Chapter 3

Σ-dual-Rickart modules

Lee et al., in [36] and [37], introduced the concept of Rickart modules and dual-Rickart modules, respectively. It is seen that the direct sum of Rickart modules may not be Rickart. In [34], Lee and Barcenas introduced the concept of Σ-Rickart modules and they called a module M, Σ-Rickart if the direct sum of arbitrary many copies of M is a Rickart module. Motivated by the notion of Σ-Rickart modules, we introduce the notion of Σ-dual-Rickart modules.

3.1 Σ-dual-Rickart Modules

Definition 3.1.1. *A module M is called a Σ-dual-Rickart module if every direct sum of copies of M is a dual-Rickart module. Equivalently, a module M is called Σ-dual-Rickart if $M^{(\Lambda)}$ is a dual-Rickart module for every non-empty arbitrary index set Λ. A ring R is called a right (left) Σ-dual-Rickart ring if R_R $({}_RR)$ is a right Σ-dual-Rickart R-module. Further, a ring R is said to be Σ-dual-Rickart if it is right as well as left Σ-dual-Rickart.*

It is easy to see that every Σ-dual-Rickart module is a dual-Rickart module. We provide an example, which shows that a dual-Rickart module need not be Σ-dual-Rickart module.

Example 3.1.2. *Let $R = \Pi_{n=1}^{\infty}\mathbb{F}_n$ where $\mathbb{F}_n = \mathbb{Z}_p$ (p is any prime) for each n. Clearly, R is a self-injective von Neumann regular ring, which is not semisimple. From [37, Proposition 2.27] for a von Neumann regular ring R, which is not semisimple Artinian, every finitely generated free R-module is a dual-Rickart module but not a dual-Baer R-module. So, $M = R$ is a dual-Rickart R-module. Also, the R-module $M^{(R)}$ satisfies the D2-condition and $End_R(M^{(R)})$ is not von Neumann regular. Also, by [37, Theorem 3.8], a module M is dual-Rickart with D2 condition if and only if the endomorphism ring of M is von Neumann regular. Therefore, $M^{(R)}$ is not a dual-Rickart R-module. Hence, M is not a Σ-dual-Rickart module.*

Now, we prove when a dual-Rickart module is a Σ-dual-Rickart module.

Proposition 3.1.3. *Let M be an R-module such that $M \trianglelefteq M^{(\Lambda)}$ for every arbitrary index set Λ. Then*

(i) M is a Rickart module if and only if M is a Σ-Rickart module.

(ii) M is a dual-Rickart module if and only if M is a Σ-dual-Rickart module.

Proof. (**i**) Let M be a Rickart module and fully invariant in $M^{(\Lambda)}$. So by [39, Proposition 2.34], $M^{(\Lambda)}$ is a Rickart module. Hence, M is a Σ-Rickart module. The converse is clear by the definition of Σ-Rickart module.

(**ii**) Assume that M is a dual-Rickart module and M is fully invariant in $M^{(\Lambda)}$. Then from [37, Proposition 5.14], $M^{(\Lambda)}$ is a dual-Rickart module. Hence, M is a Σ-dual-Rickart module. The converse is clear by the definition of Σ-Rickart module. □

Example 3.1.4. *(i) The $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ is a Σ-dual-Rickart module.*

(ii) Every injective R-module over the hereditary Noetherian ring R is a Σ-dual-Rickart module (Theorem 3.1.18).

(iii) Over the semisimple Artinian ring R every R-module is Σ-dual-Rickart module (Theorem 3.2.13).

Proposition 3.1.5. *Every direct summand of the Σ-dual-Rickart module is a Σ-dual-Rickart.*

Proof. Let M be a Σ-dual-Rickart module and $N \leq^{\oplus} M$. Then, for any non-empty index set Λ, $N^{(\Lambda)}$ is also a direct summand of $M^{(\Lambda)}$. Since $M^{(\Lambda)}$ is a dual-Rickart module, so $N^{(\Lambda)}$ is also dual-Rickart. Hence, N is a Σ-dual-Rickart module. □

Proposition 3.1.6. *The direct sum of copies of a Σ-dual-Rickart module is Σ-dual-Rickart.*

Proof. Let M be a Σ-dual-Rickart module. Then, $M^{(\Lambda_1)}$ is a dual-Rickart module, where Λ_1 is a non-empty index set. Therefore, by the definition of Σ-dual-Rickart module $(M^{(\Lambda_1)})^{(\Lambda_2)} = M^{(\Lambda_1 \times \Lambda_2)}$ is a dual-Rickart module for every index set Λ_2. Hence, $M^{(\Lambda_1)}$ is a Σ-dual-Rickart module. □

Lemma 3.1.7. *[37, Theorem 2.19]. Let M_1 and M_2 be R-modules. Then M_1 is M_2-dual-Rickart if and only if for any direct summand $N_1 \leq^{\oplus} M_1$ and any submodule $N_2 \leq M_2$, N_1 is N_2-dual-Rickart module.*

Proposition 3.1.8. *If M is a Σ-dual-Rickart module, then for any index sets Λ_1 and Λ_2 $M^{(\Lambda_1)}$ is $M^{(\Lambda_2)}$ dual-Rickart module.*

Proof. Let M be a Σ-dual-Rickart module. Then $M^{(\Lambda_1\times\Lambda_2)}$ is a dual-Rickart module. It is clear that $M^{(\Lambda_1)}$ is a direct summand of $M^{(\Lambda_1\times\Lambda_2)}$ and $M^{(\Lambda_2)}$ is a submodule of $M^{(\Lambda_1\times\Lambda_2)}$. So, by Lemma 3.1.7 $M^{(\Lambda_1)}$ is $M^{(\Lambda_2)}$-dual-Rickart module. □

Proposition 3.1.9. *Every R-module M is a Σ-dual-Rickart module if and only if R is a semisimple Artinian ring.*

Proof. Let T be the right ideal of R. Clearly, R_R is a right R-module. As every R-module is Σ-dual-Rickart, so $R^{(R)}$ is a dual-Rickart module. Now for the right ideal T of R, there exists a free module F_R and an epimorphism π such that $\pi(F_R) = T$. Since $F_R \leq^{\oplus} R^{(R)}$, F_R is dual-Rickart. Therefore, $\pi(F_R) = T \leq^{\oplus} R_R$ and thus by modularity, $T \leq^{\oplus} R_R$. Hence, R is a semisimple Artinian ring.
Conversely, assume that R is a semisimple Artinian ring and M is an R-module. Now from [37, Theorem 2.24], every module over a semisimple Artinian ring is a dual-Rickart module. So, every R-module is dual-Rickart. Therefore, for every index set Λ, $M^{(\Lambda)}$ is also a dual-Rickart module. Hence, M is Σ-dual-Rickart R-module. □

In the following proposition, we prove that every cohereditary module over a Noetherian ring is a Σ-dual-Rickart module.

Proposition 3.1.10. *Every cohereditary module M over a Noetherian ring R is a Σ-dual-Rickart R-module.*

Proof. Let M be a cohereditary module and R be a Noetherian ring. Since the direct sum of cohereditary modules over the Noetherian ring is cohereditary [50, Remark (i)], so $M^{(\Lambda)}$ is a cohereditary module for every arbitrary index set Λ. Thus, for every $\psi \in End_R(M^{(\Lambda)})$, $M^{(\Lambda)}/Ker(\psi) \cong Im(\psi)$ is an injective module. Therefore, the exact sequence $0 \to Im(\psi) \to M^{(\Lambda)} \to Coker(\psi) \to 0$ splits, which implies that $Im(\psi)$ is a direct summand of $M^{(\Lambda)}$. Hence, M is a Σ-dual-Rickart module. □

Corollary 3.1.11. *Let M be an R-module such that M is the generator in the category of right R-modules. If the ring R is semisimple, then M is a Σ-dual-Rickart module.*

Proof. Since M is a generator in the category of right R-modules and the ring R is semisimple, so from [57, Corollary 2.5] M is a cohereditary R-module. It is well known that a semisimple ring R is Noetherian. Therefore, from Proposition 3.1.10 M is a Σ-dual-Rickart module. □

Let M and U be R-modules. Then M is said to be (finitely) generated by U if and only if there exists an epimorphism $\varphi : U^{(I)} \to M$ for some (finite) arbitrary index set I [5, page 105]. Now, we generalize the concept of M-cogenerated modules as strongly M-cogenerated modules.

Definition 3.1.12. *Let $\mathcal{M}$ be a non-empty class of R-modules. We call a module N strongly cogenerated by $\mathcal{M}$ if there is a monomorphism $\sigma : N \to \bigoplus_{\lambda\in\Lambda} M_\lambda$, where $M_\lambda \in \mathcal{M}$ and Λ is a non-empty index set. An R-module N is said to be strongly cogenerated by a module M (or strongly M-cogenerated), if there exists a monomorphism $\sigma : N \to M^{(\Lambda)}$ for every non-empty arbitrary index set Λ.*

Example 3.1.13. *(i) Every finitely M-cogenerated module is strongly M-cogenerated.*

(ii) Every strongly M-cogenerated module is M-cogenerated module, while the converse need not be true. Since every torsion-free abelian group is cogenerated by $\mathbb{Q}$ [5, Example 8.3], therefore $\mathbb{Z}^{\mathbb{N}}$ is also cogenerated by $\mathbb{Q}$ but it is not strongly cogenerated by $\mathbb{Q}$. In fact, $\mathbb{Z}^{\mathbb{N}}$ is not embedded in $\mathbb{Q}^{(\mathbb{N})}$.

Lemma 3.1.14. *The direct sum of two strongly M-cogenerated modules is strongly M-cogenerated.*

Proof. Let M_1 and M_2 be two strongly M-cogenerated modules. Then for some non-empty index sets Λ_1 and Λ_2, the maps $\psi_1 : M_1 \to M^{(\Lambda_1)}$ and $\psi_2 : M_2 \to M^{(\Lambda_2)}$ are monomorphisms. Therefore, by [60, 9.2], $\psi_1 \oplus \psi_2 : M_1 \oplus M_2 \to M^{(\Lambda_1)} \oplus M^{(\Lambda_2)}$ is also a monomorphism. Hence, $M_1 \oplus M_2$ is also strongly M-cogenerated. □

Lemma 3.1.15. *Let $\psi : M \to N$ be a monomorphism and $L \leq M$. If $\psi(L)$ is a direct summand of N, then L is a direct summand of M.*

Proof. Suppose that $N = \psi(L) \oplus K$ for some $K \leq N$. Since ψ is a monomorphism from M to N, so $M = L + \psi^{-1}[K]$, where $\psi^{-1}[K]$ is the inverse image of K. If $x \in L \cap \psi^{-1}[K]$, then there exists $y \in K$, such that $\psi(x) = y$. Since $x \in L$, so $y \in \psi(L)$. Thus $y \in \psi(L) \cap K = 0$, which implies $x = 0$. Hence, $M = L \oplus \psi^{-1}[K]$. □

For an R-module M, $Add(M)$ [61] denotes the class of all R-modules, which are isomorphic to a direct summand of the direct sum of copies of M.

Proposition 3.1.16. *Let M be a Σ-dual-Rickart module and $U \in Add(M)$. Then the sum of two strongly M-cogenerated submodules of U is strongly M-cogenerated.*

Proof. Let U_1 and U_2 be two strongly M-cogenerated submodules of U and $U \in Add(M)$. Consider the short exact sequence:

$$0 \to U_1 \cap U_2 \xrightarrow{f} U_1 \oplus U_2 \xrightarrow{g} U_1 + U_2 \to 0$$

where for any $a \in U_1 \cap U_2$, $f(a) = (a, a)$ and $g(u_1, u_2) = u_1 + u_2$ where $u_1 \in U_1$, $u_2 \in U_2$. Since U_1 and U_2 are strongly M-cogenerated, so by Lemma 3.1.14, $U_1 \oplus U_2$ is also strongly M-cogenerated. Therefore, there exists a monomorphism $\varphi : U_1 \oplus U_2 \to M^{(\Lambda)}$ for some index set Λ. Since M is a Σ-dual-Rickart module, $M^{(\Lambda)}$ is dual-Rickart. Thus, $Im(\varphi f)$ is a direct summand of $M^{(\Lambda)}$. Now from

Lemma 3.1.15, $Im(f) \leq^{\oplus} U_1 \oplus U_2$ because $Im(\varphi f) = \varphi(Im(f))$, where φ is a monomorphism. Therefore, $(U_1 \oplus U_2)/Im(f)$ is strongly M-cogenerated. Since $U_1 \cap U_2 \cong (U_1 \oplus U_2)/Im(f)$, so $U_1 + U_2$ is strongly M-cogenerated. □

In the following proposition, we show that when a module is Σ-dual-Rickart.

Proposition 3.1.17. *An R-module M is a Σ-dual-Rickart if and only if every module in Add(M) is a Σ-dual-Rickart.*

Proof. Let $N \in Add(M)$ be arbitrary. Then there exists a submodule $L \leq^{\oplus} M^{(\Lambda)}$ such that $N \cong L$ for an index set Λ. From Proposition 3.1.5 and Proposition 3.1.6, L and $M^{(\Lambda)}$ are Σ-dual-Rickart modules, respectively. Hence, N is also a Σ-dual-Rickart module. The converse follows from the definition of the Σ-dual-Rickart module. □

In the following theorem, we find conditions under which an injective module is a Σ-dual-Rickart module.

Theorem 3.1.18. *Let R be the Noetherian ring. Then the following conditions are equivalent:*

(i) Every injective R-module is a Σ-dual Rickart module;

(ii) R is a right hereditary ring.

Proof. (i) ⇒ (ii) Let M be an injective R-module and N be a submodule of M. Clearly, M and $E(M/N)$ are both injective modules, so $M \oplus E(M/N)$ is also an injective module. By hypothesis $M \oplus E(M/N)$ is a Σ-dual-Rickart module. So $M \oplus E(M/N)$ is a dual-Rickart module. Thus, from Lemma 3.1.7, M is $E(M/N)$-dual-Rickart. Now consider a map $\psi : M \to E(M/N)$ such that $\psi(\zeta) = \zeta + N$ for every $\zeta \in M$. Then $Im(\psi) = M/N$ is a direct summand of $E(M/N)$. So M/N is an injective module. Hence, R is a right hereditary ring.

$(ii) \Rightarrow (i)$ Let R be a Noetherian ring and M be an injective R-module. Then $M^{(\Lambda)}$ is also an injective module. Now suppose that $\varphi \in End_R(M^{(\Lambda)})$. Since by assumption R is a hereditary ring, $Im(\varphi) \cong M^{(\Lambda)}/Ker(\varphi)$ is an injective module. Therefore, $Im(\varphi)$ is a direct summand of M^{Λ}. Hence, M is a Σ-dual-Rickart module. □

3.2 Σ-Rickart Modules vs. Σ-dual-Rickart Modules

In this section, we find connections between the class of Σ-Rickart modules and the class of Σ-dual-Rickart modules. Further, we show that when a Σ-dual-Rickart module is a Σ-Rickart module and vice-versa.

Now, we provide an example of Σ-Rickart module which is not a Σ-dual-Rickart module and vice-versa.

Example 3.2.1. *(i) It is clear from [37, Theorem 2.29] that for the hereditary ring R, every injective R-module is dual-Rickart. Therefore, $\mathbb{Z}_{p^\infty}^{(\Lambda)}$ is a dual-Rickart $\mathbb{Z}$-module for every arbitrary index set Λ. Thus, $\mathbb{Z}_{p^\infty}$ is a Σ-dual-Rickart $\mathbb{Z}$-module, while $\mathbb{Z}_{p^\infty}$ is not a Σ-Rickart module. In fact, $\mathbb{Z}_{p^\infty}$ is not a Rickart module (see [36, Example 2.17]).*

(ii) Since from [36, Theorem 2.26], every free (projective) module over a right hereditary ring is a Rickart module, so $\mathbb{Z}$-module $\mathbb{Z}$ is a Σ-Rickart module, while $\mathbb{Z}$ is not a $\mathbb{Z}$-dual-Rickart module. Hence, it can not be a Σ-dual-Rickart module.

Definition 3.2.2. *A module M is called a Σ-$C2$ module [3] if the direct sum of arbitrary many copies of M is a $C2$ module. Analogously, M is said to be a Σ-injective (Σ-quasi-injective) module if the direct sum of arbitrary many copies of M is an injective (quasi-injective) module (see [1]).*

Proposition 3.2.3. *If M is a Σ-$C2$ module, then every Σ-Rickart module is Σ-dual-Rickart.*

Proof. To prove M is a Σ-dual-Rickart module, it is enough to show that $Im(\psi)$ is a direct summand of $M^{(\Lambda)}$ for every $\psi \in End_R(M^{(\Lambda)})$. For it, let $\psi \in End_R(M^{(\Lambda)})$. Since M is a Σ-Rickart module, $Ker(\psi)$ is a direct summand of $M^{(\Lambda)}$ for every arbitrary index set Λ. So, for a submodule N of $M^{(\Lambda)}$, $M^{(\Lambda)} = Ker(\psi) \oplus N$. Clearly, the restriction map ψ_N is one-one. Since by assumption M is a Σ-$C2$ module, so $M^{(\Lambda)}$ is a $C2$ module. Therefore, $\psi(N)$ is a direct summand of $M^{(\Lambda)}$. Thus, $Im(\psi) = \{0\} \oplus \psi(N)$ is a direct summand of $M^{(\Lambda)}$. Hence, M is a Σ-dual-Rickart module. □

The following example shows that the condition "M is Σ-$C2$" in Proposition 3.2.3 is not superfluous.

Example 3.2.4. *The $\mathbb{Z}$-module $\mathbb{Z}$ is a Σ-Rickart module but not a Σ-dual-Rickart module and also not a Σ-$C2$ module (see Example 3.2.1). In fact, $\mathbb{Z}$-module $\mathbb{Z}$ is not a dual-Rickart module as well as not a $C2$-module.*

Corollary 3.2.5. *If R is a right Noetherian ring, then every injective Σ-Rickart R-module is a Σ-dual-Rickart module.*

Proof. Let R be a right Noetherian ring and M be an injective R-module. It is well known that over a right Noetherian ring, the direct sum of arbitrary many copies of an injective module is injective. Thus, $M^{(\Lambda)}$ is an injective. Therefore, $M^{(\Lambda)}$ is a $C2$ module. Hence, by Proposition 3.2.3 M is a Σ-dual-Rickart module. □

Corollary 3.2.6. *Every Σ-quasi-injective, Σ-Rickart module is a Σ-dual-Rickart module.*

A module M is said to be Σ-$C3$-module [3] if the direct sum of arbitrary many copies of M is a $C3$-module.

Corollary 3.2.7. *Every Σ-$C3$, Σ-Rickart module is a Σ-dual-Rickart module.*

Proof. It is clear from [3, Corollary 2.7] that a module M is Σ-$C3$ if and only if M is a Σ-$C2$. Hence, the result follows from Proposition 3.2.3. □

Definition 3.2.8. *A module M is called a Σ-$D2$ (Σ-$D3$) [62] if the direct sum of arbitrary copies of M is a $D2$-module ($D3$-module). Analogously, a module M is said to be a Σ-quasi projective if the direct sum of arbitrary copies of M is a quasi-projective module [1].*

In the following proposition, we show that when a Σ-dual-Rickart module is Σ-Rickart.

Proposition 3.2.9. *Every Σ-dual-Rickart module with Σ-$D2$ condition is Σ-Rickart.*

Proof. Let Λ be any arbitrary index set and $\psi \in End_R(M^{(\Lambda)})$. Since M is a Σ-dual-Rickart module, this implies $M^{(\Lambda)}/Ker(\psi) \cong Im(\psi) \leq^{\oplus} M^{(\Lambda)}$. By hypothesis M is a Σ-$D2$ module, so $M^{(\Lambda)}$ has $D2$-condition. Therefore, $Ker(\psi)$ is a direct summand of $M^{(\Lambda)}$. Hence, M is a Σ-Rickart module. □

The following example shows that the condition "M is Σ-$D2$" in Proposition 3.2.9 is not superfluous.

Example 3.2.10. *It is clear from Example 3.2.1 that the* $\mathbb{Z}$*-module* $\mathbb{Z}_{p^\infty}^{(\Lambda)}$ *is a dual-Rickart module for every arbitrary index set* Λ*. Thus,* Z_{p^∞} *is a* Σ*-dual-Rickart* $\mathbb{Z}$*-module, although it is not a* Σ*-*$D2$ $\mathbb{Z}$*-module as it is not a* $D2$*-module. While* Z_{p^∞} *is not a* Σ*-Rickart module because it is not a Rickart module.*

Corollary 3.2.11. *For an* R*-module* M*, the following statements hold:*

(i) Every projective Σ*-dual-Rickart module is a* Σ*-Rickart module.*

(ii) Every Σ*-quasi projective* Σ*-dual-Rickart module is a* Σ*-Rickart module.*

(iii) Every Σ*-*$D3$*,* Σ*-dual-Rickart module is a* Σ*-dual-Rickart.*

Proof. It is easy to see that every projective and every Σ-quasi projective modules are Σ-$D2$ modules. Hence, part (i) and (ii) easily follows from Proposition 3.2.9. For part (iii), let M be a Σ-$D3$ module. Since by [62, Corollary 9] direct sum of copies of M is $D3$ if and only if the direct sum of copies of M is $D2$, so M is a Σ-$D2$ module. Hence, the proof is clear from Proposition 3.2.9. □

Corollary 3.2.12. *Let* R *be a uniserial ring. Then every quasi-injective,* Σ*-dual-Rickart module over* R *is a* Σ*-Rickart module.*

Proof. The proof follows from Proposition 3.2.9 and from the fact that every quasi-injective module over a uniserial ring is a Σ-quasi projective module (see [25, Theorem 5.1]). □

In the following theorem, we characterize the semisimple Artinian ring in terms of Σ-dual-Rickart modules and Σ-Rickart modules.

Theorem 3.2.13. *The following conditions are equivalent for a ring* R*:*

(i) Every right R-module is a Σ-Rickart module;

(ii) Every right R-module is a Σ-dual-Rickart module;

(iii) R is a right semisimple Artinian ring.

Proof. $(i) \Leftrightarrow (iii)$ It is clear from [36, Theorem 2.25] that a ring R is right semisimple Artinian if and only if every right R-module is Rickart. Therefore, for any arbitrary index set Λ, $M^{(\Lambda)}$ is a Rickart module if and only if R is a right semisimple Artinian ring. Hence, R is a right semisimple Artinian ring if and only if every right R-module is Σ-Rickart.
$(ii) \Rightarrow (iii)$ Let T be a right ideal of R. Clearly, R_R is a right R-module. As every R-module is Σ-dual-Rickart, so $R^{(R)}$ is a dual-Rickart module. Thus, R is also a dual-Rickart module. Now for the right ideal T of R, there exists a free module F_R and an epimorphism π such that $\pi(F_R) = T$. Since $F_R \leq^{\oplus} R^{(R)}$, F_R is dual-Rickart. Therefore, $\pi(F_R) = T \leq^{\oplus} F_R$ and thus by modularity, $T \leq^{\oplus} R_R$. Hence, R is a right semisimple Artinian ring.
$(iii) \Rightarrow (ii)$ Now, suppose that R is a right semisimple Artinian ring and M is an R-module. So from [37, Theorem 2.24], every R-module is dual-Rickart. Therefore, for every index set Λ, $M^{(\Lambda)}$ is also a dual-Rickart module. Hence, M is a Σ-dual-Rickart R-module. □

3.3 Endomorphism Rings of Σ-dual-Rickart Modules

In this section, we study the endomorphism ring of a Σ-dual-Rickart modules and characterize the semi-hereditary rings, hereditary rings and von Neumann regular rings in terms of it.

Lemma 3.3.1. *[46, Corollary], Let R be a ring that contains an infinite direct product $\Pi_{i\in I}R_i$, where R_i is a ring with identity e_i for $i \in I$. Then R is not a (right) hereditary ring.*

The following example shows that the endomorphism ring of a Σ-dual-Rickart module need not be a hereditary ring.

Example 3.3.2. *From [37, Theorem 2.29], it is clear that for a right hereditary ring R, $E(M)$ is a dual-Rickart R-module for any right R-module M. Therefore, $\mathbb{Q}^{(\Lambda)}$ is a dual-Rickart $\mathbb{Z}$-module for every arbitrary index set Λ. Hence, $\mathbb{Q}^{(\Lambda)}$ is also a Σ-dual-Rickart module, while $End_{\mathbb{Z}}(\mathbb{Q}^{(\Lambda)})$ is not a hereditary ring. In fact, $End_{\mathbb{Z}}(\mathbb{Q}^{(\Lambda)}) \cong CFM_{\Lambda}(\mathbb{Q})$ contains $\Pi_{\lambda\in\Lambda}R_\lambda$ where $R_\lambda = \mathbb{Q}$ for each $\lambda \in \Lambda$. Therefore, by Lemma 3.3.1, $End_{\mathbb{Z}}(\mathbb{Q}^{(\Lambda)})$ is not a right hereditary ring.*

Proposition 3.3.3. *If M is a Σ-dual-Rickart R-module, then the endomorphism ring $S = End_R(M)$ is a left semi-hereditary.*

Proof. Let M be a Σ-dual-Rickart module. Then $M^{(n)}$ is a dual-Rickart module for every $n \in \mathbb{N}$. From [37, Proposition 3.1], $End_R(M^{(n)}) \cong Mat_n(S)$ is a left Rickart ring for all $n \in \mathbb{N}$. Hence, S is a left semi-hereditary from [52, Proposition]. □

Lemma 3.3.4. *[60, 47.7(2)]. Let M be a right R-module with $S = End_R(M)$. Then ${}_SM$ is FP-injective S-module if and only if for every homomorphism $\phi : M^{(n_1)} \to M^{(n_2)}$ with $n_1, n_2 \in \mathbb{N}$, $Coker(\phi)$ is a M-cogenerated module.*

Lemma 3.3.5. *If M is a right Σ-dual-Rickart module and $S = End_R(M)$, then ${}_SM$ is an FP-injective S-module.*

Proof. Let M be a Σ-dual-Rickart module and $\phi : M^{(n_1)} \to M^{(n_2)}$ be any homomorphism. Then $Im(\phi) \leq^{\oplus} M^{(n_2)}$ which implies that $Coker(\phi) = M^{(n_2)}/Im(\phi) \cong N \leq^{\oplus} M^{(n_2)}$ for some $N \leq M^{(n_2)}$. Therefore, $Coker(\phi)$ is M-cogenerated. Hence, from Lemma 3.3.4 $_SM$ is FP-injective S-module. □

Proposition 3.3.6. *If M is a finitely generated Σ-dual-Rickart module with endomorphism ring $S = End_R(M)$, then S is a left hereditary ring and $_SM$ is a FP-injective S-module.*

Proof. Since M is a finitely generated module, $End_R(M^{(\Lambda)}) \cong End_S(S^{(\Lambda)})$ for any non-empty arbitrary index set Λ. By hypothesis, M is a Σ-dual-Rickart module, so $M^{(\Lambda)}$ is a dual-Rickart. From [37, Proposition 3.1], $End_R(M^{(\Lambda)})$ is a left Rickart ring. Thus, $End_S(S^{(\Lambda)}) = CFM_\Lambda(S)$ is a left Rickart ring. Hence, S is a left hereditary ring from [39, Proposition 3.20]. □

The following Proposition illustrates when the endomorphism ring of a Σ-dual-Rickart module is a von Neumann regular.

Proposition 3.3.7. *Let M be a projective Σ-dual-Rickart module. Then $S = End_R(M)$ is a von Neumann regular ring.*

Proof. Let M be a Σ-dual-Rickart module and $f \in End(M^{(\Lambda)})$ be an endomorphism. Then $M^{(\Lambda)}$ is a dual-Rickart module (where Λ is an index set). Therefore, $M/Ker(f) \cong Im(f) \leq^{\oplus} M^{(\Lambda)}$. Since M is a projective module, $Ker(f) \leq^{\oplus} M^{(\Lambda)}$. Hence, $End(M^{(\Lambda)}) \cong CFM_\Lambda(S)$ is a von Neumann regular, which implies that S is a von Neumann regular ring. □

Recall from [40] that a module M is endoregular if the endomorphism ring $S = End_R(M)$ is a von Neumann regular ring.

Corollary 3.3.8. *Every projective* Σ*-dual-Rickart module is an endoregular module.*

Proposition 3.3.9. *If* M *is an* R*-module such that* $M^{(\Lambda)}$ *is an endoregular module for every arbitrary index set* Λ*, then* M *is a* Σ*-dual-Rickart module.*

Proof. The Proof is clear. □