Chapter 5

Purely Extending Modules and their Generalizations

In this chapter, we study some properties of purely extending modules. We also introduce purely essentially Baer modules that generalize purely extending modules as well as purely Baer modules. We provide examples and counterexamples which delineate our results.

5.1 Examples and Results on Purely Extending Modules

Definition 5.1.1. *According to J. Clark [15], a module M is called purely extending if every submodule of M is essential in a pure submodule of M; equivalently, every closed submodule of M is a pure submodule of M. A ring R is called a right (left) purely extending ring if R_R ($_RR$) is a purely extending R-module.*

Every extending module is purely extending and every module over a regular ring is purely extending [15]. The following examples show that a purely extending module need not be an extending module.

Example 5.1.2. *(i) By [26, Example 13.8], there exists a commutative continuous regular ring F such that $R = M_{2\times 2}(F)$ is neither left nor right continuous ring. Since F is a regular ring, R is also a regular ring. So R_R is a purely extending right R-module while R_R is not a right extending R-module. In fact, by [43, Proposition A.14], a regular ring is right continuous if and only if it is the right extending ring. Therefore, R_R is neither left nor right extending R-module.*

(ii) Let $\mathbb{F}$ be a field and $F_n = \mathbb{F}$ for every $n \in \mathbb{N}$. Consider $R_1 = \Pi_{n=1}^{\infty} F_n$ and $R = \{(x_n)_{n=1}^{\infty} \in R_1 : x_n$ is constant eventually$\}$, where R is a subring of R_1. Clearly, R is a regular ring but not a Baer ring. So, R is a purely extending ring but not an extending ring (see Example 3.1.14, [10]). In fact, a nonsingular extending ring is a Baer ring, but R is not a Baer ring (see Lemma 4.1.17, [10]). Hence, R_R is a purely extending R-module which is not an extending R-module.

Now, we discuss when a purely extending module will be extending module.

Theorem 5.1.3. *(i) A finitely generated flat module M over a Noetherian ring is a purely extending module if and only if it is an extending module.*

(ii) A module M over a pure semisimple ring R is purely extending if and only if it is extending.

(iii) A pure split module M is purely extending if and only if it is extending.

Proof. (i) Let N be a submodule of a purely extending module M. Then there exists a pure submodule P of M such that $N \leq^e P$. So, by Lemma 1.0.42, M/P is flat. Since M/P is finitely generated and R is a Noetherian ring, therefore M/P is finitely presented. Hence, M/P is projective by Proposition 1.0.41. Thus, $P \leq^{\oplus} M$. Hence, M is an extending module. The converse is obvious.

(ii) Let R be a pure semisimple ring and N be a submodule of a purely extending module M. Then there exists a pure submodule L of M such that $N \leq^e L$. Since R is a pure semisimple ring so for any right R-module P, the pure exact sequence $0 \to L \otimes K \to M \otimes K \to P \otimes K \to 0$ splits for every left R-module K. Therefore, the exact sequence $0 \to L \to M \to P \to 0$ is also split. So $L \leq^{\oplus} M$. Hence, M is an extending module. The converse is clear.

(iii) It follows from the fact that an R-module M is pure split if every pure submodule of M is a direct summand of M.

□

In general, submodules of a purely extending module need not be purely extending.

Example 5.1.4. *Let* $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$, *then* R_R *is a finitely generated and Noetherian R-module which is not extending (see Example 2.2, [14]). Since by Lemma 1.0.46 over the Noetherian ring R every pure submodule of a finitely generated R-module is a direct summand, therefore* R_R *is not a purely extending R-module because it is not extending. But the injective hull* $E(R_R)$ *of* R_R *is purely extending as it is injective while* R_R *is not purely extending.*

Now we provide the condition under which the submodules of a purely extending module are purely extending.

Proposition 5.1.5. *Let M be a purely extending module and N be a submodule of M. If for each pure submodule P of M, $N \cap P$ is a pure submodule of N, then N is purely extending.*

Proof. Let V be a submodule of N. Then there exists a pure submodule P of M such that $V \leq^e P$ which implies $V \leq^e P \cap N$. Since $P \cap N$ is a pure submodule of N, so $V \leq^e N \cap P \leq^p N$. Hence, N is a purely extending submodule of M. □

Proposition 5.1.6. *Let M be a module, N be a purely extending submodule of M and P be a pure submodule of M. If $P + N$ is nonsingular, then $P \cap N$ is a pure submodule of M.*

Proof. Let P be a pure submodule of M and $V = P \cap N$. Since $V \leq N$ and N is purely extending, there exists a pure submodule Q of N such that V is essential in Q. Assume that $V \neq Q$, then $P \neq P + Q$. Let $p \in P$ and $q \in Q$ such that $p + q \in P + Q$ and $p + q \notin P$ then $q \neq 0$. Thus, there exists an essential right ideal S of R such that $0 \neq qS \subseteq V$. Since P is nonsingular, $0 \neq (p + q)S \subseteq P$. Thus, P is essential in $P + Q$, which is a contradiction. Therefore, we get $V = Q$. □

Corollary 5.1.7. *If M is nonsingular and N is a purely extending submodule of M and P is a pure submodule of M, then $P \cap N$ is a pure submodule of N.*

Lemma 5.1.8. *[6, Theorem 3]. A nonsingular purely extending module is a purely Baer module.*

Proposition 5.1.9. *A direct summand of a purely extending module is purely extending.*

Proof. Let M be a purely extending module and $N \leq^{\oplus} M$. To prove N is purely extending, it suffices to prove that every closed submodule of N is a pure submodule of N. Let V be a closed submodule of N. Since every direct summand is closed, $V \leq^c N \leq^c M$. Thus, by Proposition 1.0.21(iv) $V \leq^c M$. Since M is a purely extending module, V is a pure submodule of M. Hence, by Lemma 1.0.43 V is a pure submodule of N. □

Now we give an example which shows that the direct sum of purely extending modules need not be purely extending.

Example 5.1.10. *Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_p \oplus \mathbb{Z}_{p^3}$ (where p is any prime number). Then M is not an extending R-module while $\mathbb{Z}_p$ and $\mathbb{Z}_{p^3}$ are extending R-modules. Since $\mathbb{Z}$ is a Noetherian ring and M is finitely generated, so by Lemma 1.0.46 pure submodules of M are direct summand. But from [22] M is not an extending module. Therefore, $\mathbb{Z}$-module $\mathbb{Z}_p \oplus \mathbb{Z}_{p^3}$ is not purely extending while $\mathbb{Z}_p$ and $\mathbb{Z}_{p^3}$ both are purely extending modules as these are extending.*

Now we discuss when the direct sum of purely extending modules is a purely extending module.

Proposition 5.1.11. *Let $M = \bigoplus_{i\in I} M_i$ be direct sum of R-modules M_i $(i \in I)$ for an index set $|I| \geqslant 2$. Then the following statements are equivalent:*

(i) *M is purely extending;*

(ii) *There exist $i, j \in I$, $i \neq j$ such that every closed submodule W of M with $W \cap M_i = 0$ or $W \cap M_j = 0$ is a pure submodule of M;*

(iii) *There exist $i, j \in I$, $i \neq j$, such that every complement of M_i or of M_j in M is purely extending and a pure submodule of M.*

Proof. $(i) \Rightarrow (ii)$. It is clear.

$(ii) \Rightarrow (iii)$ Let N be the complement of M_i in M, so by the hypothesis, N is a pure submodule of M. Now, to prove N is purely extending, it suffices to show that every closed submodule of N is a pure submodule of N. Let $L \leq^c N$ then $L \leq^c M$ and clearly $L \cap M_i = 0$. Therefore, L is a pure submodule of M. Hence, by Lemma 1.0.43 L is a pure submodule of N.

$(iii) \Rightarrow (i)$ Let $N \leq^c M$, so there exists a closed submodule L of N such that $N \cap M_i \leq^e L$ which implies that $L \cap M_j = 0$. By Zorn's Lemma, there exists a complement H of M_j in M such that $L \leq H$. From which it follows that $L \leq^c M$ and hence $L \leq^c H$. Applying (iii), we see that L is a pure submodule of H and H is a pure submodule of M. Applying lemma 1.0.43, $L \leq^p M$ and so $L \leq^p N$. Since $L \subseteq N \subseteq M$, by lemma 1.0.43 $N/L \leq^p M/L$. Thus, we get $L \leq^p M$ and $N/L \leq^p M/L$. Hence, $N \leq^p M$. □

Theorem 5.1.12. *Let $M = \bigoplus_{i \in I} M_i$ be direct sum of R-modules M_i $(i \in I)$, where I is an index set such that $|I| \geqslant 2$. Then M is extending module if and only if there exists a subset $\{i_1, i_2, ..., i_n\}$ of I such that every closed submodule N with either $N \cap M_{i_k} \leq^e N$ for some i_k, $1 \leq k \leq n$ or $N \cap M_{i_k} = 0$ for all k, $1 \leq k \leq n$ is a pure submodule of M.*

Proof. The only if part is trivial.

To prove if part, it is enough to prove that there exists $i \neq j \in I$ such that every closed submodule N of M with $N \cap M_i = 0$ or $N \cap M_j = 0$ is a pure submodule. To prove it, let N be a closed submodule with $N \cap M_{i_1} = N \cap M_{i_2} = \ldots = N \cap M_{i_n} = 0$. If $N \cap M_{i_1} = 0$, then by assumption N is a pure submodule of M. Now we consider $N \cap M_{i_1} \neq 0$ and L is a closed submodule of N such that $N \cap M_{i_1} \leq^e L$. Since $L \leq^c N \leq^c M$, so by Proposition 1.0.21 $L \leq^c M$. Therefore, $L \cap M_{i_1} = N \cap M_{i_1} \leq^e L$. So by hypothesis, L is a pure submodule of M. Applying Lemma 1.0.43, $L \leq^p N$ and $N/L \leq^p M/L$, so again by Lemma 1.0.43, $N \leq^p M$. Continuing in a similar way, we can prove that whenever N is a closed submodule of M with $N \cap M_{i_n} = 0$, then N is a pure submodule of M. Now there exists $i_1 \neq i_n \in I$ such that for every closed submodule N of M with $N \cap M_{i_1} = 0$ or $N \cap M_{i_n} = 0$ is pure submodule of M. Hence, M is a purely extending module. □

Now we show when finitely generated torsion-free modules and finitely generated flat modules are purely extending.

Proposition 5.1.13. *Every finitely generated torsion-free module over a principal ideal domain is purely extending.*

Proof. Let M be a finitely generated torsion-free module over a principal ideal domain R and $N \leq M$. Then M/N is either a torsion-free submodule or a torsion submodule of M. Assume first that M/N is torsion-free, then $M/N \cong R^n$ for some $n \in \mathbb{N}$, which implies M/N is projective. So M/N is flat, and hence N is a pure submodule of M. Now, we suppose that M/N is not torsion-free, then there exists a submodule $L \leq M$ containing N such that M/L is torsion-free and L/N is torsion. Since M/L is torsion-free and finitely generated R-module, therefore M/L is projective, which implies that M/L is flat. Hence, L is a pure submodule of M. Now we show that $N \leq^e L$. For it, let $l \in L \backslash N$ and $r_1 \in R$ with $lr_1 \neq 0$. Also, let $\phi : L \to L/N$ be the natural map. Since L/N is torsion submodule of M and $\phi(l)$ is non zero in L/N, so there exists a $0 \neq r_2 \in R$ such that $\phi(l)r_2 = \phi(lr_2) = 0 \in L/N$ which implies that $lr_2 \in N$. Therefore, $N \leq^e L$ and L is pure submodule of M. Hence, M is a purely extending module. □

Proposition 5.1.14. *Finitely generated flat R-module M over a principal ideal domain is purely extending.*

Proof. It follows from Proposition 5.1.13 and by the fact that a module over the principal ideal domain is flat if and only if it is torsion-free. □

Proposition 5.1.15. *Every finitely generated torsion-free module over a Prüfer ring is purely extending.*

Proof. Let M be a finitely generated torsion-free module over a Prüfer ring R and N be a closed submodule of M. Then M/N is also torsion-free. In fact, if M/N is not torsion-free, then there exists $m \in M \backslash N$ such that $mr \in N$ for some $0 \neq r \in R$, which contradicts that N is a closed submodule of M. Since M/N is finitely generated torsion-free and R is Prüfer ring, therefore M/N is flat (see, Proposition 4.20, [32]). Hence, N is a pure submodule of M, which proves that M is purely extending. □

Corollary 5.1.16. *Every finitely generated flat module over a Prüfer ring is a purely extending module.*

Proposition 5.1.17. *A nonsingular ring R is a purely extending if and only if every torsionless right R-module is flat.*

Proof. Since nonsingular purely extending ring R is purely Baer ring, therefore R is purely extending if and only if every cyclic torsionless right R-module is flat (see Theorem 1, [6]) □

Lemma 5.1.18. *[54, Lemma 3.1] Let N be a submodule of M. If $Cl_M(0) \subseteq N$, then $Cl_M(N)$ is a closed submodule M.*

The following proposition tells about the behavior of closures of submodules of a module with purely extending property.

Proposition 5.1.19. *Let N be a submodule of the purely extending R-module M. Then*

(i) $Cl(Cl(N))$ is always a purely extending module.

(ii) $Cl(N)$ *is purely extending if* $N \supseteq Cl(0)$.

Proof. (i) Since $Cl(N) \supseteq Cl(0)$ therefore by Lemma 5.1.18, $Cl(Cl(N))$ is always closed in M. Thus, $Cl(Cl(N))$ is a pure submodule of M, hence a purely extending module.

(ii) Since under the given conditions $Cl(N)$ is closed, which implies that $Cl(N)$ is pure submodule of M, hence a purely extending module.

□

The following example shows that the endomorphism ring of a purely extending module need not be purely extending.

Example 5.1.20. *[14, Example 2.3]. Let* $R = \begin{pmatrix} \mathbb{C} & \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{R} & \mathbb{C} \\ 0 & 0 & \mathbb{C} \end{pmatrix}$ *and* $e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. *Note that* R_R *is extending* R*-module. Therefore,* R_R *is purely extending* R*-module. Take* $M = eR$*, then* $S = End_R(M) \cong \begin{pmatrix} \mathbb{C} & \mathbb{C} & 0 \\ 0 & \mathbb{R} & 0 \\ 0 & 0 & 0 \end{pmatrix}$ *Since* M *is a direct summand of* R_R*, so* M *is purely extending. But* S *is not right purely extending ring. In fact, it is easy to show that closed right ideal* $\begin{pmatrix} 0 & \mathbb{C} & 0 \\ 0 & \mathbb{R} & 0 \\ 0 & 0 & 0 \end{pmatrix}$ *is not essential in any pure right ideal of* S_S.

In the following proposition, we find conditions under which the endomorphism ring of a purely extending module is purely extending.

Proposition 5.1.21. *(i) If* M *is a finitely generated projective right* R*-module over a regular ring, then* $S = End_R(M)$ *is purely extending.*

(ii) If M is finitely cogenerated right R-module over a right V-ring, then $S = End_R(M)$ is purely extending.

Proof. (i) From [26, Theorem 1.7], the endomorphism ring S of a finitely generated projective R-module M is von Neumann regular. Therefore, S is a purely extending ring.

(ii) If M is finitely cogenerated right R-module over V-ring R then by [40, Proposition 2.14], M is endoregular. Therefore, S is a von Neumann regular ring, so S is purely extending.

□

Proposition 5.1.22. *Let R be a semisimple Artinian ring. Then the endomorphism ring of every right R-module M is purely extending.*

Proof. Let R be a semisimple Artinian ring and M be an R-module with endomorphism ring S. Since over a semisimple Artinian ring R every R-module is endoregular (see [40, Proposition 2.17]), so the R-module M is endoregular. Therefore S is a von Neumann regular ring. Hence, S is purely extending. □

5.2 Purely Essentially Baer Modules

Definition 5.2.1. *An R-module M is called a purely essentially Baer module if for every left ideal I of $S = End_R(M)$, $Ann^r_M(I)$ is essential in a pure submodule of M. Further, R is right purely essentially Baer ring if R_R is a purely essentially Baer R-module.*

Proposition 5.2.2. *Consider the following statements for a right R-module M.*

(i) M is a purely Baer module.

(ii) M is a purely extending module.

(iii) M is a purely essentially Baer module.

Then $(i) \Rightarrow (iii)$ and $(ii) \Rightarrow (iii)$ but in general, these implications are not reversible.

Proof. $(i) \Rightarrow (iii)$ Let M be an R-module, $S = End_R(M)$ and I be a left ideal of S. By (i) $Ann^r_M(I)$ is pure submodule of M Therefore, M is an purely essentially Baer module.

$(ii) \Rightarrow (iii)$ It is clear that $Ann^r_M(I) \leq M$ for every left ideal I of S and M is purely extending module which implies that $Ann^r_M(I)$ is essential in a pure submodule of M.

$(ii) \not\Rightarrow (i)$ The $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ (where p is prime) is purely essentially Baer module while $\mathbb{Z}_{p^\infty}$ is not purely Baer $\mathbb{Z}$-module.

$(iii) \not\Rightarrow (ii)$ Let $R = \begin{pmatrix} \mathbb{F} & 0 & \mathbb{F} \\ 0 & \mathbb{F} & \mathbb{F} \\ 0 & 0 & \mathbb{F} \end{pmatrix}$ the $\mathbb{F}$-subalgebra of the ring $M_{3\times 3}(\mathbb{F})$. R is the left and right Artinian hereditary ring. Hence, R is the left and right nonsingular ring. So from [6, Theorem 5], R is purely Baer ring and so R_R is purely Baer R-module. Hence, R_R is a purely essentially Baer R-module, however, R_R is not purely extending. In fact, if R_R is purely extending then R_R is extending R-module but R_R is not extending (see [13, Example 5.5]). □

In the following proposition, we prove that when a purely essentially Baer module is purely Baer.

Proposition 5.2.3. *Let M be a nonsingular right R-module with $S = End_R(M)$. If M is a purely essentially Baer module, then M is a purely Baer module.*

Proof. Let M be a purely essentially Baer module and I be a left ideal S. Then $Ann^r_M(I) \leq^e P$, where P is a pure submodule of M. Let $U = \{r \in R : pr \in Ann^r_M(I) \text{ for } p \in P\}$. Then, $U \leq^e R_R$ and $pU \subseteq Ann^r_M(I)$, so for each $\phi \in I$, $\phi(pU) = \phi(p)U = 0$. Since M is nonsingular, $\phi(p) = 0$ for each $\phi \in I$. Therefore, $Ann^r_M(I) = P$ is a pure submodule of M. Hence, M is a purely Baer module. □

In the following proposition, we show when notions of purely essentially Baer modules and essentially Baer modules are equivalent to each other.

Proposition 5.2.4. *(i) Let M be a pure split module with $S = End_R(M)$. Then M is purely essentially Baer module if and only if M is essentially Baer module.*

,(ii) Let R be right Noetherian ring and M be a finitely generated flat right R-module. Then M is purely essentially Baer module if and only if it is essentially Baer module.

(iii) Let R be a right pure semisimple ring. Then a right R-module M is purely essentially Baer module if and only if M is essentially Baer module.

Proof. (i) Let M is purely essentially Baer module and I be left ideal of S. Then $Ann^r_M(I) \leq^e P$ for some pure submodule P of M. Since M is a pure split module, P is a direct summand of M. Hence, M is an essentially Baer module. The converse is clear.

(ii) Let M be purely essentially Baer module and $I \leq S$. Then $Ann^r_M(I) \leq^e P$ for any pure submodule P of M. By Lemma 1.0.42, M/P is flat. Since by hypothesis M/P is finitely generated and R is right Noetherian, therefore by Proposition 1.0.41, M/P is projective, which implies $P \leq^{\oplus} M$. Hence, M is an essentially Baer module. The converse is clear from the definition.

(iii) The proof follows from the fact that for a pure semisimple ring R, every pure exact sequence of R-modules is split.

□

The following proposition shows when purely essentially Baer modules are closed under summands.

Proposition 5.2.5. *A direct summand of a purely essentially Baer module M is purely essentially Baer module if each pure submodule of M is fully invariant.*

Proof. Let $M = M_1 \oplus M_2$ be an R-module. Then $S = End_R(M) = \begin{pmatrix} S_1 & S_{12} \\ S_{21} & S_2 \end{pmatrix}$, where $S_i = End_R(M_i)$ for $i = 1, 2$ and $S_{ij} = Hom_R(M_j, M_i)$ for $i \neq j$, $i, j = 1, 2$. Let I be a left ideal of S_1 and J=$\{\sum_{i=1}^{n} f_i g_i$: $f_i \in S_{21}$ and $g_i \in I$ for all $n \in \mathbb{N}\}$, then $T = \begin{pmatrix} I & 0 \\ J & 0 \end{pmatrix}$ is clearly a left ideal of S. Since M is the purely essentially Baer module, $Ann^r_M(T) \leq^e N$ for some pure submodule N of M. By assumption, every pure submodule of M is fully invariant, so N is a fully invariant in M. Therefore by [48, Lemma 1.10] $N = N_1 \oplus N_2$ such that $N_1 \trianglelefteq M_1$ and $N_2 \trianglelefteq M_2$, where $N_i = N \cap M_i$ for $i = 1, 2$. Now, for any $(m_1 + m_2) \in M$, where $m_1 \in M_1$ and $m_2 \in M_2$, the element $m_1 + m_2 \in Ann^r_M(I)$ if and only if $m_1 \in Ann^r_{M_1}(I)$. Therefore, $Ann^r_M(T) = Ann^r_{M_1}(I) \oplus M_2 \leq^e N_1 \oplus N_2$ which implies that $Ann^r_{M_1}(I) \leq^e N_1$. Since N_1 is a direct summand of N and N is a pure submodule of M, so $N_1 \trianglelefteq^p N \trianglelefteq^p M$. Thus by Lemma 1.0.43 N_1 is a pure submodule of M. Hence, M_1 is a purely essentially Baer module. □

In the following theorem, we characterize von Neumann regular rings in terms of purely essentially Baer modules.

Theorem 5.2.6. *Let M be an R-module with $S = End_R(M)$. Then the following are equivalent:*

(i) Every purely essentially Baer R-module is purely Baer;

(ii) Every purely extending R-module is purely Baer;

(iii) R is von Neumann regular ring.

Proof. $(i) \Rightarrow (ii)$ Let M be a purely extending module and I be a left ideal of S. Then $Ann^r_M(I)$ is essential in a pure submodule X of M, which implies that M is purely essentially Baer module. Therefore, from (i), M is purely Baer module.

$(ii) \Rightarrow (iii)$ Let M be a R-module and $E(M)$ is injective hull of M. The homomorphism $\phi : E(M) \to E(E(M)/M)$ defined by $\phi(h) = h + M$ for each $h \in E(M)$, can be extended by the endomorphism $\bar{\phi}$ of $E(M) \oplus E(E(M)/M)$ and $Ker(\bar{\phi}) = M$. Since $E(M) \oplus E(E(M)/M)$ is a purely extending module, so by (ii) it is a purely Baer module. Hence, M is pure in $E(M) \oplus E(E(M)/M)$, which implies that M is pure in $E(M)$. Therefore, M is absolutely pure R-module. Hence, R is a regular ring.

$(iii) \Rightarrow (i)$ Let R be a von Neumann regular ring and M be a purely essentially Baer R-module. It is clear from [6, Theorem 4] that every module over a von Neumann regular ring is purely Baer. Therefore, M is a purely Baer module. □

In the following proposition, we show when the direct sum of purely essentially Baer modules is purely essentially Baer.

Proposition 5.2.7. *Let $M = \bigoplus_{\lambda \in \Lambda} M_\lambda$ (where Λ is an index set) be such that $Hom(M_\lambda, M_\mu) = 0$ for every $\lambda \neq \mu \in \Lambda$. Then M is purely essentially Baer if and only if each M_λ $(\lambda \in \Lambda)$ is purely essentially Baer.*

Proof. If part is clear from proposition 5.2.5.

For only if part, let each M_λ is purely essentially Baer module and $S = End(M)$. Since $Hom(M_\lambda, M_\mu) = 0$ for every $\lambda \neq \mu \in \Lambda$ therefore S viewed as a diagonal matrix with S_λ $(\lambda \in \Lambda)$ on its diagonal, where $S_\lambda = End(M_\lambda)$. Let T be left ideal of S, then $Ann^r_M(T) = \bigoplus_{\lambda\in\Lambda} Ann^r_{M_\lambda}(T \cap S_\lambda)$. As each M_λ is purely essentially Baer, therefore $Ann^r_{M_\lambda}(T \cap S_\lambda) \leq^e X_\lambda$ for a pure submodule X_λ of M_λ. So we get $Ann^r_M(T) \leq^e \bigoplus_{\lambda\in\Lambda} X_\lambda$. Since each X_λ is pure in M_λ therefore $\bigoplus_{\lambda\in\Lambda} X_\lambda$ is pure in $\bigoplus_{\lambda\in\Lambda} M_\lambda$. Hence, M is purely essentially Baer module. □

Proposition 5.2.8. *Let N be a submodule of a purely essentially Baer module M. If $N \cap X$ is a pure submodule of N for every pure submodule X of M, then N is a purely essentially Baer.*

Proof. Let $T = End_R(N)$ and I be a left ideal of T. As M is a purely essentially Baer module, so $Ann^r_M(I) \leq^e X$, where X is a pure submodule of M. Now $Ann^r_N(I) = N \cap Ann^r_M(I)$ which is clearly essential in X. From the assumption $N \cap Ann^r_M(I)$ is a pure submodule of N. Hence, N is a purely essentially Baer. □

Proposition 5.2.9. *A finitely generated $\mathbb{Z}$-module M is a purely essentially Baer module if M is semisimple or torsion-free module.*

Proof. If M is a semisimple module, then it is purely essentially Baer. If M is a finitely generated torsion-free $\mathbb{Z}$-module, then $M \cong \mathbb{Z}^n$, $n \in \mathbb{N}$, which is clearly purely essentially Baer module. □

The Converse of proposition 5.2.9 need not be true.

Example 5.2.10. *Let $M = \mathbb{Z} \oplus \mathbb{Z}_p$ be a $\mathbb{Z}$-module, where p is prime. Clearly, M is a purely essentially Baer module, but M is neither torsion free nor semisimple.*

Proposition 5.2.11. *For a finitely generated projective R-module M, the following are equivalent:*

(i) M is purely essentially Baer module;

(ii) The endomorphism ring of M is a left purely Baer ring.

Proof. $(i) \Rightarrow (ii)$ Let M be a purely essentially Baer module and S be the endomorphism ring of M. It is well known that the endomorphism ring of a finitely generated projective module is von Neumann regular. Therefore, S is a purely Baer ring.

$(ii) \Rightarrow (i)$ From [6, Proposition 5] if the endomorphism ring of a finitely generated projective module M is purely Baer, then M is purely Baer module. Hence, M is purely essentially Baer module. □