Chapter 2

Principally Quasi-dual-Baer Modules

In 2010, Tutuncu and Tribak [56] introduced the notion of dual-Baer modules as a dual notion of Baer modules. Further, in 2013 Amouzegar and Talebi [4] dualized the concept of quasi-Baer modules as quasi-dual-Baer modules which is a generalization of the notion of dual-Baer modules. In this chapter, we introduce the notion of principally quasi-dual-Baer modules. We discuss various properties of principally quasi-dual-Baer modules as direct summand, direct sum, etc. We also study endomorphism rings of principally quasi-dual-Baer modules.

2.1 Principally Quasi-dual-Baer Modules

Definition 2.1.1. *Let M be an R-module and $S = \mathrm{End}_R(M)$. Then M is said to be a principally quasi-dual-Baer module (or, in short, PQ-dual-Baer module) if for every cyclic S-submodule N of M, $D_S(N) = \{\psi \in S : Im(\psi) \subseteq N\}$ is a direct* *summand of S.*

Further, a ring R is called right (left) PQ-dual-Baer if the R-module R_R $({}_RR)$ is PQ-dual-Baer.

Example 2.1.2. *(i) The $\mathbb{Z}$-modules $\mathbb{Q}$ and $\mathbb{Z}_{p^\infty}$ are PQ-dual-Baer modules.*

(ii) Every injective indecomposable module is PQ-dual-Baer module.

(iii) Every dual-Baer and quasi-dual-Baer modules are PQ-dual-Baer module.

In the following lemma, we show that the idempotent element $f \in S$ such that $D_S(Sm) = Sf$ is a right semi-central if M is a PQ-dual-Baer module.

Lemma 2.1.3. *If M is a PQ-dual-Baer module and $S = End_R(M)$ then for any $m \in M$, there exists a right semi-central element $f \in \mathbb{S}_r(S)$ such that $D_S(Sm) = Sf$.*

Proof. Let M be a PQ-dual-Baer module and $m \in M$. Then there exists $f^2 = f \in S$ such that $D_S(Sm) = Sf$. Since Sm is a fully invariant S-submodule of M as for every $f \in S$, $f(Sm) \subseteq Sm$, so $Sf\varphi(Sm) \subseteq Sf(Sm) \subseteq Sm$ for every $\varphi \in S$. Therefore, $Sf\varphi \subseteq D_S(Sm) = Sf$, which implies that $f\varphi = f\varphi f$. Hence $f \in \mathbb{S}_r(S)$. □

Proposition 2.1.4. *The following statements are equivalent for an R-module M:*

(i) M is a PQ-dual-Baer module;

(ii) For every cyclic submodule $P \leq M$, there exists a decomposition $M = P_1 \oplus P_2$ with $P_1 \leq^{\oplus} P$ and $Hom(M,\ P \cap P_2) = 0$.

Proof. $(i) \Rightarrow (ii)$ Let P be a cyclic S-submodule of M and $S = End_R(M)$. Then by hypothesis, there exists $e^2 = e \in S$ such that $D_S(P) = Se$. Suppose that $P_1 = eM$

and $P_2 = (1-e)M$, then $M = P_1 \oplus P_2$. Also $E_M(D_S(P)) = E_M(Se) = eM = P_1$ and P_1 is a direct summand of P. Therefore, $P = P_1 \oplus (P \cap P_2)$. Now take $\varphi \in S$ be such that $\varphi(M) \subseteq P \cap P_2$, then $\varphi \in D_S(P)$. So, there exists $\psi \in S$ such that $\varphi = \psi e$. Therefore, $\varphi(M) \subseteq P_1$ so $\varphi = 0$ as $\varphi(M) \subseteq P_2$ and $P_1 \cap P_2 = 0$. Hence, $Hom_R(M,\ P \cap P_2) = 0$.

$(ii) \Rightarrow (i)$ Let $P = Sm$ for some $m \in M$. Clearly, P is a cyclic S-submodule of M, so there exists a decomposition of M such that $M = P_1 \oplus P_2$, $P_1 \leq^{\oplus} P$ and $Hom(M,\ P \cap P_2) = 0$. Let $P_1 = eM$ for some idempotent $e^2 = e \in S$, then $Se = D_S(P_1) \subseteq D_S(P)$. Now assume that $\varphi \in D_S(P)$ and π be a projection map from P to $P \cap P_2$. Then $\pi\varphi = 0$ which implies that $\varphi(M) \subseteq e(M)$. Thus, $\varphi(1-e) = 0 \Rightarrow \varphi = \varphi e \in Se$ which gives $D_S(P) \subseteq Se$. Therefore, $D_S(P) = Se$. Hence, M is a PQ-dual-Baer module. □

Corollary 2.1.5. *If every cyclic submodule of M is a direct summand of M, then M is a PQ-dual-Baer module.*

Corollary 2.1.6. *If R is von Neumann regular, then R is a PQ-dual-Baer R-module.*

Proof. Let R be a von Neumann regular ring and I be a principal ideal of R. Then, from Proposition 1.0.69 I is a direct summand of R. So every principal ideal is a direct summand of R. Hence, by Proposition 2.1.4 R is a PQ-dual-Baer R-module. □

Corollary 2.1.7. *For an indecomposable module M, the following are equivalent:*

(i) M is a PQ-dual-Baer module;

(ii) For every cyclic submodule P of M, $Hom(M,\ P) = 0$.

Proof. $(i) \Rightarrow (ii)$ It follows from Proposition 2.1.4.

$(ii) \Rightarrow (i)$ Let P be a cyclic S-submodule of M and $S = End_R(M)$. Since M is an indecomposable module and $Hom(M, P) = 0$, so $D_S(M) = S$ and $D_S(P) = 0$. Therefore, $D_S(P) \leq^{\oplus} S$. Hence, M is a PQ-dual-Baer module. □

It is clear from the definition that the following hierarchy is true in general.

Dual-Baer module ⇒ Quasi-dual-Baer module ⇒ PQ-dual-Baer module.

We provide some examples which show that the converse of the above implications need not be true.

Example 2.1.8. *(i) [55, Example 2.9(iii)] Let J be a simple domain that is not a division ring (we can take $J = A_n(\mathbb{F})$, the nth Weyl algebra over a field $\mathbb{F}$ of characteristic zero). Consider the ring $R = \begin{pmatrix} J & K/J \\ 0 & J \end{pmatrix}$, where K is the classical ring of quotients of J. Take the idempotent $E = \begin{pmatrix} 1 & \bar{0} \\ 0 & 0 \end{pmatrix}$ and the right R-module $M = ER = \begin{pmatrix} J & K/J \\ 0 & 0 \end{pmatrix}$. In [55], Tribak et al. proved that the module M is a quasi-dual-Baer module that is not a dual-Baer module.*

(ii) Let $\mathbf{P}$ be the set of all primes. Also, let $R = \mathbb{Z}$ and $M = \prod_{p \in \mathbf{P}} \mathbb{Z}_p$ be an R-module. It is clear that every cyclic submodule of M is a direct summand of M. Therefore, $\mathrm{D}_S(N)$ is a direct summand of S for every cyclic submodule of M. Hence, M is a PQ-dual-Baer R-modules, while from [55, Example 3.2], M is not a quasi-dual-Baer module.

In the following proposition, we show that when a PQ-dual-Baer module is a quasi-dual-Baer module.

Proposition 2.1.9. *An R-module M is a quasi-dual-Baer if and only if M is a PQ-dual-Baer and the endomorphism ring of M has FI-SSSP.*

Proof. Let M be a quasi-dual-Baer module and $S = End_R(M)$. Since every quasi-dual-Baer module is PQ-dual-Baer. So for sufficient condition, it only remains to prove that the left S-module ${}_SS$ has FI-SSSP. For it, let $T = \Sigma_{i\in\Lambda}Se_i$ and each $e_i \in \mathbb{S}_r(S)$. Then, $\Sigma_{i\in\Lambda}Se_i = \Sigma_{i\in\Lambda}D_S(e_iM) = D_S(\Sigma_{i\in\Lambda}e_iM) = Se$ for some $e \in \mathbb{S}_r(S)$. Therefore, the left S-module ${}_SS$ has FI-SSSP.
Conversely, assume that N is a fully invariant submodule of M. Since $D_S(N) = \Sigma_{n\in N}D_S(Sn)$ and M is a PQ-dual-Baer module, there exists a right semi-central element $e_i \in \mathbb{S}_r(S)$ such that $D_S(Sn) = Se_i$ for every $i \in I$, where I is an index set. By hypothesis ${}_SS$ has FI-SSSP, so $D_S(N) = \Sigma_{i\in I}Se_i \leq^{\oplus} Se$ for some $e \in \mathbb{S}_r(S)$. Hence, M is a quasi-dual-Baer module. □

Proposition 2.1.10. *Let M be a PQ-dual-Baer module and ${}_SS$ has SSP. Then for every finitely generated submodule N of M, $D_S(N)$ is a direct summand of S.*

Proof. Let $N = \sum_{i=1}^{n} Sm_i$ be a finitely generated submodule of M, where $m_i \in M$ for each $1 \leq i \leq n$ and $n \in \mathbb{N}$. It is clear that $D_S(N) = D_S(\Sigma_{i=1}^{n}Sm_i) = \Sigma_{i=1}^{n}D_S(Sm_i)$. Since M is a PQ-dual-Baer module, so from Lemma 2.1.3, there exists $e_i^2 = e_i \in \mathbb{S}_r(S)$ such that $D_S(Sm_i) = Se_i$ for every $1 \leqslant i \leqslant n$. Thus, $D_S(N) = \Sigma_{i=1}^{n}Se_i$. Since ${}_SS$ has SSP, $\Sigma_{i=1}^{n}Se_i$ is a direct summand of S. □

Corollary 2.1.11. *Let R be a principal ideal domain, M be a finitely generated R-module and $S = End_R(M)$ has SSP as a left S-module. Then the following are equivalent:*

(i) M is a dual-Baer module;

(ii) M is a quasi-dual-Baer module;

(iii) M is a PQ-dual-Baer module.

Proof. $(i) \Rightarrow (ii) \Rightarrow (iii)$ follows from the definitions of dual-Baer, quasi-dual-Baer and PQ-dual-Baer modules.
$(iii) \Rightarrow (i)$ It follows from Proposition 2.1.10. □

In the following proposition, we show that PQ-dual-Baer modules are closed under direct summands.

Proposition 2.1.12. *Every direct summand of a PQ-dual-Baer module is a PQ-dual-Baer.*

Proof. Let M be a PQ-dual-Baer module with endomorphism ring S, N be a direct summand of M and $n \in N$. Then, there exists $e^2 = e \in S$ such that $N = eM$ and $T = End_R(N) \cong eSe$. Since M is a PQ-dual-Baer module, there exists $\varphi \in \mathbb{S}_r(S)$ such that $I = D_S(Sn) = S\varphi$. From [4, Lemma 1.3] $I \trianglelefteq S$, so $eIe = eSe \cap I$. Since $\varphi \in \mathbb{S}_r(S)$, $\varphi e = \varphi e \varphi$. Thus, $eIe = eS\varphi e = eS\varphi e\varphi = (eS\varphi e)(e\varphi)$, which implies $eIe \leq^{\oplus} eSe$. Now, we claim that $D_T(Tn) = eIe$. For it, let $\psi \in I$, $e\psi e(M) = e\psi(eM) = e\psi(N) \subseteq e(Sn) \subseteq (eSe)n = Tn$, which yields $e\psi e \in D_T(Tn)$. Thus, $eIe \subseteq D_T(Tn)$. Now assume that $0 \neq e\theta e \in eSe$ such that $e\theta e(N) \subseteq Tn$ where $\theta \in S$. Since $N = eM$, $e\theta e(M) = e\theta e(N) \subseteq Tn \subseteq Sn$, so $e\theta e \in D_S(Sn) = I$. But $e\theta e = ee\theta ee = e(e\theta e)e \in eIe$. Therefore, $D_T(Tn) = eIe$ for all $n \in N$. Hence, N is a PQ-dual-Baer module. □

Proposition 2.1.13. *The following statements are equivalent for a ring R:*

(i) Every R-module is PQ-dual-Baer;

(ii) Every projective R-module is PQ-dual-Baer;

(iii) The free module $R^{(R)}$ is PQ-dual-Baer;

(iv) R is a right semisimple Artinian ring.

Proof. The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ are easy to verify.
$(iii) \Rightarrow (iv)$ Let J be a right ideal of R. Then there exists a free R-module K and an epimorphism π for which $\pi(K) = J$. Since $R^{(R)}$ is a PQ-dual-Baer R-module and $K \leq^{\oplus} R^{(R)}$ implies K_R is a PQ-dual-Baer module. Thus, $\pi(K_R) = J \leq^{\oplus} K_R$, which gives $J \leq^{\oplus} R_R$. Hence, R is a right semisimple Artinian ring.
$(iv) \Rightarrow (i)$ Let R be a semisimple Artinian ring and M be an R-module. It is clear from [56, Corollary 2.10] that every R-module is dual-Baer if R is a semisimple ring. Therefore, M is a dual-Baer R-module. Hence, M is a PQ-dual-Baer R-module. □

Now, we characterize PQ-dual-Baer modules over regular rings.

Proposition 2.1.14. *For a ring R, the following are equivalent:*

(i) Each finitely generated free (projective) right R-module is PQ-dual-Baer;

(ii) The free R-module $R^{(n)}$ is PQ-dual-Baer module, where $n \in \mathbb{N}$;

(iii) R is a regular ring.

Proof. $(i) \Rightarrow (ii) \Rightarrow (iii)$ It is clear.
$(iii) \Rightarrow (i)$. It is well known that $End(R^{(n)}) \cong Mat_n(R)$ for every $n \in \mathbb{N}$, where $Mat_n(R)$ is a matrix of order n over R. Since R is a regular ring, so $Mat_n(R)$ is also a regular ring. Hence, $R^{(n)}$ is PQ-dual-Baer R-module. □

The following proposition provides examples of PQ-dual-Baer modules which are not dual-Baer.

Proposition 2.1.15. *Let R be a von Neumann regular ring, which is not semisimple Artinian. Then every finitely generated free R-module is a PQ-dual-Baer module but not a dual-Baer module.*

Proof. From Proposition 2.1.14, every finitely generated free R-module M is a PQ-dual-Baer module. Since from hypothesis, R is not semisimple, so by [56, Corollary 2.10] the R-module M is not a dual-Baer module. □

Example 2.1.16. *The ring $J = \Pi_{i=1}^{\infty}\mathbb{Z}_p$ (where p is a prime) is von Neumann regular ring which is not semisimple Artinian. Hence, from Proposition 2.1.15 every finitely generated free R-module is a PQ-dual-Baer module that is not a dual-Baer module.*

Now, we give an example which shows that the direct sum of PQ-dual-Baer modules need not be PQ-dual-Baer.

Example 2.1.17. *The $\mathbb{Z}$-modules $\mathbb{Z}_{p^\infty}$ and $\mathbb{Z}_p$, where p is a prime, are PQ-dual-Baer modules because these are dual-Baer modules [56]. From [37, Example 2.10] $M = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_p$, the direct sum of $\mathbb{Z}_{p^\infty}$ and $\mathbb{Z}_p$ is not a dual-Rickart module. Therefore, M can not be a PQ-dual-Baer module.*

In the following theorem, we discuss when the direct sum of two PQ-dual-Baer modules is PQ-dual-Baer.

Proposition 2.1.18. *If M_1 and M_2 are PQ-dual-Baer modules such that $Hom(M_i, M_j) = 0$ for every $i \neq j$, $i, j = 1, 2$, then $M_1 \oplus M_2$ is a PQ-dual-Baer module.*

Proof. Let $M = M_1 \oplus M_2$ with $S_1 = End_R(M_1)$ and $S_2 = End_R(M_2)$. Since $Hom_R(M_i,\ M_j) = 0$ for every $i \neq j$, $S = End_R(M) = S_1 \oplus S_2$. Therefore, for every $m = (m_1,\ m_2) \in M$, $D_S(Sm) = D_{S_1}(S_1m_1) \oplus D_{S_2}(S_2m_2)$. From hypothesis M_i is a PQ-dual-Baer module, so there exists $e_i^2 = e_i \in S_i$ such that $D_{S_i}(S_i m_i) = S_i e_i$ for each i. Thus, $D_S(Sm) = S_1 e_1 \oplus S_2 e_2 \leq^{\oplus} S$. Hence, M is a PQ-dual-Baer module. □

Now, we study when the direct sum of arbitrary many copies of a PQ-dual-Baer module is PQ-dual-Baer.

Theorem 2.1.19. *Let M be a PQ-dual-Baer module and $S = End_R(M)$. Then the direct sum of copies of M is PQ-dual-Baer if ${}_SS$ has SSSP.*

Proof. Let M be a PQ-dual-Baer module and $M^{(\mathbf{I})} = \bigoplus_{\mathbf{I}} M$ be the direct sum of $\mathbf{I}$ copies of M, where $\mathbf{I}$ is an arbitrary index set. First, we assume $\mathbf{I} = \mathbb{N}$. Let $m = (m_i)_{i\in\mathbf{I}} \in M^{(\mathbf{I})}$ and E_{ij} denote an $(\mathbf{I}\times\mathbf{I})$ matrix of $\mathbf{H} = End(M^{(\mathbf{I})})$ with 1_S (identity element of S) at $(i, j)th$ position and 0 elsewhere. Clearly, $E_{ij}(m)$ is an element of $M^{(\mathbf{I})}$ such that m_j is at i-th position and 0 elsewhere. So there exists $n \in \mathbb{N}$ such that for each $l > n$, $m_l = 0$, that is $E_{ll}(m) = 0$, which implies that $m = \Sigma_{i=1}^{n} E_{ii}(m)$. Then from the claim of [33, Theorem 3.8], we get $\mathbf{H}(m) = \bigoplus_{j\in\mathbf{I}}(\Sigma_{i=1}^{n} S_{ji}(m_i))$, where $S_{ji} = Hom(M_i,\ M_j) = S$. If we consider $N_j = \Sigma_{i=1}^{n} S_{ji}(m_i)$ for every $j \in \mathbf{I}$, then it is clear that $D_S(N_j) = D_S(\Sigma_{i=1}^{n} S_{ji}(m_j)) = \Sigma_{i=1}^{n} D_S(S_{ji}(m_j))$. Since M is a PQ-dual-Baer module and ${}_SS$ has SSSP, from Proposition 2.1.10 $D_S(N_j) = Se$ for some $e^2 = e \in S$. Let $1_{\mathbf{H}}$ be the identity of $\mathbf{H}$ and take $e1_{\mathbf{H}} = diag[e, e, ..., e...] \in \mathbf{H}$. Then $e1_{\mathbf{H}}$ is an idempotent element of $\mathbf{H}$. Since $e1_{\mathbf{H}}(\bigoplus_{j\in\mathbf{I}}(N_j)) = \bigoplus_{j\in\mathbf{I}} e(N_j) \subseteq \bigoplus_{j\in\mathbf{I}}(N_j)$. Therefore, $\mathbf{H}e1_{\mathbf{H}} \subseteq D_{\mathbf{H}}(\bigoplus_{j\in\mathbf{I}} N_j)$. Again let $\psi = [\psi_{kj}] \in D_{\mathbf{H}}(\bigoplus_{j\in\mathbf{I}} N_j)$, then $\psi(\bigoplus_{j\in\mathbf{I}} N_j) \subseteq \bigoplus_{j\in\mathbf{I}} N_j$ which implies that $\psi_{kj}(N_j) \subseteq N_j$ for all $j, k \in \mathbf{I}$. So $\psi_{kj} \in D_S(N_j) = Se$ for some idempotent $e \in S$ because M is PQ-dual-Baer module. Therefore, $\psi_{kj} = \psi_{kj}e$ for all $j, k \in \mathbf{I}$. Hence, $D_{\mathbf{H}}(\bigoplus_{j\in\mathbf{I}} N_j) \subseteq \mathbf{H}e1_{\mathbf{H}}$. So we get $D_{\mathbf{H}}(\bigoplus_{j\in\mathbf{I}} N_j) = \mathbf{H}e1_{\mathbf{H}}$. Thus, $D_{\mathbf{H}}(\mathbf{H}m) = \mathbf{H}e1_{\mathbf{H}}$. When $\mathbf{I}$ is an arbitrary index set, then the proof is similar to the above case. □

2.2 Endomorphism Rings of Principally Quasi-dual-Baer Modules

In this section, we study the endomorphism ring of a PQ-dual-Baer module.
The following proposition shows that the endomorphism ring of a PQ-dual-Baer module is a left principally quasi-Baer ring.

Proposition 2.2.1. *The endomorphism ring of a PQ-dual-Baer module is a left PQ-Baer ring.*

Proof. Let M be a PQ-dual-Baer module, $m \in M$ and T be a principal ideal of S. Then there exists $f^2 = f \in S$ such that $D_S(Tm) = Sf$. For every $g \in T$, $Im(g) \subseteq \Sigma_{g \in D_S(Tm)} Im(g) = \Sigma_{g \in Sf} Im(g) = E_M(Sf) = fM$. So for every $g \in T$, $(1-f)gM = 0$, which implies that $(1-f)g = 0$. Therefore, $(1-f) \in l_S(T)$. Now to show that S is a PQ-Baer ring, it is enough to prove that $l_S(T) = S(1-f)$. Let $h \in l_S(T)$ then $h(D_S(Tm)) = 0 \Rightarrow (Sf)h = 0 \Rightarrow fh = 0$. Therefore, $h = (1-f)h \in S(1-f)$. Thus, $l_S(T) \subseteq S(1-f)$. Now, assume that $h \in S(1-f)$ then for every $m \in M$, $hT(m) = h(1-f)T(m) \subseteq h(1-f)(fM)$ because for every $h \in T$, $Im(h) \in fM$. So, $hT(m) = 0$ for every $m \in M \Rightarrow hT = 0 \Rightarrow h \in l_S(T)$. Thus, $l_S(T) = S(1-f)$. Hence, S is a left PQ-Baer ring. □

The converse of the above proposition need not be true. In fact, a $\mathbb{Z}$-module $\mathbb{Z}$ is not a PQ-dual-Baer, while $End_{\mathbb{Z}}\mathbb{Z} \cong \mathbb{Z}$ is a PQ-Baer ring.
In the next proposition, we find the condition under which the endomorphism ring of a PQ-dual-Baer module is a PQ-dual-Baer ring.

Proposition 2.2.2. *Let M be a finitely generated PQ-dual-Baer module and the endomorphism ring S of M has SSP. Then S is the PQ-dual-Baer ring.*

Proof. Let M be a PQ-dual-Baer module with $S = \mathrm{End}_R(M)$ and $f \in S$. Assume that M is generated by $m_1, m_2, ..., m_n$ where each $m_i \in M$ and $n \in \mathbb{N}$. It is clear that, for every $\psi \in D_S(S\varphi)$, $\psi(S\varphi) \subseteq S\varphi$ and $\psi(S\varphi)M \subseteq S\varphi M$. Thus, $\psi(S\varphi)(m_i) \subseteq S\varphi(m_i)$ for all $1 \leqslant i \leqslant n$. Therefore, $\psi \in D_S(S(\varphi(m_i)))$ for each i. Since M is a PQ-dual-Baer module, so there exists $e_i \in \mathbb{S}_r(S)$ such that $D_S(S(\varphi(m_i))) = Se_i$ for each $1 \leqslant i \leqslant n$. Hence, $\psi \in \Sigma_{i=1}^n Se_i$, so $D_S(S\varphi) \subseteq \Sigma_{i=1}^n Se_i$. Now, let $f \in \Sigma_{i=1}^n Se_i$ and $m \in M$ be arbitrary. Then for $r_i \in R$, $f(S\varphi(m)) = f(\Sigma_{i=1}^n S\varphi(m_i r_i)) = f(\Sigma_{i=1}^n (S\varphi(m_i) r_i$ where $1 \leqslant i \leqslant n$. Clearly $\Sigma_{i=1}^n (S\varphi(m_i) r_i$ is a finitely generated submodule of M. It is clear that $f(\Sigma_{i=1}^n (S\varphi(m_i) r_i) \subseteq \Sigma_{i=1}^n (S\varphi(m_i) r_i$ for each i. Thus, $f(S\varphi) \subseteq S\varphi$ that implies $f \in D_S(S\varphi)$. Therefore, $\Sigma_{i=1}^n Se_i = D_S(S\varphi)$. Since ${}_SS$ has summand sum property, $D_S(S\varphi) \leq^{\oplus} S$. Hence, S is a PQ-dual-Baer ring. □

Proposition 2.2.3. *If the endomorphism ring of every direct sum of copies of a PQ-dual-Baer M is left PQ-dual-Baer, then $S = End_R(M)$ is a quasi-dual-Baer ring.*

Proof. Let M be a PQ-dual-Baer module and $T \trianglelefteq S$. Consider $\mathbf{I} = |T|$ and $\mathbf{H} = End(M^{(\mathbf{I})})$. Clearly $CFM_S \subseteq \mathbf{H} \subseteq Mat_{\mathbf{I}}(S)$. Set $\psi = diag[\psi_1, \psi_2, ..., \psi_i, ...]_{i \in \mathbf{I}} \in \mathbf{H}$. We claim that $D_{\mathbf{H}}(\mathbf{H}\psi) = \mathbf{H} \cap Mat_{\mathbf{I}}(\Sigma_{\psi_i \in T} D_S(S\psi_i))$. Let $\varphi = [\varphi_{ij}] \in D_{\mathbf{H}}(\mathbf{H}\psi)$ be arbitrary. Then $\varphi(\mathbf{H}\psi) \subseteq \mathbf{H}\psi$. Denote by E_{ii} a unit matrix in $\mathbf{H}$ with 1_S at (i,i)-th position and 0 elsewhere. Then $E_{ii}\varphi E_{jj}(\mathbf{H}E_{kk}\psi E_{kk}) \subseteq \mathbf{H}E_{kk}\psi E_{kk}$ that implies $\varphi_{ij}(S\psi_k) \subseteq S\psi_k$ for all $i,j,k \in \mathbf{I}$. Thus, $\varphi_{ij} \in \Sigma_{\psi_k \in T} D_S(S\psi_k)$ for every $i,j \in \mathbf{I}$. Therefore, $\varphi \in \mathbf{H} \cap Mat_{\mathbf{I}}(\Sigma_{\psi_k \in T} D_S(S\psi_k))$. For the reverse inclusion, let $\theta = [\theta_{ij}] \in \mathbf{H} \cap Mat_{\mathbf{I}}(\Sigma_{\psi_k \in T} D_S(S\psi_k))$ be arbitrary. Then $\theta_{ij} \in \Sigma_{\psi_k \in T} D_S(S\psi_k)$ for every $i,j \in \mathbf{I}$. Thus, $\theta_{ij}(S\psi_k) \subseteq S\psi_k$ for all $i,j,k \in \mathbf{I}$. Therefore, $\theta(\mathbf{H}\psi) \subseteq \mathbf{H}\psi$. Hence, $\theta \in D_{\mathbf{H}}(\mathbf{H}\psi)$, which proves our claim. Now assume that $P = \Sigma_{\psi_k \in T} D_S(S\psi_k)$. So from our claim $\mathbf{H} \cap Mat_{\mathbf{I}}(P) = D_{\mathbf{H}}(\mathbf{H}\psi)$. Since from assumption $\mathbf{H}$ is PQ-dual-Baer ring, there exists $F^2 = F = [F_{ij}] \in \mathbf{H}$ such that $D_{\mathbf{H}}(\mathbf{H}\psi) = \mathbf{H}F$. Note that $E_{ii}FE_{ii} = F_{ii}E_{ii}$ is a right semi-central idempotent of $E_{ii}\mathbf{H}E_{ii}$. Thus, $PE_{ii} = E_{ii}(\mathbf{H} \cap Mat_{\mathbf{I}}(P))E_{ii} = E_{ii}\mathbf{H}FE_{ii} = E_{ii}\mathbf{H}FE_{ii}FE_{ii}$. Thus, $P = PF_{ii} \subseteq SE_{ii}$ for all $i \in \mathbf{I}$. Since $\mathbf{H}F = \mathbf{H} + Mat_{\mathbf{I}}(P)$, $SF_{ii} \subseteq P$. Hence, $P = SE_{ii}$ with F_{ii} is a right semi-central idempotent of S. Therefore, S is a quasi-dual-Baer ring. $\square$