0, \quad i = m+1, m+2, ..., b-1, \tag{2.36}$$

where $H^n(x)$ is the n^{th} derivative of $H(z)$ at $z = x$, $(n \geq 1)$ and $H^0(x) = H(x)$. Hence, from (2.35) and (2.36) one can derive $(b-1)$ linearly independent homogeneous equations in b unknowns $P_n^*(0 \leq n \leq b-1)$. Solving them $P_n^* = \xi_n P_0^*$ $(0 \leq n \leq b-1)$ are obtained, where each ξ_n is known constants.

Hence, it concludes that all P_n^* $(0 \leq n \leq b-1)$ will be known once P_0^* is known, which is derived in the following section.

2.3.1.1 Derivation of P_0^*

Since $H(z)$ is polynomial of degree $(b+1)$ and z_i $(1 \leq i \leq b)$ are b zeros of $H(z)$, assume that all $(b+1)$ zeros of $H(z)$ are $\alpha_1, z_i(1 \leq i \leq b)$.

Result 1: The zeros z_0 of $L(z)$ and α_1 of $H(z)$ are given by,

$$z_0 = \beta - \sum_{i=1}^{b} z_i, \quad \beta = 1 + \frac{\mu}{\lambda}, \tag{2.37}$$

$$\alpha_1 = \frac{\sigma}{\omega} - \sum_{i=1}^{b} z_i, \tag{2.38}$$

where

$$\sigma = \begin{cases} (\lambda+\nu)\mu \sum\limits_{r=a}^{b-1} \xi_r + (\lambda+\nu)\nu \sum\limits_{n=0}^{b-1} \sum\limits_{k=0}^{min(n,a-1)} \frac{\mu\xi_k}{\lambda\alpha^{n-k+1}} + \mu\nu \sum\limits_{k=0}^{a-1} \frac{\xi_k}{\alpha^{b-k}} + \lambda\mu\xi_{b-1}, & \text{for SV,} \\ \alpha\nu \sum\limits_{j=a}^{b-1} \sum\limits_{k=0}^{a-1} \alpha^{k-j} S_k^{[k]} + \alpha\mu \sum\limits_{j=a}^{b-1} \xi_j + \mu\xi_{b-1} + \nu \sum\limits_{k=0}^{a-1} \alpha^{k-b+1} S_k^{[k]}, & \text{for MV,} \end{cases} \tag{2.39}$$

$$\omega = \begin{cases} \lambda\mu \sum\limits_{r=a}^{b-1} \xi_r + \nu\mu \sum\limits_{n=0}^{b-1} \sum\limits_{k=0}^{min(n,a-1)} \frac{\xi_k}{\alpha^{n-k+1}}, & \text{for SV,} \\ \nu \sum\limits_{j=a}^{b-1} \sum\limits_{k=0}^{a-1} \alpha^{k-j} S_k^{[k]} + \mu \sum\limits_{j=a}^{b-1} \xi_j, & \text{for MV,} \end{cases} \tag{2.40}$$

and $S_0^{[0]} = \frac{\mu}{\lambda}$, $S_k^{[k]} = \frac{\mu}{\lambda}\xi_k + \frac{\nu}{\lambda} \sum\limits_{j=0}^{k-1} \alpha^{j-k} S_j^{[j]}$, $\quad 1 \leq k \leq a-1$.

Proof. Since $L(z)$ and $H(z)$ are polynomials in z of degree $(b+2)$ and $(b+1)$, respectively, hence,

$$\text{sum of all the zeros of } L(z) = \alpha + \sum_{i=1}^{b} z_i + z_0 = -\frac{\text{coefficient of } z^{b+1} \text{ of } L(z)}{\text{coefficient of } z^{b+2} \text{ of } L(z)}. \tag{2.41}$$

$$\text{sum of all the zeros of } H(z) = \sum_{i=1}^{b} z_i + \alpha_1 = -\frac{\text{coefficient of } z^{b} \text{ in } H(z)}{\text{coefficient of } z^{b+1} \text{ in } H(z)}. \tag{2.42}$$

which finally led to the derived results (2.37), (2.38) after some algebraic manipulation. ☐

Lemma 2.4. *Sum of the zeros of $L(z)$, which lies inside and on the unit circle $\{z \in \mathbb{C} : |z| = 1\}$, can never be zero, i.e., $\sum_{i=1}^{b} z_i \neq 0$.*

Proof. From (2.37), $z_0 = \beta$ if $\sum_{i=1}^{b} z_i = 0$ which means $f(z) + g(z)$ vanishes at $z = \beta$ which is not possible. ☐

Hence from (2.37) and Lemma (2.4), it is found that $z_0 \neq \beta$.

Lemma 2.5. *The value of P_0^* is given by*

$$P_0^* = \begin{cases} \dfrac{(\alpha-1)(z_0-1)\lambda^2\alpha_1}{(\alpha_1-1)\alpha z_0\lambda^2+\mu\alpha_1\lambda(z_0-1)\sum\limits_{k=0}^{a-1}\xi_k+\alpha_1(\alpha-1)(z_0-1)\nu\mu\sum\limits_{n=0}^{a-1}\sum\limits_{r=0}^{n}\sum\limits_{k=0}^{r}\frac{\xi_k}{\alpha^{r-k+1}}}, & \textit{for } SV, \\ \dfrac{(\alpha-1)(z_0-1)\lambda\alpha_1}{\lambda(\alpha_1-1)\alpha z_0+\mu\alpha_1(z_0-1)\sum\limits_{k=0}^{a-1}\xi_k+\alpha_1(z_0-1)\nu\sum\limits_{k=0}^{a-1}\sum\limits_{j=0}^{k}\alpha^{j-k}S_j^{[j]}}, & \textit{for } MV. \end{cases} \tag{2.43}$$

Proof. As $H(z)$ and $L(z)$ are the polynomials of degree $(b+1)$ and $(b+2)$, respectively, due to analyticity of (2.33) in $|z| \leq 1$, corresponding to each zero z_i $(1 \leq i \leq b)$ of $L(z)$, both $H(z)$ and $L(z)$ must have common factors of the form $z - z_i$ $(1 \leq i \leq b)$. On canceling these common factors from $H(z)$ and $L(z)$, and using Case 1 (or Case 2), $P^*(z)$ can be rewritten as,

$$P^*(z) = \frac{\eta P_0^*(z-\alpha_1)}{(z-\alpha)(z-z_0)}, \quad |z| \leq 1, \tag{2.44}$$

where η is any constant. Using $P^*(0)=P_0^* > 0$ in (2.44) after some algebraic manipulation one can obtain

$$\eta = -\frac{\alpha z_0}{\alpha_1}. \tag{2.45}$$

Using (2.45) in (2.44) one can get

$$P^*(z) = -\frac{\alpha z_0 P_0^*(z-\alpha_1)}{\alpha_1(z-\alpha)(z-z_0)}, \quad |z| \leq 1. \tag{2.46}$$

Using (2.46), (2.32), Lemma 2.1 (for SV), Lemma 2.2 (or Lemma 2.3) and the relation $P_n^* = \xi_n P_0^*$ $(0 \leq n \leq b-1)$ in the normalizing condition $P^*(1) + Q^*(1) + (1-\delta)R(1)$=1, after some algebraic manipulation desired result (2.43) is obtained. □

2.3.2 Determination of the steady state joint probabilities

In this section, the closed form expression for all the required joint probabilities (except $P_{n,b}$, $n \geq 0$ which is obtained in Section 2.3.3) have been obtained in terms of P_0^*. As P_0^* is already known from Lemma 2.5, these joint probabilities can be obtained in known terms in Theorem 2.6 and Theorem 2.7.

Theorem 2.6. *For the case of single vacation (SV) the steady state joint probabilities* R_n $(0 \leq n \leq a-1)$, $Q_n^{[k]}$ $(0 \leq k \leq a-1, n \geq k)$ *and* $P_{n,r}$ $(n \geq 0, a \leq r \leq b-1)$ *are given by,*

$$Q_n^{[k]} = \frac{\mu \xi_k P_0^*}{\lambda \alpha^{n-k+1}}, \quad 0 \leq k \leq a-1 \text{ and } n \geq k, \tag{2.47}$$

$$R_n = \frac{\nu\mu}{\lambda^2} \sum_{r=0}^{n} \sum_{k=0}^{r} \frac{P_0^* \xi_k}{\alpha^{r-k+1}}, \quad 0 \leq n \leq a-1, \tag{2.48}$$

$$P_{n,a} = \frac{\nu \sum_{r=0}^{a} \sum_{k=0}^{min(r,a-1)} \frac{\mu \xi_k P_0^*}{\lambda \alpha^{r-k+1}} + \mu \xi_a P_0^*}{\lambda \beta^{n+1}}, \quad n \geq 0 \tag{2.49}$$

$$P_{n,r} = \frac{\sum_{k=0}^{a-1} \frac{\mu \xi_k P_0^*}{\lambda \alpha^{r-k+1}} + \mu \xi_r P_0^*}{\lambda \beta^{n+1}}, \quad a+1 \leq r \leq b-1, \text{ where } \beta = 1 + \frac{\mu}{\lambda}, \; n \geq 0. \tag{2.50}$$

Proof. Using the relation $P_n^* = \xi_n P_0^*$ $(0 \leq n \leq b-1)$ in Lemma 2.2 equation (2.47) is obtained. Using (2.24) in Lemma 2.1 one have,

$$R_n = \frac{\nu}{\lambda} \sum_{r=0}^{n} \sum_{k=0}^{r} Q_r^{[k]}, \quad 0 \leq n \leq a-1. \tag{2.51}$$

Using (2.47) in (2.51) desired result (2.48) is obtained.
Using (2.19) in (2.26) comparing the coefficient of y^r $(a \le r \le b-1)$ one can get

$$\sum_{n=0}^{\infty} P_{n,a} z^n = \frac{\nu \sum\limits_{j=0}^{a} Q_j^* + \mu P_a^*}{\lambda \beta^{n+1}} \sum_{n=0}^{\infty} z^n, \tag{2.52}$$

$$\sum_{n=0}^{\infty} P_{n,r} z^n = \frac{\nu Q_r^* + \mu P_r^*}{\lambda \beta^{n+1}} \sum_{n=0}^{\infty} z^n, \quad a+1 \le r \le b-1. \tag{2.53}$$

Now collecting the coefficients of z^n from both the side of (2.52) and (2.53) one can obtain

$$P_{n,a} = \frac{\nu \sum\limits_{j=0}^{a} Q_j^* + \mu P_a^*}{\lambda \beta^{n+1}}, \quad n \ge 0, \tag{2.54}$$

and

$$P_{n,r} = \frac{\nu Q_r^* + \mu P_r^*}{\lambda \beta^{n+1}}, \quad n \ge 0, \quad a+1 \le r \le b-1. \tag{2.55}$$

Using (2.24) and (2.47) and the relation $P_n^* = \xi_n P_0^*$ $(0 \le n \le b-1)$ in (2.54) and (2.55), respectively, (2.49) and (2.50) are obtained, respectively. □

Theorem 2.7. *For the case of multiple vacation (MV) the steady state joint probabilities $Q_n^{[k]}$ $(0 \le k \le a-1, n \ge k)$ and $P_{n,r}$ $(n \ge 0, a \le r \le b-1)$ are given by,*

$$Q_n^{[k]} = \frac{P_0^* \left(\mu \xi_k + \nu \sum\limits_{j=0}^{k} \alpha^{j-k} S_j^{[j]} \right)}{\lambda \alpha^{n-k+1}}, \quad n \ge k, \; 0 \le k \le a-1, \tag{2.56}$$

$$P_{n,r} = \frac{\left(\nu \sum\limits_{k=0}^{a-1} \alpha^{k-r} S_k^{[k]} + \mu \xi_r \right) P_0^*}{\lambda \beta^{n+1}}, \quad n \ge 0, \; a \le r \le b-1. \tag{2.57}$$

Proof. Using the relation $P_n^* = \xi_n P_0^*$ $(0 \le n \le b-1)$ in Lemma 2.3 after some algebraic manipulation desired result (2.56) is obtained.
Using (2.19) in (2.26) collecting the coefficients of y^r $(a \le r \le b-1)$ the following expression is obtained,

$$\sum_{n=0}^{\infty} P_{n,r} z^n = \frac{\nu Q_r^* + \mu P_r^*}{\lambda \beta^{n+1}} \sum_{n=0}^{\infty} z^n, \quad a \le r \le b-1. \tag{2.58}$$

Collecting the coefficients of z^n from both the side of (2.58) and using (2.24), Lemma 2.3 and the relation $P_n^* = \xi_n P_0^*$ $(0 \le n \le b-1)$ after some simplification desired result (2.57) is obtained. □

2.3.3 Determination of unknown probabilities $P_{n,b}$ $(n \geq 0)$

In previous section $P_{n,r}$ $(n \geq 0,\ a \leq r \leq b-1)$ and $Q_n^{[k]}$ $(0 \leq k \leq a-1,\ n \geq k)$ have been successively obtained. Now the main task is to obtain the remaining steady state joint probabilities $P_{n,b}$ $(n \geq 0)$. To get these probabilities, substituting (2.19) in (2.26) and comparing the coefficients of y^b the following expression is obtained,

$$\sum_{n=0}^{\infty} P_{n,b} z^n = \frac{\nu\big(Q^*(z) - \sum\limits_{n=0}^{b-1} Q_n^* z^n\big) + \mu\big(P^*(z) - \sum\limits_{n=0}^{b-1} P_n^* z^n\big)}{z^b(\lambda + \mu - \lambda z)}, \quad |z| \leq 1. \tag{2.59}$$

Using equations (2.46), (2.32) and Lemma 2.2 (or Lemma 2.3) in (2.59) then using the analyticity of (2.59) in $|z| \leq 1$, after some algebraic manipulation, the following expression is obtained,

$$\sum_{n=0}^{\infty} P_{n,b} z^n = \frac{\tilde{\varepsilon} M(z)}{-\lambda^2 \alpha_1 N(z)}, \quad |z| \leq 1, \tag{2.60}$$

where $M(z) = z - \gamma$,
$\gamma = \alpha + z_0 - (1-\delta)\dfrac{\nu \sum\limits_{k=0}^{a-1} \frac{\mu \xi_k}{\lambda \alpha^{b-k-1}} + \mu \xi_{b-2}}{\nu \sum\limits_{k=0}^{a-1} \frac{\mu \xi_k}{\lambda \alpha^{b-k}} + \mu \xi_{b-1}} - \delta \dfrac{\nu \sum\limits_{k=0}^{a-1} \alpha^{k-b+2} S_k^{[k]} + \mu \xi_{b-2}}{\nu \sum\limits_{k=0}^{a-1} \alpha^{k-b+1} S_k^{[k]} + \mu \xi_{b-1}},$
$N(z) = (z-\alpha)(z-\beta)(z-z_0)$, and $\tilde{\varepsilon} = -\lambda \alpha_1 (\nu Q_{b-1}^* + \mu P_{b-1}^*)$.
Now two cases arise which are discussed below.

Case (I): When $z_0 \neq \alpha \neq \beta$ for this case,

$$\frac{M(z)}{N(z)} = \frac{A}{(z-\alpha)} + \frac{B}{(z-\beta)} + \frac{C}{(z-z_0)}, \tag{2.61}$$

where A, B and C are constants. Using the residue theorem the value of A, B and C are given by,
$A = \big[\frac{M(z)}{N'(z)}\big]_{z=\alpha}$, $B = \big[\frac{M(z)}{N'(z)}\big]_{z=\beta}$, $C = \big[\frac{M(z)}{N'(z)}\big]_{z=z_0}$.
Now using (2.61) and the value of $\tilde{\varepsilon}$ in (2.60) collecting the coefficients of z^n the following expression is obtained

$$P_{n,b} = \frac{-\big(\nu Q_{b-1}^* + \mu P_{b-1}^*\big)}{\lambda}\left(\frac{A}{\alpha^{n+1}} + \frac{B}{\beta^{n+1}} + \frac{C}{z_0^{n+1}}\right), \quad n \geq 0. \tag{2.62}$$

Using (2.24) Lemma 2.2 (or Lemma 2.3) and the relation $P_n^* = \xi_n P_0^*$ $(0 \leq n \leq b-1)$ in (2.62) $P_{n,b}$ $(n \geq 0)$ is obtained.

Case (II): Define a variable x such that,

$$x = \begin{cases} \beta & if\ \alpha = z_0, \\ z_0 & if\ \alpha = \beta. \end{cases}$$

Hence, $\frac{M(z)}{N(z)}$ can be written as,

$$\frac{M(z)}{N(z)} = \frac{A}{(z-x)} + \sum_{i=1}^{2} \frac{A_i}{(z-\alpha)^i}. \tag{2.63}$$

$A = \left[\frac{M(z)}{N'(z)}\right]_{z=x}$, $A_1 = \left[M(z)\frac{d}{dz}\left(\frac{\frac{d^2}{dz^2}(z-\alpha)^2}{\frac{d^2}{dz^2}N(z)}\right) + \left(\frac{\frac{d^2}{dz^2}(z-\alpha)^2}{\frac{d^2}{dz^2}N(z)}\right)\left(\frac{d}{dz}M(z)\right)\right]_{z=\alpha}$, $A_2 = \left[M(z)\frac{\frac{d^2}{dz^2}(z-\alpha)^2}{\frac{d^2}{dz^2}N(z)}\right]_{z=\alpha}$.

Using (2.63) and the value of $\tilde{\varepsilon}$ in (2.60) collecting the coefficients of z^n the following expression is obtained,

$$P_{n,b} = -\frac{\left(\nu Q_{b-1}^* + \mu P_{b-1}^*\right)}{\lambda}\left\{\frac{A}{x^{n+1}} + \frac{A_1}{\alpha^{n+1}} - \frac{A_2(n+1)}{\alpha^{n+2}}\right\}, \quad n \geq 0. \tag{2.64}$$

Using (2.24) Lemma 2.2 (or Lemma 2.3) and the relation $P_n^* = \xi_n P_0^*$ $(0 \leq n \leq b-1)$ in (2.64) $P_{n,b}$ $(n \geq 0)$ is obtained.

2.4 Marginal Probabilities

In this section, the important marginal probabilities have been presented which can be derived from the steady state join probabilities obtained in previous section.

1. Probability of the queue length when server is busy is given by $P_n^* = \sum\limits_{r=a}^{b} P_{n,r}$, $n \geq 0$.

2. Probability of the queue length when server is in vacation is given by $Q_n^* = \sum\limits_{k=0}^{min(n,a-1)} Q_n^{[k]}$, $n \geq 0$.

3. Queue length distribution is given by

$$P_n^{queue} = \begin{cases} (1-\delta)R_n + P_n^* + Q_n^*, & 0 \leq n \leq a-1, \\ P_n^* + Q_n^*, & n \geq a. \end{cases}$$

4. Probability that server is in dormant state is given by $P^{dor}=\sum_{n=0}^{a-1} R_n$.
5. Probability that there are r $(a \leq r \leq b)$ customers with the server is given by $P_r^{ser}=\sum_{n=0}^{\infty} P_{n,r}$.
6. Probability that the server is in k^{th} $(0 \leq k \leq a-1)$ type of vacation is given by $Q_{vac}^{[k]}=\sum_{n=k}^{\infty} Q_n^{[k]}$.
7. Probability that the server is busy is given by $P_{busy}=\sum_{r=a}^{b}\sum_{n=0}^{\infty} P_{n,r}$.
8. Probability that the server is in vacation is given by $Q_{vac}=\sum_{k=0}^{a-1}\sum_{n=k}^{\infty} Q_n^{[k]}$.
9. Probability that the server is idle is given by $P_{idle}=(1-\delta)P^{dor}+Q_{vac}$.

2.5 Performance measure

Since all the steady state probabilities are known, in this section some important performance measures of the model under consideration are presented.

1. The expected number of customers in the queue is given by $L_q= (1-\delta)\sum_{n=0}^{a-1} nR_n + \sum_{n=0}^{\infty}\sum_{r=a}^{b} nP_{n,r} + \sum_{k=0}^{a-1}\sum_{n=k}^{\infty} nQ_n^{[k]}=(1-\delta)\sum_{n=0}^{a-1} nP_n^{queue} + \sum_{n=a-\delta a}^{\infty} nP_n^{queue}$.
2. The expected number of customers in the system is given by $L_s=(1-\delta)\sum_{n=0}^{a-1} nR_n + \sum_{n=0}^{\infty}\sum_{r=a}^{b}(n+r)P_{n,r} + \sum_{k=0}^{a-1}\sum_{n=k}^{\infty} nQ_n^{[k]}$.
3. The expected waiting time of a customer in the queue is given by $W_q=\frac{L_q}{\lambda}$.
4. The expected waiting time of a customer in the system is given by $W_s=\frac{L_s}{\lambda}$.
5. Expected number of customers with the server when server is busy is given by $L^{ser}=\sum_{r=a}^{b}(rP_r^{ser}/P_{busy})$.
6. Expected type of vacation taken by server when server is in vacation is given by $L^{vac} = \sum_{k=0}^{a-1}(kQ_{vac}^{[k]}/Q_{vac})$.

2.6 Numerical results

In this section, some numerical observations are presented in order to validate the analytical results by means of graphs and tables. First, one real life example associated with sugarcane-juice production are presented, by which the reader may easily connect the insight of the possible application of the considered model.

Example: A practical situation for the proposed model may be observed in the sugarcane juice production of the sugar mill industries wherein the attention is focused on the sugarcane juice machine. The machine operator produces the sugarcane juice by machining operations like cane cutting, cleaning, peeling, and then producing juice. Suppose that the machine (server), in a sugar mill, which extracts the juice, can take a minimum of three packets and a maximum of six packets of sugarcane for producing juice (service). After extraction of the juice if the server finds at least r (≥ 3) packets in the queue then it takes minimum $(r, 6)$ packets for the service, as per the GBS rule, otherwise, it takes either 0^{th} type or 1^{th} type or 2^{th} type of vacation. In the case of 0^{th} type of vacation, the server removes waste, checks all the machinery parts, and purifies the extracted juice assembled in the containers. In the case of 1^{th} or 2^{th} type of vacation, either it checks the machinery parts (*viz.*, greasing, fuel) or purifies the extracted juice which has collected in the containers. Such a model may be analyzed as a bulk service queue with SV and MV. For this example, assume that sugarcane packets are arriving in the system with rate 5.5 following Poisson process and the server provides service with rate 2.5, and the server takes 0^{th} or 1^{th} or 2^{th} type of vacation with rate 1.3. The service and vacation time follow an exponential distribution. For this particular example, it is observed that on an average, 4.616 {5.017} packets are in sugarcane machine (i.e., with the server) when the server is busy for SV {MV}, and when the server is on vacation, then an average type of vacation is 0.84^{th} for SV and 1.01^{th} for MV. It is also observed that the average 4.295 {4.928} packets are waiting in the queue for service, and the average waiting time in queue is 0.781 {0.896} for SV {MV}. Assume that every service is costing the same amount, then the manager would like to provide the server six packets each time for producing sugarcane juice. The information on joint probabilities helps the manager to observe the expected number of packets with the server during the busy period. The information on the expected type of vacation may help the manager to arrange the minimum number of packets for service during the vacation period. As a result, it may modulate queue length and waiting time of packets for service, by which sugarcane can also be saved from drying.

Table 2.1 presents the steady state joint probabilities R_n $(0 \leq n \leq a-1)$, $P_{n,r}$ $(n \geq 0, a \leq r \leq b)$, $Q_n^{[k]}$ $(0 \leq k \leq a-1, n \geq k)$ for $M/M^{(5,10)}/1$ queue with SV. The other input parameters are taken as λ=120, μ=15 and ν=5. The description of Table 2.1 is given as follows:

- The first column presents the number of customer present in the queue (excluding the number in service).
- The second column presents the probability that the system is in state $(n, 0)$, i.e., the system size is n $(0 \leq n \leq a-1)$.
- 3rd to 8th column present the probability that the system is in state (n, r), i.e., the system size is $n+r$ $(n \geq 0, a \leq r \leq b)$.
- The 9th column presents the probability of the queue length when server is busy.
- 10th to 14th column present the server is in state (n, k), i.e., system size is n $(\geq k)$ when server is in k^{th} type of vacation.
- 15th column presents the probability of the queue length when server is in vacation.
- The last column presents the queue length distribution.
- The last row presents performance measures defined in section 2.5 and the second last row presents few more marginal probabilities defined in section 2.4.

Table 2.2 and Table 2.3 present steady state joint probabilities $P_{n,r}$ $(n \geq 0, a \leq r \leq b)$ and $Q_n^{[k]}$ $(0 \leq k \leq a-1, n \geq k)$ for $M/M^{(10,19)}/1$ queue with MV. The other input parameters for Table 2.2 and Table 2.3 are taken as λ=135, μ=9, ν=4. Table 2.2 and Table 2.3 are self explanatory as similar notation has been used as used in Table 2.1.

For the richer understanding of queueing models graphical representation is very much needed. Figure 2.3 to Figure 2.12 show the effect of the key parameters, i.e., λ, μ and ν, on some important performance measures for SV (MV). These graphs are presented here to understand sensitivity of the system performance for considered model. For this purpose, $M/M^{(10,25)}/1$ queue is considered.

In Figure 2.3 to Figure 2.6 the effect of μ on performance measures L_q, L_s, W_q, W_s, L^{ser} and L^{vac} are displayed for SV and MV, keeping the value of λ and ν fixed at 240 and 15, respectively. It is observed from the figures that as μ increases from 12 to 30 the values of L_q, L_s, W_q, W_s, L^{ser} and L^{vac} decrease. This behavior of the considered performance measures is quite obvious as for fixed λ=240 and ν=15, the value of ρ varies from 0.8 to

0.32. As traffic intensity decreases it will obviously decrease the value of the performance measures L_q, L_s, W_q, W_s, L^{ser} and L^{vac}.

The effect of ν on the performance measures L_q, L_s, W_q, and W_s are presented in Figure 2.7 and Figure 2.8 for SV and MV, respectively. In these figures the value of the parameter ν varies from 4 to 22 and the values of the other parameters, i.e., λ and μ, are kept fixed at 240 and 12, respectively. It is observed from Figure 2.7 and Figure 2.8 that as ν increases from 4 to 22, the value of L_q, L_s, W_q, and W_s decreases significantly. This type of behavior is observed because when ν increases it eventually decreases the mean vacation time of the server. As a result server became available to the system more frequently for providing service, which eventually decreases the mean queue (system) length and mean waiting time of a customer in the queue (system).

Figure 2.9 to Figure 2.12 present the effect of λ on the performance measures L_q, L_s, W_q, W_s, L^{ser} and L^{vac}. In these figures λ varies from 11 to 20 and the values of μ and ν are kept fixed at μ=1.1 and ν=15. As λ varies from 11 to 20 the values of ρ varies from 0.4 to 0.727. Due to increase in ρ the expected queue (system) length and expected waiting time in queue (system) will obviously increase . As $L_q(L_s)$ is increasing it will eventually increase L^{ser} and L^{vac}. Hence, one can conclude here that the behavior of the graphs as presented in Figure 2.9 to Figure 2.12 are on its expected direction.

Figure 2.13 to Figure 2.15 present a comparison between SV and MV model. In these figures the effect of vacation rate, i.e., ν is shown on some important probabilities, *viz.*, Q_{vac}, P_{busy} and P_{idle}, respectively. For this purpose $M/M^{(15,31)}/1$ is considered queue with SV and MV. The values of the parameters λ and μ are kept fixed at λ=150 and μ=6, and ν varies from 4 to 22. It is observed from the figures that as ν increases, P_{busy} increases, however, P_{idle} and Q_{vac} decreases significantly for both the cases SV and MV. Increase in ν, decrease the mean vacation time taken by the server, which eventually increase the chance of the fraction of the time that server is busy, i.e., P_{busy}, and decrease the chance of the fraction of the time that the server is in vacation or idle state, i.e., Q_{vac} or P_{idle}, and this behavior is well reflected in Figure 2.13 to Figure 2.15. It is also observed in Figure 2.13 to Figure 2.15 that Q_{vac} and P_{idle} is less for SV, in comparison to MV, and P_{busy} is more for SV, in comparison to MV, which is quite expected.

TABLE 2.1: The steady state joint probabilities of the queue and the server content, and queue content and the type of vacation taken by the server for $M/M^{(5,10)}/1$ queue with SV, λ=120, μ=15 and ν=5.

n	R_n	$P_{n,5}$	$P_{n,6}$	$P_{n,7}$	$P_{n,8}$	$P_{n,9}$	$P_{n,10}$	P_n^*	$Q_n^{[0]}$	$Q_n^{[1]}$	$Q_n^{[2]}$	$Q_n^{[3]}$	$Q_n^{[4]}$	Q_n^*	P_n^{queue}
0	0.000053	0.002335	0.001642	0.001655	0.001665	0.001671	0.001674	0.010643	0.001277					0.001277	0.011973
1	0.000160	0.002076	0.001459	0.001471	0.001480	0.001486	0.003163	0.011135	0.001226	0.001336				0.002562	0.013857
2	0.000320	0.001845	0.001297	0.001308	0.001316	0.001321	0.004483	0.011570	0.001177	0.001283	0.001388			0.003848	0.015738
3	0.000534	0.001845	0.001153	0.001162	0.001169	0.001174	0.005652	0.011951	0.001130	0.001231	0.001333	0.001434		0.005128	0.017613
4	0.000800	0.001458	0.001024	0.001033	0.001039	0.001043	0.006683	0.012282	0.001085	0.001182	0.001279	0.001377	0.001474	0.006397	0.019480
5		0.001296	0.000911	0.000918	0.000924	0.000927	0.007590	0.012567	0.001041	0.001135	0.001228	0.001322	0.001415	0.006141	0.018708
6		0.001152	0.000810	0.000816	0.000821	0.000824	0.008385	0.012809	0.000100	0.001089	0.001179	0.001268	0.001358	0.005895	0.018704
7		0.001024	0.000720	0.000726	0.000730	0.000733	0.009078	0.013010	0.000960	0.001046	0.001132	0.001218	0.001304	0.005660	0.018670
8		0.000910	0.000640	0.000645	0.000649	0.000651	0.009679	0.013174	0.000921	0.001004	0.001087	0.001169	0.001252	0.005433	0.018608
9		0.000809	0.000569	0.000573	0.000577	0.000579	0.010197	0.013304	0.000884	0.000964	0.001043	0.001122	0.001202	0.005216	0.018520
10		0.000719	0.000505	0.000510	0.000513	0.000515	0.010640	0.013401	0.000849	0.000925	0.001002	0.001078	0.001154	0.005008	0.018409
11		0.000639	0.000449	0.000453	0.000456	0.000458	0.011014	0.013469	0.000815	0.000888	0.000961	0.001035	0.001108	0.004807	0.018276
12		0.000568	0.000399	0.000403	0.000405	0.000407	0.011327	0.013509	0.000782	0.000853	0.000923	0.000993	0.001063	0.004615	0.018124
13		0.000505	0.000355	0.000358	0.000360	0.000361	0.011584	0.013524	0.000751	0.000819	0.000886	0.000953	0.001020	0.004430	0.017954
14		0.000449	0.000316	0.000318	0.000320	0.000321	0.011791	0.013515	0.000721	0.000786	0.000851	0.000915	0.000980	0.004253	0.017768
15		0.000399	0.000280	0.000283	0.000284	0.000286	0.011952	0.013485	0.000692	0.000754	0.000817	0.000879	0.000941	0.004083	0.017568
50		0.000007	0.000005	0.000005	0.000005	0.000005	0.007407	0.007432	0.000166	0.000181	0.000196	0.000210	0.000225	0.000978	0.008410
75		0.000000	0.000000	0.000000	0.000000	0.000000	0.003724	0.003726	0.000060	0.000065	0.000070	0.000076	0.000081	0.000352	0.004078
100		0.000000	0.000000	0.000000	0.000000	0.000000	0.001710	0.001710	0.000022	0.000023	0.000025	0.000027	0.000029	0.000127	0.001837
150		0.000000	0.000000	0.000000	0.000000	0.000000	0.000314	0.000314	0.000003	0.000003	0.000003	0.000004	0.000004	0.000016	0.000330
200		0.000000	0.000000	0.000000	0.000000	0.000000	0.000052	0.000052	0.000000	0.000000	0.000000	0.000000	0.000000	0.000002	0.000054
300		0.000000	0.000000	0.000000	0.000000	0.000000	0.000001	0.000001	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000001
$\geq$ 400		0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
Total	0.001868	0.021019	0.014774	0.014897	0.014985	0.015042	0.744672	0.825389	0.031929	0.033405	0.034709	0.035853	0.036847	0.172743	1.000000
	P^{dor}	P_5^{ser}	P_6^{ser}	P_7^{ser}	P_8^{ser}	P_9^{ser}	P_{10}^{ser}	P_{busy}	$Q_{vac}^{[0]}$	$Q_{vac}^{[1]}$	$Q_{vac}^{[2]}$	$Q_{vac}^{[3]}$	$Q_{vac}^{[4]}$	Q_{vac}	

L_q=37.954, L_s=45.957, W_q=0.31628, W_s=0.38298, L^{ser}= 9.692, L^{vac}=2.0711

TABLE 2.2: The steady state joint probabilities of queue content and server content for $M/M^{(10,19)}/1$ queue with MV, λ=135, μ=9 and ν=4.

n	$P_{n,10}$	$P_{n,11}$	$P_{n,12}$	$P_{n,13}$	$P_{n,14}$	$P_{n,15}$	$P_{n,16}$	$P_{n,17}$	$P_{n,18}$	$P_{n,19}$	P_n^*
0	0.0006267	0.0006290	0.0006307	0.0006318	0.0006325	0.0006326	0.0006323	0.0006316	0.0006305	0.0006291	0.0063068
1	0.0005876	0.0005897	0.0005912	0.0005923	0.0005929	0.0005931	0.0005928	0.0005922	0.0005911	0.0012169	0.0065398
2	0.0005508	0.0005528	0.0005543	0.0005553	0.0005559	0.0005560	0.0005558	0.0005552	0.0005542	0.0017659	0.0067561
3	0.0005164	0.0005183	0.0005196	0.0005206	0.0005211	0.0005213	0.0005210	0.0005205	0.0005195	0.0022780	0.0069564
4	0.0004841	0.0004859	0.0004872	0.0004881	0.0004886	0.0004887	0.0004885	0.0004879	0.0004871	0.0027553	0.0071413
5	0.0004539	0.0004555	0.0004567	0.0004575	0.0004580	0.0004581	0.0004579	0.0004574	0.0004566	0.0031997	0.0073115
9	0.0003506	0.0003519	0.0003528	0.0003534	0.0003538	0.0003539	0.0003538	0.0003534	0.0003527	0.0046823	0.0078586
10	0.0003287	0.0003299	0.0003308	0.0003314	0.0003317	0.0003318	0.0003316	0.0003313	0.0003307	0.0049870	0.0079648
13	0.0002708	0.0002718	0.0002725	0.0002730	0.0002733	0.0002734	0.0002733	0.0002730	0.0002725	0.0057652	0.0082188
14	0.0002539	0.0002548	0.0002555	0.0002560	0.0002562	0.0002563	0.0002562	0.0002559	0.0002555	0.0059834	0.0082836
15	0.0002380	0.0002389	0.0002395	0.0002400	0.0002402	0.0002403	0.0002402	0.0002399	0.0002395	0.0061827	0.0083392
50	0.0000249	0.0000250	0.0000250	0.0000251	0.0000251	0.0000251	0.0000251	0.0000251	0.0000250	0.0068073	0.0070326
75	0.0000050	0.0000050	0.0000050	0.0000050	0.0000050	0.0000050	0.0000050	0.0000050	0.0000050	0.0049393	0.0049842
100	0.0000010	0.0000010	0.0000010	0.0000010	0.0000010	0.0000010	0.0000010	0.0000010	0.0000010	0.0032566	0.0032656
150	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0012366	0.0012369
200	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0004275	0.0004276
300	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000447	0.0000447
400	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000043	0.0000043
450	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000013	0.0000013
500	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000004	0.0000004
550	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000001	0.0000001
≥600	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
	0.0100	0.0101	0.0101	0.0101	0.0101	0.0101	0.0101	0.0101	0.0101	0.7225	0.8134
	P_{10}^{ser}	P_{11}^{ser}	P_{12}^{ser}	P_{13}^{ser}	P_{14}^{ser}	P_{15}^{ser}	P_{16}^{ser}	P_{17}^{ser}	P_{18}^{ser}	P_{19}^{ser}	P_{busy}

TABLE 2.3: The steady state joint probabilities of queue content and type of vacation taken by the server for $M/M^{(10,19)}/1$ queue with MV, λ=135, μ=9 and ν=4.

n	$Q_n^{[0]}$	$Q_n^{[1]}$	$Q_n^{[2]}$	$Q_n^{[3]}$	$Q_n^{[4]}$	$Q_n^{[5]}$	$Q_n^{[6]}$	$Q_n^{[7]}$	$Q_n^{[8]}$	$Q_n^{[9]}$	Q_n^*	P_n^{queue}
0	0.0004205										0.0004205	0.0067273
1	0.0004084	0.0004481									0.0008564	0.0073963
2	0.0003966	0.0004352	0.0004751								0.0013069	0.0080630
3	0.0003852	0.0004227	0.0004614	0.0005014							0.0017706	0.0087270
4	0.0003741	0.0004105	0.0004481	0.0004869	0.0005270						0.0022467	0.0093880
5	0.0003633	0.0003987	0.0004352	0.0004729	0.0005119	0.0005521					0.0027341	0.0100457
6	0.0003529	0.0003872	0.0004227	0.0004593	0.0004971	0.0005362	0.0005765				0.0032320	0.0106998
7	0.0003427	0.0003761	0.0004105	0.0004461	0.0004828	0.0005208	0.0005599	0.0006004			0.0037394	0.0113500
8	0.0003329	0.0003653	0.0003987	0.0004333	0.0004689	0.0005058	0.0005438	0.0005831	0.0006237		0.0042554	0.0119961
9	0.0003233	0.0003547	0.0003872	0.0004208	0.0004554	0.0004912	0.0005282	0.0005663	0.0006057	0.0006464	0.0047793	0.0126379
10	0.0003140	0.0003445	0.0003761	0.0004087	0.0004423	0.0004771	0.0005130	0.0005500	0.0005883	0.0006278	0.0046418	0.0126066
11	0.0003049	0.0003346	0.0003653	0.0003969	0.0004296	0.0004634	0.0004982	0.0005342	0.0005713	0.0006097	0.0045082	0.0125681
14	0.0002794	0.0003066	0.0003346	0.0003636	0.0003936	0.0004245	0.0004564	0.0004894	0.0005234	0.0005586	0.0041301	0.0124137
15	0.0002713	0.0002977	0.0003250	0.0003532	0.0003823	0.0004123	0.0004433	0.0004753	0.0005084	0.0005425	0.0040112	0.0123505
50	0.0000976	0.0001072	0.0001170	0.0001271	0.0001376	0.0001484	0.0001595	0.0001711	0.0001830	0.0001952	0.0014436	0.0084762
75	0.0000471	0.0000516	0.0000564	0.0000613	0.0000663	0.0000715	0.0000769	0.0000824	0.0000882	0.0000941	0.0006957	0.0056799
100	0.0000227	0.0000249	0.0000272	0.0000295	0.0000319	0.0000345	0.0000371	0.0000397	0.0000425	0.0000453	0.0003353	0.0036008
150	0.0000053	0.0000058	0.0000063	0.0000069	0.0000074	0.0000080	0.0000086	0.0000092	0.0000099	0.0000105	0.0000779	0.0013148
200	0.0000012	0.0000013	0.0000015	0.0000016	0.0000017	0.0000019	0.0000020	0.0000021	0.0000023	0.0000024	0.0000181	0.0004456
300	0.0000001	0.0000001	0.0000001	0.0000001	0.0000001	0.0000001	0.0000001	0.0000001	0.0000001	0.0000001	0.0000010	0.0000456
400	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000001	0.0000043
450	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000013
500	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000004
550	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000001
$\geq$600	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
	0.0146 $Q_{vac}^{[0]}$	0.0156 $Q_{vac}^{[1]}$	0.0165 $Q_{vac}^{[2]}$	0.0174 $Q_{vac}^{[3]}$	0.0183 $Q_{vac}^{[4]}$	0.0192 $Q_{vac}^{[5]}$	0.0200 $Q_{vac}^{[6]}$	0.0209 $Q_{vac}^{[7]}$	0.0217 $Q_{vac}^{[8]}$	0.0225 $Q_{vac}^{[9]}$	0.1866 Q_{vac}	1.0000000

L_q=58.516, L_s=73.480, W_q=0.4334 , W_s=0.5443, L^{ser}=18.441, $L^{vac} = 4.885$

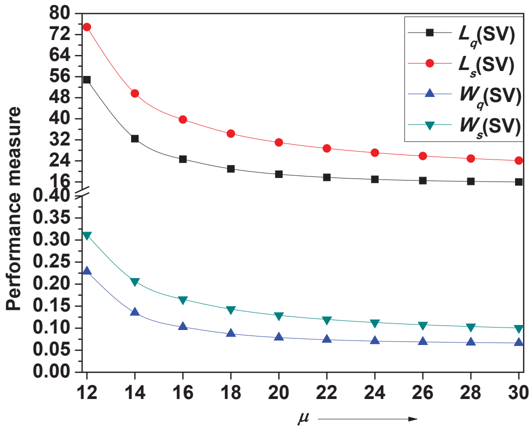

FIGURE 2.3: Effect of μ on performance measure

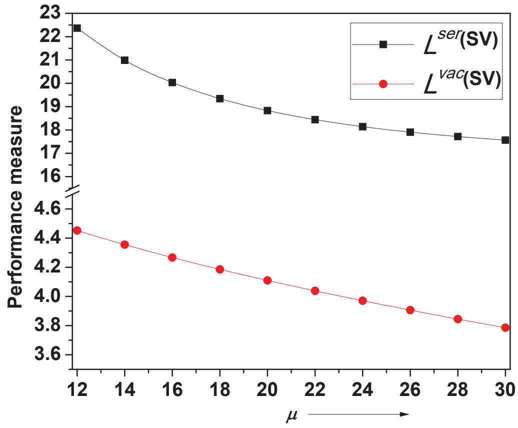

FIGURE 2.4: Effect of μ on L^{ser} and L^{vac}

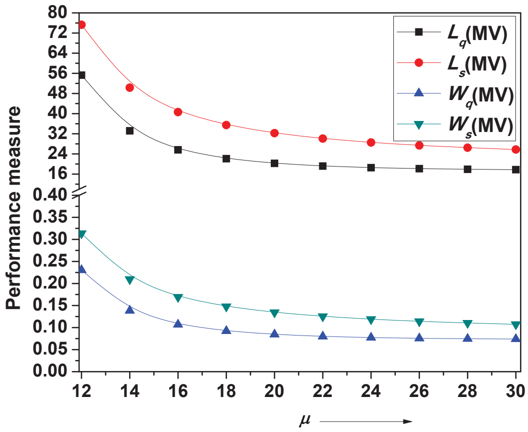

FIGURE 2.5: Effect of μ on performance measure

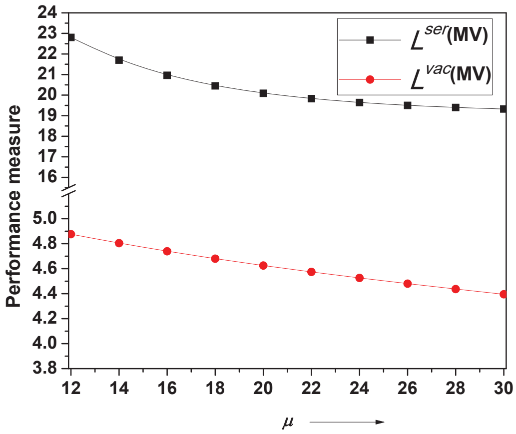

FIGURE 2.6: Effect of μ on L^{ser} and L^{vac}

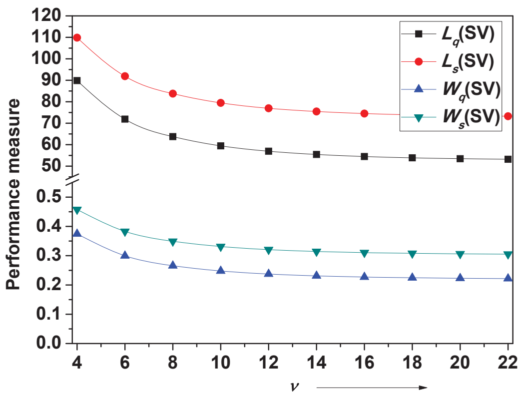

FIGURE 2.7: Effect of ν on performance measure

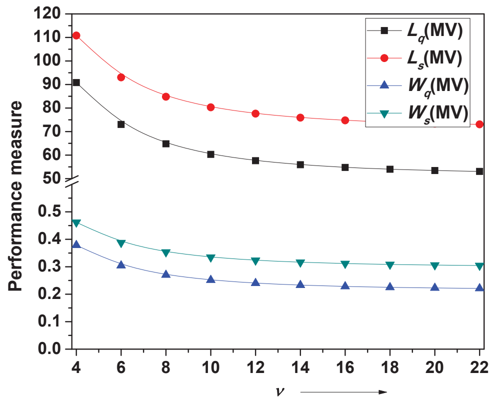

FIGURE 2.8: Effect of ν on performance measure

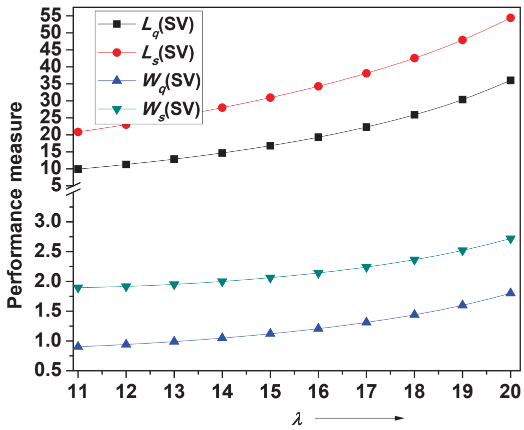

FIGURE 2.9: Effect of λ on performance measure

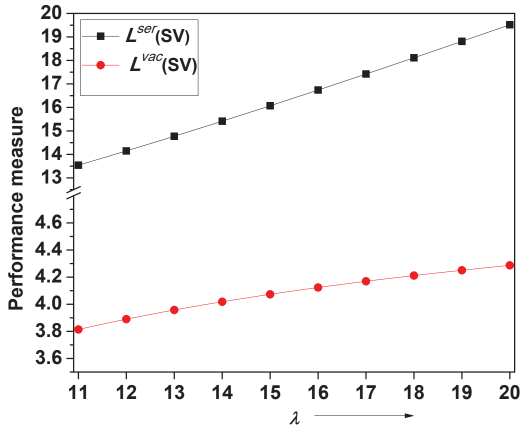

FIGURE 2.10: Effect of λ on L^{ser} and L^{vac}

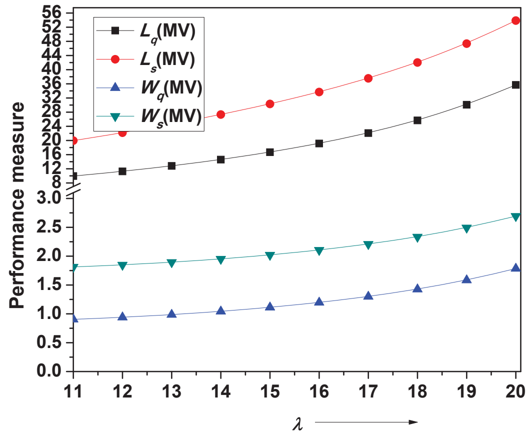

FIGURE 2.11: Effect of λ on performance measure

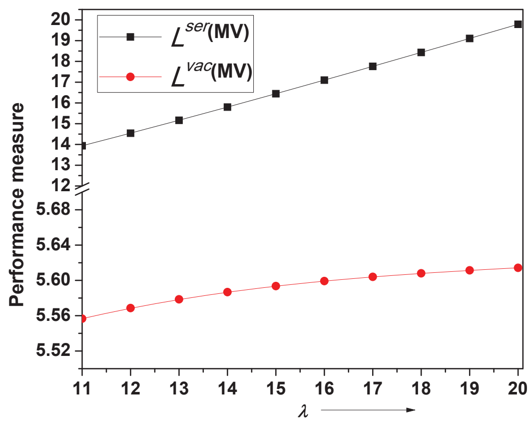

FIGURE 2.12: Effect of λ on L^{ser} and L^{vac}

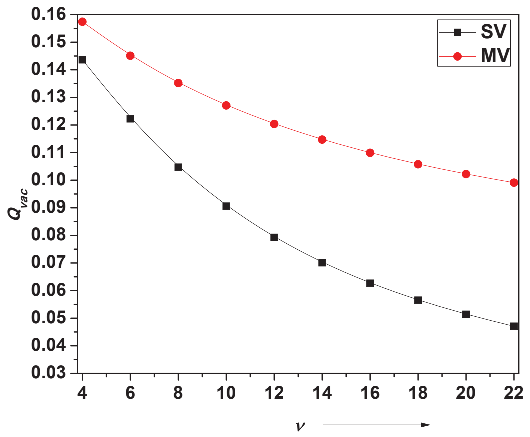

FIGURE 2.13: Effect of ν on Q_{vac}

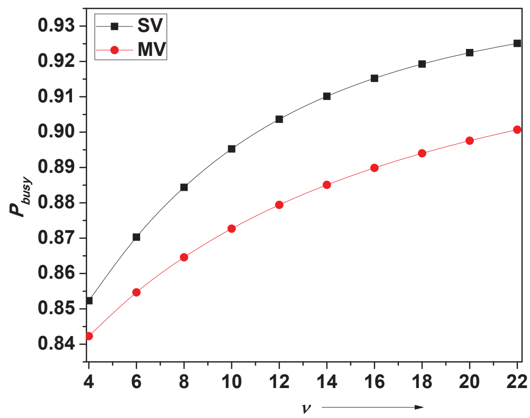

FIGURE 2.14: Effect of ν on P_{busy}

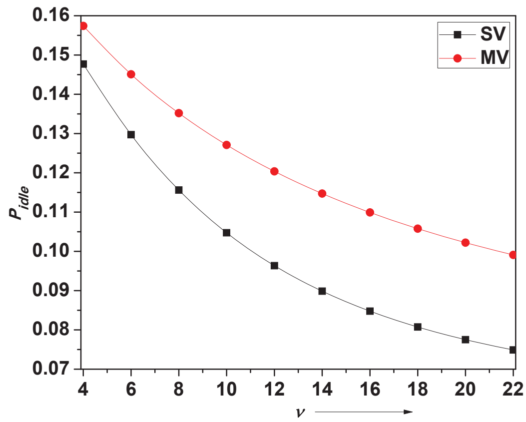

FIGURE 2.15: Effect of ν on P_{idle}

2.6.1 Cost model

The objective of this section is to optimize the minimum threshold of the GBS rule (a, b) for the queueing model under study. Keeping this in mind, a cost function is formulated for the long run system. Define the following notations for the considered cost model.
$C_{st}\equiv$ Startup cost per customer per unit time.
$C_{hol}\equiv$ Holding cost per customer per unit time.
$C_o\equiv$ Operating cost per customer per unit time.
Thus, in the long run, the total system cost (TSC) is given by:

TSC=λC_{st}+$C_{hol}L_q$+C_oL^{ser}.

The minimum threshold limit a is optimized by keeping b fixed at 10, i.e., b=10 and for different service rates μ. The values of total system cost (TSC) for different values of a and μ are presented in Table 2.4 for SV and MV. The other input parameters are taken as, $\lambda = 1.5$, $\nu = 1.3$, $C_{st} = 0.15$, $C_{hol} = 0.90$, $C_o = 0.333$.

In Table 2.4 the first column represents the value of a. The 2nd, 3rd, and 4th columns represent the value of TSC for the MV and for μ=0.3, 0.25, 0.22, respectively. Similarly, 5th, 6th and 7th columns of Table 2.4 represent the value of TSC for different values of μ, i.e., μ=0.3, 0.25, 0.22 and for SV. In Table 2.4 the minimum TSC, corresponding to each μ, and for MV (SV) are mentioned by bold letter in each column, which results in obtaining the corresponding optimum value of a, for example, in 4th column the minimum TSC is 13.28201 and the corresponding value of a is 6. Hence, one can conclude from this observation that for b=10, MV, and μ=0.22, the minimum value of TSC is 13.28201, and the optimum value of a is 6. Similar observation can be made for each column. The

TABLE 2.4: TSC for the fixed value of b=10 for the different values of a and μ

a	μ=0.3 (MV)	μ=0.25 (MV)	μ=0.22 (MV)	μ=0.30 (SV)	μ=0.25 (SV)	μ=0.22 (SV)
1	7.68418	10.57045	14.08282	7.71110	10.60578	14.12252
2	7.56123	10.39092	13.87081	7.58379	10.44922	13.94946
3	**7.52192**	10.24694	13.66072	7.49760	10.29843	13.75895
4	7.58406	10.16446	13.47838	**7.47903**	10.17719	13.56967
5	7.75175	**10.16059**	13.34564	7.54493	10.10624	13.40109
6	8.02187	10.24661	**13.28201**	7.70159	**10.10019**	13.27019
7	8.38945	10.43184	13.30788	7.94733	10.16727	13.19031
8	8.85219	10.72852	13.44979	8.27571	10.31065	**13.17123**
9	9.41617	11.15925	13.74932	8.67817	10.53008	13.22002

graphical representation of Table 2.4 is presented in Figure 2.16 (for MV) and Figure 2.17 (for SV) and the corresponding minimum value of TSC for each μ (=0.3, 0.25, 0.22) are indicated by arrow sign in the figures.

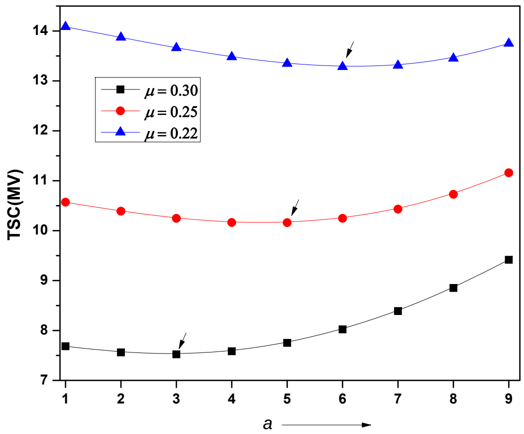

FIGURE 2.16: Effect of a on TSC for MV

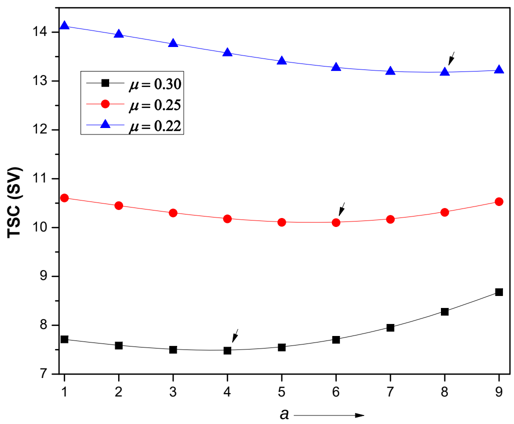

FIGURE 2.17: Effect of a on TSC for SV

2.7 Conclusion

In this chapter, an $M/M^{(a,b)}/1$ queue with single and multiple vacation has been analyzed and steady state joint distribution of the queue content and server content (when server is busy) and the joint distribution of the queue content and the type of vacation taken by the server (when server is in vacation) are obtained by using bivariate probability generating function (PGF) method. Various performance measures, such as, the average number in the queue (system), the average waiting time in queue (system) are presented. A cost model is also presented in which the minimum threshold limit for GBS rule is numerically optimized, which eventually minimizes the total system cost of the model for a particular example. The proposed analysis may be helpful to analyze infinite buffer batch size dependent bulk service vacation queueing models with general arrival and/or service and general vacation time distribution and are left for the future study. Such models may be very useful for controlling the congestion in the real life phenomenon.