Chapter 4

Finite Σ-dual-Rickart Modules

In [35], Lee and Barcenas introduced the notion of finite Σ-Rickart modules. Motivated by the notion of finite Σ-Rickart modules, we introduce the notion of finite Σ-dual-Rickart modules. The class of finite Σ-dual-Rickart modules generalizes the class of Σ-Rickart modules. In this chapter, we study some properties of finite Σ-dual-Rickart modules and give a comparison between finite Σ-Rickart modules and finite Σ-dual-Rickart modules. In the last section, we study endomorphism rings of finite Σ-dual-Rickart modules.

4.1 Finite Σ-dual-Rickart Modules

Definition 4.1.1. *An R-module M is said to be a finite Σ-dual-Rickart if the direct sum of finitely many copies of M is a dual-Rickart module. Equivalently, a module M is said to be a finite Σ-dual-Rickart module if $M^{(n)} = \bigoplus_{i=1}^{n} M_i$ ($M_i = M$ for each i) is a dual-Rickart for every $n \in \mathbb{N}$. Further, a ring R is called a right (left) finite Σ-dual-Rickart if R_R (${}_RR$) is a finite Σ-dual-Rickart module.*

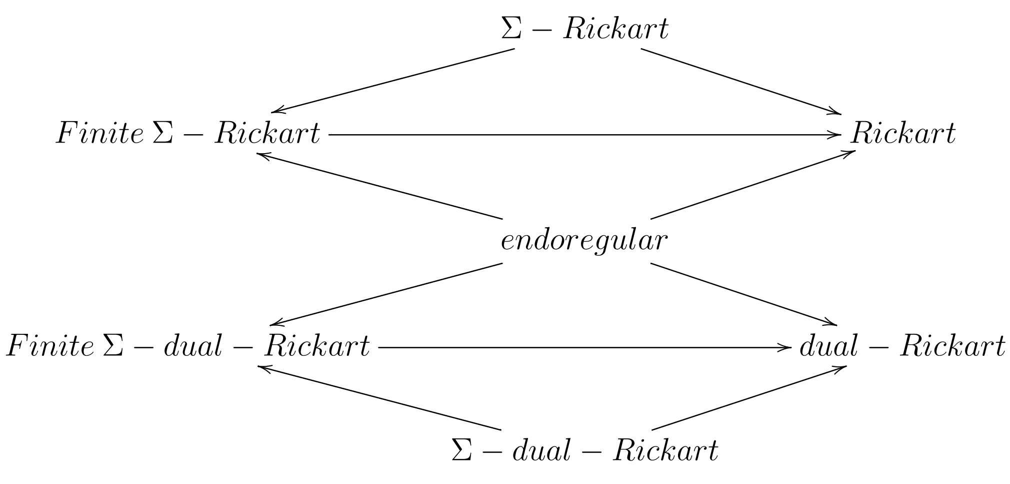
Example 4.1.2. *(i) Every dual-Rickart module with D2-condition is a finite Σ-dual-Rickart module as from [37, Corollary 5.12] the direct sum of finitely many copies of a dual-Rickart module with D2-condition is dual-Rickart.*

(ii) Every indecomposable Hopfian dual-Rickart module is a finite Σ-dual-Rickart module [37, Remark 5.13].

(iii) Every injective module over a right hereditary ring is a finite Σ-dual-Rickart module, as every injective module over a right hereditary ring is a dual-Rickart module [37, Theorem 2.29] and finite direct sum of injective modules is injective.

(iv) A ring R is a finite Σ-dual-Rickart if R is a von Neumann regular ring.

We have the following diagram for a right R-module M:



The reverse of any arrows in the above diagram need not be true. The following example illustrate it.

Example 4.1.3. *(i) Let $R = \Pi_{n=1}^{\infty} F_n$, where $F_n = \mathbb{Z}_2$ for each n. It is easy to see that the ring R is a von Neumann regular. From [37, Proposition 2.25], it is clear that for a von Neumann regular, the direct sum of finite copies* *of R is a dual-Rickart R-module. Therefore R_R is a dual-Rickart as well as finite Σ-dual-Rickart R-module. Since, $M = R^{(R)}$ satisfies D2-condition and $End_R(M)$ is not a von Neumann regular ring. Therefore, from [37, Theorem 3.8] M is not a dual-Rickart R-module. Hence, M can not be a Σ-dual-Rickart R-module.*

(ii) Let $M = \mathbb{Z}_{p^\infty}$ (Prüfer p-group for a prime p) and $R = \mathbb{Z}$. It is clear from [37, Theorem 2.29] that every injective R-module is a dual-Rickart module if and only if R is a hereditary ring. Therefore, $M^{(\Lambda)}$ is a dual-Rickart R-module for every non-empty index set Λ. Thus, M is Σ-dual-Rickart module as well as finite Σ-dual-Rickart module, while M is not an endoregular module (see [40, Remark 4.28]).

(iii) From [35, Example 2.3], the converse of "$\Sigma - Rickart \Rightarrow Finite\,\Sigma - Rickart$" and "$endoregular \Rightarrow Finite\ \Sigma - Rickart \Rightarrow Rickart$" need not be true.

(iv) On the basis of Corollary 5.12 and Remark 5.13 of [37] theoretically, it seems that the finite direct sum of copies of a dual-Rickart module without an extra condition (like D2-condition, indecomposable Hopfian module) need not be a dual-Rickart module. Therefore, we can say that a dual-Rickart module need not be a finite Σ-dual-Rickart module.

The following proposition shows that the finite Σ-dual-Rickart module is closed under direct summand.

Proposition 4.1.4. *The direct summand of a finite Σ-dual-Rickart module is finite Σ-dual-Rickart.*

Proof. Let M be a finite Σ-dual-Rickart module and N be a direct summand of M. Then, $M^{(n)}$ is a dual-Rickart module for each $n \in \mathbb{N}$. It is easy to see that $N^{(n)}$ is a direct summand of $M^{(n)}$. Since the direct summand of a dual-Rickart module is dual-Rickart, so $N^{(n)}$ is a dual-Rickart module. Hence, N is a finite Σ-dual-Rickart module. □

Proposition 4.1.5. *If M is a finite Σ-dual-Rickart module, then $M^{(n)}$ is a finite Σ-dual-Rickart module for every positive integer n.*

Proof. Let M be a finite Σ-dual-Rickart module. Then $M^{(n)}$ is a dual-Rickart module. Thus, $(M^{(n)})^{(m)} = M^{(nm)}$ is also a dual-Rickart module for any positive integer m. Hence, $M^{(n)}$ is a finite Σ-dual-Rickart module. □

Lemma 4.1.6. *If M is a finite Σ-dual-Rickart module, then $M^{(n_1)}$ is $M^{(n_2)}$ dual-Rickart for every positive integers n_1 and n_2.*

Proof. Let $\psi : M^{(n_1)} \to M^{(n_2)}$ be any homomorphism. Since M is a finite Σ-dual-Rickart module, $M^{(n_1 n_2)}$ is a dual-Rickart module. So, $M^{(n_1 n_2)}$ is $M^{(n_1 n_2)}$-dual-Rickart. Hence, from [37, Theorem 2.19] $M^{(n_1)}$ is $M^{(n_2)}$-dual-Rickart. □

For a module M, $add(M)$ [61] denotes the class of all right R-modules, which are isomorphic to a direct summand of $M^{(n)}$ for a positive integer n.

Proposition 4.1.7. *A module M is a finite Σ-dual-Rickart module if and only if every module $N \in add(M)$ is a finite Σ-dual-Rickart module.*

Proof. Let $N \in add(M)$. Then, there exists a module K such that $N \cong K \leq^{\oplus} M^{(n)}$, for a positive integer n. So, by Proposition 4.1.4 and Proposition 4.1.5, K and $M^{(n)}$ are finite Σ-dual-Rickart modules, respectively. Hence, N is a finite Σ-dual-Rickart module. The converse follows from the fact that the module M also lies in $add(M)$. □

Proposition 4.1.8. *Every cohereditary module is finite Σ-dual-Rickart.*

Proof. Let M be a cohereditary module. By [50, Proposition 3.3′], $M^{(n)}$ is also a cohereditary module for each positive integer n. So, for every $\psi \in End_R(M^{(n)})$, $M^{(n)}/Ker(\psi) \cong Im(\psi)$ is injective.
Therefore, the short exact sequence $0 \to Im(\psi) \to M^{(n)} \to Coker(\psi) \to 0$ splits. So, $Im(\psi)$ is a direct summand of $M^{(n)}$. Thus, $M^{(n)}$ is a dual-Rickart module. Hence, M is a finite Σ-dual-Rickart module. □

The following example illustrates that the converse of Proposition 4.1.8 is not true in general.

Example 4.1.9. *Let $R = \Pi_{\lambda\in\Lambda} R_\lambda$, where Λ is an infinite index set and $R_\lambda = \mathbb{F}$ for every $\lambda \in \Lambda$, where $\mathbb{F}$ is any field. Clearly, R is a von Neumann regular ring. Therefore, from [37, Proposition 2.25] right R-module $R^{(n)}$ is dual-Rickart for every $n \in \mathbb{N}$. Hence, R_R is a finite Σ-dual-Rickart R-module, while R_R is not a cohereditary R-module (see Example 3.1, [57]).*

Proposition 4.1.10. *For an R-module M, the following statements are true:*

(i) Every finite Σ-dual-Rickart module has SSP.

(ii) Every finite Σ-dual-Rickart module with $D3$ condition has SIP.

Proof. (i) Let M be any finite Σ-dual-Rickart module. Then for every positive integer n, $M^{(n)}$ must be a dual-Rickart module. Since every dual-Rickart module has SSP [37, Proposition 2.11], so $M^{(n)}$ has SSP. Hence, the module M also has SSP.
(ii) Let M be a finite Σ-dual-Rickart module with $D3$-condition. Since from [2, Lemma 19] a $D3$-module with SSP has SIP. Hence, the module M has SIP. □

Corollary 4.1.11. *A quasi-projective finite* Σ*-dual-Rickart module has SIP.*

Proof. Since every quasi-projective module has $D3$-condition, so the proof follows from Proposition 4.1.10(ii) □

Note 4.1.12. *The converse of both the statements of Proposition 4.1.10 need not be true. It can be seen in the following example.*
The $\mathbb{Z}$*-module* $\mathbb{Z}_4$ *satisfies the SIP and the SSP, but it is neither Rickart nor dual-Rickart. So,* $\mathbb{Z}_4$ *can not be a finite* Σ*-dual-Rickart* $\mathbb{Z}$*-module.*

In the following Proposition, we characterize von Neumann regular rings in terms of the finite Σ-dual-Rickart modules.

Proposition 4.1.13. *The following statements are equivalent for a ring* $quasi-projective$

(i) Every finitely generated free (projective) R*-module is a finite* Σ*-dual-Rickart;*

(ii) The free R*-module* $R^{(2)}$ *has SSP;*

(iii) R *is a von Neumann regular ring.*

Proof. $(i) \Rightarrow (ii)$ Clearly, $R^{(2)}$ is a finitely generated free R-module. Therefore, by hypothesis $R^{(2)}$ is a finite Σ-dual-Rickart ring. So, from Proposition 4.1.10, $R^{(2)}$ has summand sum property.
$(ii) \Rightarrow (iii)$ It is clear from [37, Proposition 2.25].
$(iii) \Rightarrow (i)$ Let M be a finitely generated free R-module, then $M \cong R^{(n)}$ for some $n \in \mathbb{N}$. From hypothesis, R is a von Neumann regular ring, so for every $k \in \mathbb{N}$ $Mat_k(M) \cong Mat_k(R^{(n)}) \cong End_R(R^{(n\times k)})$ is von Neumann regular ring. Hence, $M^{(k)}$ is a dual-Rickart module, which implies that M is finite Σ-dual-Rickart. □

Proposition 4.1.14. *A projective and quasi-injective right R-module M over right hereditary ring R is a finite Σ-dual-Rickart module.*

Proof. Let M be a projective and quasi-injective module. Then for every $n \in \mathbb{N}$, $M^{(n)}$ is projective and continuous. Therefore, $M^{(n)}$ is a projective continuous right R-module. Since projective continuous modules over the hereditary ring are dual-Rickart [37, Proposition 2.4], so $M^{(n)}$ is a dual-Rickart module as by assumption R is hereditary. Hence, M is a finite Σ-dual-Rickart module. □

In the following Proposition, we characterize finite Σ-dual-Rickart modules in terms of the hereditary rings.

Proposition 4.1.15. *A ring R is hereditary if and only if every injective R-module is a finite Σ-dual-Rickart module.*

Proof. Let M be an injective module. Then for every $n \in \mathbb{N}$, $M^{(n)}$ is an injective module. Now, let $\psi \in End_R(M^{(n)})$ be arbitrary. Since R is a right hereditary ring, $M^{(n)}/Ker(\psi) \cong Im(\psi)$ is an injective module. Therefore, $Im(\psi) \leq^{\oplus} M^{(n)}$. Thus, $M^{(n)}$ is a dual-Rickart module. Hence, M is a finite Σ-dual-Rickart module.
Conversely, let M be an injective module and N be any submodule of M. Then $M \oplus E(M/N)$ is an injective module. By hypothesis, $M \oplus E(M/N)$ is a finite Σ-dual-Rickart module. So, $M \oplus E(M/N)$ is also dual-Rickart module. From Lemma 3.1.7, M is $E(M/N)$-dual-Rickart. Now define a homomorphism $\phi : M \to E(M/N)$ such that $\phi(m) = m + N$. Then $Im(\phi) = M/N \leq^{\oplus} E(M/N)$. Therefore, M/N is an injective module. Hence, R is a right hereditary ring. □

In the following proposition, with the help of Lemma 4.1.16, we characterize finite Σ-dual-Rickart modules in terms of finitely M-cogenerated modules. Recall from

[5] that an R-module N is finitely M-cogenerated if there exists a monomorphism $\rho : N \to M^{(n)}$ for every $n \in \mathbb{N}$.

Lemma 4.1.16. *[5, Proposition 10.8]. The direct sum of two finitely M-cogenerated modules is finitely M-cogenerated.*

Proposition 4.1.17. *If M is a finite Σ-dual-Rickart module, then the sum of two finitely M-cogenerated submodules of $K \in add(M)$ is finitely M-cogenerated.*

Proof. (i) Let $K \in add(M)$ and K_1, K_2 be two finitely M-cogenerated submodules of K. Consider the exact sequence, $0 \to K_1 \cap K_2 \xrightarrow{f} K_1 \oplus K_2 \xrightarrow{g} K_1 + K_2 \to 0$, where for any $k \in K_1 \cap K_2$, $f(k) = (k, k)$ and for any $(k_1, k_2) \in K_1 \oplus K_2$, $g(k_1, k_2) = k_1 + k_2$. Since K_1 and K_2 are finitely M-cogenerated, from Lemma 4.1.16 $K_1 \oplus K_2$ is finitely M-cogenerated. So, there exists a monomorphism $h : K_1 \oplus K_2 \to M^{(n)}$ for some positive integer n. Since M is a finite Σ-dual-Rickart and $K \in add(M)$, so by Lemma 4.1.6 K is $M^{(n)}$ dual-Rickart. Therefore, $Im(hf)$ is a direct summand of $M^{(n)}$. It is clear that $Im(hf) = h(Imf)$. Therefore, from Lemma 3.1.15, $Im(f)$ is a direct summand of $K_1 \oplus K_2$. Thus, $(K_1 \oplus K_2)/Im(f)$ is finitely M-cogenerated. Since $K_1 + K_2 \cong (K_1 \oplus K_2)/Im(f)$, hence $K_1 + K_2$ is finitely M-cogenerated. □

4.2 Finite Σ-dual-Rickart Modules vs. Finite Σ-Rickart Modules

The notion of finite Σ-dual-Rickart modules is dual of the notion of finite Σ-Rickart modules. In this section, we study when a finite Σ-Rickart module implies a finite Σ-dual-Rickart module and vice-versa. Also, we discuss when these notions are equivalent to each other.

The following example illustrates that the class of finite Σ-Rickart modules and the class of finite Σ-dual-Rickart modules are independent of each other.

Example 4.2.1. *As every injective right R-module over a right hereditary ring R is a dual-Rickart module [37, Theorem 2.29]. Therefore, $\mathbb{Z}_{p^\infty}^{(\mathscr{I})}$ is a dual-Rickart $\mathbb{Z}$-module for any finite index set $\mathscr{I}$. So, $\mathbb{Z}_{p^\infty}$ is a finite Σ-dual-Rickart $\mathbb{Z}$-module while it is not a Rickart $\mathbb{Z}$-module [36]. Therefore, $\mathbb{Z}_{p^\infty}$ is not a finite Σ-Rickart module. Further, $\mathbb{Z}$ considered as $\mathbb{Z}$-module is a finite Σ-Rickart module (see [35]), but $\mathbb{Z}$ considered as $\mathbb{Z}$-module is not a finite Σ-dual-Rickart as $\mathbb{Z}$ is not a dual-Rickart module (see [37]).*

Proposition 4.2.2. *Let $M = M_1 \oplus M_2$, where M_1 and M_2 are submodules of M. Then the following statements hold:*

(i) If M is a $C4$-module and M_1 is M_2-Rickart then M_1 is M_2-dual-Rickart.

(ii) If M is a $D4$-module and M_1 is M_2-dual-Rickart then M_1 is M_2-Rickart.

Proof. (i) Let M_1 be M_2-Rickart module and $\psi : M_1 \to M_2$ be any homomorphism. Then $Ker(\psi) \leq^{\oplus} M_1$. Since M has $C4$-condition, $Im(\psi) \leq^{\oplus} M_2$. Hence, M_1 is M_2-dual-Rickart module.

(ii) Let M_1 be M_2-dual-Rickart module and $\psi : M_1 \to M_2$ be any homomorphism. Then $Im(\psi) \leq^{\oplus} M_2$. Since M has $D4$-condition, so $Ker(\psi) \leq^{\oplus} M_1$. Hence, M_1 is M_2-Rickart module. □

Corollary 4.2.3. *Let $M = M_1 \oplus M_2$ for submodules M_1 and M_2 of M. Then the following statements hold:*

(i) If M is a $C2$ (or $C3$) module and M_1 is M_2-Rickart module then M_1 is M_2-dual-Rickart module.

(ii) *If M is a $D2$ (or $D3$) module and M_1 is M_2-dual-Rickart module then M_1 is M_2-Rickart module.*

Proof. The proof is clear because every $C2$ and $C3$ modules are $C4$ module. □

The following examples show that the $C4$-condition and $D4$-condition in Proposition 4.2.2 are not superfluous.

Example 4.2.4. (i) *Let $M = \mathbb{Z} \oplus \mathbb{Z}$. It is clear that $\mathbb{Z}$ is a $\mathbb{Z}$-Rickart module. Since $\mathbb{Z}$ is not a $C2$ module, by [20, Proposition 2.15] M is not a $C4$ module. Further, $\mathbb{Z}$ is not a Σ-dual-Rickart module as it is not a dual-Rickart module [37].*
(ii) *Let $M = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty}$ be a $\mathbb{Z}$-module. It is easy to see that $\mathbb{Z}_{p^\infty}$ is a $\mathbb{Z}_{p^\infty}$-dual-Rickart module while $\mathbb{Z}_{p^\infty}$ is not a $\mathbb{Z}_{p^\infty}$-Rickart module [36]. Also, $\mathbb{Z}_{p^\infty}$ is not a $D2$ module. Therefore, from [21, Proposition 2.11] M is not a $D4$ module.*

Proposition 4.2.5. *For an R-module M, the following statements hold:*

(i) *Every dual-Rickart module with $D2$-condition is a Rickart module.*

(ii) *Every Rickart module with $C2$-condition is a dual-Rickart module.*

Proof. (i) Let M is a dual-Rickart module and $\psi \in End_R(M)$ be an endomorphism. Since , $M/Ker(\psi) \cong Im(\psi) \leq^{\oplus} M$. Thus, by $D2$-condition, $Ker(\psi) \leq^{\oplus} M$. Hence, M is a Rickart module.
(ii) Let M be a Rickart module and $\varphi \in End_R(M)$ be an endomorphism. Then there exists submodule $K \leq M$ such that $M = Ker(\varphi) \oplus K$. Since the restriction map $\varphi|_K$ is a monomorphism, so by $C2$-condition $\varphi(K) \leq^{\oplus} M$. Thus, $Im(\varphi) = \{0\} \oplus \varphi(K)$ is a direct summand of M. Hence, M is a dual-Rickart module. □

Proposition 4.2.6. *Every quasi-injective finite Σ-Rickart module is a finite Σ-dual-Rickart module.*

Proof. Let M be a quasi-injective finite Σ-Rickart module. Then, $M^{(n)}$ is a Rickart module for every positive integer n. Since M is a quasi-injective R-module, therefore $M^{(n)}$ is also a quasi-injective module. Thus, $M^{(n)}$ is a Rickart module with $C2$-condition. Hence, by Proposition 4.2.5(ii) M is a finite Σ-dual-Rickart module. □

The following example shows that the quasi-injectivity in proposition 4.2.6 is not superfluous.

Example 4.2.7. *The $\mathbb{Z}$-module $M = \mathbb{Q} \oplus \mathbb{Z}$ is not quasi-injective, but M is a finite Σ-Rickart module (see [35, Example 2.3]), while M is not a finite Σ-dual-Rickart $\mathbb{Z}$-module. In fact, M is not a dual-Rickart module [36].*

Proposition 4.2.8. *Every projective finite Σ-dual-Rickart module is a finite Σ-Rickart.*

Proof. Let M be a projective finite Σ-dual-Rickart module. Then, $M^{(n)}$ is a dual-Rickart module for every positive integer n. Since M is a projective R-module, $M^{(n)}$ is also a projective module. Thus, $M^{(n)}$ is a Rickart module with $D2$-condition. Hence, M is a finite Σ-Rickart module. □

Corollary 4.2.9. *If R is a semisimple ring, then every finite Σ-dual-Rickart R-module is finite Σ-Rickart.*

Projectivity in Proposition 4.2.8 is not superfluous. We provide an example that illustrates it.

Example 4.2.10. *It is easy to see that $\mathbb{Z}_{p^\infty}$ is a finite Σ-dual-Rickart $\mathbb{Z}$-module while $\mathbb{Z}_{p^\infty}$ is not a projective $\mathbb{Z}$-module. Also, $\mathbb{Z}_{p^\infty}$ is not a finite Σ-Rickart $\mathbb{Z}$-module as $\mathbb{Z}_{p^\infty}$ is not a Rickart module [36].*

In the following proposition, we discuss the conditions under which a module is finite Σ-Rickart as well as finite Σ-dual-a Rickart.

Proposition 4.2.11. *The following statements hold for an R-module M:*

(i) If R is a right V-ring, then every finitely cogenerated R-module is finite Σ-dual-Rickart and finite Σ-Rickart module.

(ii) Every finitely cogenerated right R-module over a right SSI-ring is finite Σ-Rickart and finite Σ-dual-Rickart module.

(iii) If M is an endoregular module, then M is a finite Σ-Rickart and finite Σ-dual-Rickart module.

Proof. (i) Let $\psi \in End_R(M^{(n)})$ be arbitrary, where n is a positive integer. As M is a finitely cogenerated right R-module, $M^{(n)}$ is also finitely cogenerated. Since every finitely cogenerated module over a right V-ring is endoregular [40, Proposition 2.14], so $M^{(n)}$ is an endoregular module. Therefore, $Im(\psi)$ and $Ker(\psi)$ are direct summand of $M^{(n)}$. Hence, M is a finite Σ-Rickart module as well as a finite Σ-dual-Rickart module.

(ii) It follows from part (i) and from the fact that every SSI-ring is right Noetherian and right V-ring.

(iii) Since M is an endoregular module, so $M^{(n)}$ is also an endoregular module [40, Corollary 3.15]. Hence, M is a finite Σ-Rickart and finite Σ-dual-Rickart module. □

Theorem 4.2.12. *The following conditions are equivalent for a ring R:*

(i) R is semisimple Artinian ring;

(ii) Every R-module is finite Σ-Rickart module;

(iii) Every R-module is finite Σ-dual-Rickart module.

Proof. $(i) \Leftrightarrow (ii)$ From [36, Theorem 2.25], the ring R is semisimple Artinian if and only if every R-module is Rickart. So, for any $n \in \mathbb{N}$, $M^{(n)}$ is Rickart R-module if and only if R is semisimple Artinian ring. Hence, the result follows.

$(i) \Leftrightarrow (iii)$ It is clear from[37, Theorem 2.24], the ring R is semisimple Artinian if and only if every R-module is dual-Rickart. So, for any $n \in \mathbb{N}$, $M^{(n)}$ is dual-Rickart R-module if and only if R is semisimple Artinian ring. □

4.3 Endomorphism Rings of Finite Σ-dual-Rickart Modules

In this section, we study some properties of the endomorphism ring of a finite Σ-dual-Rickart modules. We characterize von Neumann regular rings, coherent rings and hereditary rings with the help of finite Σ-dual-Rickart modules.

Theorem 4.3.1. *(i) The endomorphism ring of every finite Σ-dual-Rickart projective R-module is von Neumann regular.*

(ii) The endomorphism ring of finite Σ-dual-Rickart free R-module is von Neumann regular.

Proof. (i) Let M be a finite Σ-dual-Rickart module. Then for every finite index set $\mathscr{I}$, $M^{(\mathscr{I})}$ is a dual-Rickart module. Now, let $\phi \in End(M^{(\mathscr{I})})$ be arbitrary. So, $M^{(\mathscr{I})}/Ker(\varphi) \cong Im(\varphi) \leq^{\oplus} M^{(\mathscr{I})}$. Since M is a projective module, $M^{(\mathscr{I})}$ is a projective module. So, $M^{(\mathscr{I})}$ satisfies $D2$ condition. Thus, $Ker(\varphi)$ is a direct summand of $M^{(\mathscr{I})}$. Therefore, from [47, Theorem 4] $End_R(M^{(\mathscr{I})}) = Mat_{\mathscr{I}}(End_R(M))$ is a von Neumann regular ring. Hence, $End_R(M)$ is a von Neumann regular ring.

(ii) The proof follows from part (i). □

Corollary 4.3.2. *Let M be an R-module and $S = End_R(M)$. If M is a finite Σ-dual-Rickart and projective module, then M is intrinsically injective.*

Proof. Let M be a finite Σ-dual-Rickart projective module. Then by Theorem 4.3.1 S is a von Neumann regular ring. Hence, from [59, 6.11(3)], M is an intrinsically injective module. □

Proposition 4.3.3. *The endomorphism ring of a finite Σ-dual-Rickart module is left semi-hereditary. Conversely, if $S = End_R(M)$ is a left semi-hereditary ring with $C2$-condition as a left S-module, then M is a finite Σ-dual-Rickart module.*

Proof. Let M be a finite Σ-dual-Rickart module. Then for every positive integer n, $M^{(n)}$ is a dual-Rickart module. It is clear that $End_R(M^{(n)}) = Mat_n(End_R(M)) = Mat_n(S)$. So by [37, Proposition 3.1], $Mat_n(S)$ is a left Rickart ring. Therefore, from [52, Proposition], S is a left semi-hereditary ring.

For the converse part, let S be a left semi-hereditary ring with $C2$ condition. Then S is a left Rickart ring with $C2$-condition. So from [36, Corollary 3.18], S is a von Neumann regular ring. Therefore, $Mat_n(S) = End_R(M^{(n)})$ is a von Neumann regular ring for each positive integer n. Thus, $M^{(n)}$ is a dual-Rickart. Hence, M is a finite Σ-dual-Rickart module. □

Corollary 4.3.4. *Let M be a finite Σ-dual-Rickart R-module and $S = End_R(M)$. If ${}_SM$ is an FP-injective module, then M is an intrinsically injective module.*

Proof. Let M be a finite Σ-dual-Rickart module. Then by Proposition 4.3.3, S is a left semi-hereditary ring. Since by hypothesis ${}_SM$ is FP-injective module, from [59, 6.11(2)] M is an intrinsically injective module. □

Proposition 4.3.5. *The endomorphism ring of a finitely generated finite Σ-dual-Rickart module is a left hereditary.*

Proof. The proof follows from [37, Remark 3.3]. □

Proposition 4.3.6. *The endomorphism ring of a finite Σ-dual-Rickart module is a left coherent ring.*

Proof. Let M be a finite Σ-dual-Rickart module and $S = End_R(M)$. Then, from Proposition 4.3.3 S is a left semi-hereditary ring. Since from [32, Proposition 4.47] every semi-hereditary ring is left coherent ring, so S is left coherent. □