Chapter 1

Preliminaries

This chapter is mainly devoted to the collection of definitions and basic results, which are used in the subsequent chapters of the thesis. Throughout the thesis, unless otherwise indicated, all rings are considered as associative rings with unity and all modules are unital right R-modules denoted by M_R or M for short. For a right R-module M, $S = End_R(M)$ denotes the endomorphism ring of M. In this case, we take M as left S-module and right R-module. For undefined definitions and notations if any, we refer to [5] and [60].

Rings and Modules

Definition 1.0.1. *An algebraic structure (R, +, .), where R is a non-empty set together with two binary operations + and . is said to be a ring if the following conditions are satisfied.*

(1) $(R, +)$ *is an abelian group.*

(2) $(R, .)$ *is a semi-group.*

(3) The binary operation ' . ' distributes over ' + ' from the left as well as from the right, i.e., $\forall\ r, s, t \in R$

(i) $r.(s+t) = r.s + r.t$

(ii) $(r+s).t = r.t + s.t$

A ring $(R, +, .)$ is called commutative if $r.s = s.r\ \forall\ r, s \in R$. Further, R is said to be a ring with unity if there exists $1 \in R$ such that $1.r = r.1 = r\ \forall\ r \in R$.

Definition 1.0.2. *(1) Let R be a ring. An additive abelian group $(M, +)$ is called a right R-module if there exists a mapping from $M \times R$ to M defined by $(m, r) \to mr$, $\forall\ m \in M$, $r \in R$ satisfying the following conditions:*

(i) $(m+n)r = mr + nr$ *for every* $m, n \in M$ *and* $r \in R$.

(ii) $m(r+s) = mr + ms$ *for every* $m \in M$ *and* $r, s \in R$.

(iii) $m(rs) = (mr)s$ *for every* $m \in M$ *and* $r, s \in R$.

A left R-module can be defined by taking action of the ring R from left.

(2) Further, if $m.1 = m$ for all $m \in M$, where 1 is the unity of R, then R is called an unital right R-module.

(3) A non-empty subset N of a module M is called a submodule of M if N is also an R-module and we denote it by $N \leq M$.

Definition 1.0.3. *Let M and N be R-modules. Then*

(1) A mapping $\varphi : M \to N$ is called a module homomorphism or R-homomorphism or simply homomorphism if φ satisfies the following conditions:

(i) $\varphi(m_1 + m_2) = \varphi(m_1) + \varphi(m_2)$ *for all* $m_1, m_2 \in M$.

(ii) $\varphi(mr) = \varphi(m)r$ *for all* $m \in M$ *and* $r \in R$.

The set of all R-homomorphism from M to N is denoted by $\text{Hom}_R(M, N)$.

(2) *For any* $\varphi \in \text{Hom}_R(M, N)$, *the* kernel *and the* image *of* φ *are defined as follows* $\text{Ker}(\varphi) = \{m \in M : \varphi(m) = 0\}$ *and* $\text{Im}(\varphi) = \{\varphi(m) \in N : m \in M\}$.

(3) *An R-homomorphism from M to M is called an endomorphism and the set of all endomorphisms is denoted by* $\text{End}_R(M)$.

Theorem 1.0.4. *(Fundamental theorem of module homomorphisms) Let M and N be R-modules. If* $\varphi : M \to N$ *be any R-homomorphism, then* $\varphi(M) \cong M/\text{Ker}(\varphi)$.

Direct Sums, Direct Products and Direct Summands

Definition 1.0.5. *Let* $\{M_i\}_{i\in I}$ *be a family of R-modules, where I is an arbitrary index set. Then*

(i) $\Pi_{i\in I} M_i = \{(m_i)_{i\in I} : m_i \in M_i \; \forall \, i \in I\}$ *denotes the direct product of the family of R-modules* $\{M_i\}_{i\in I}$.

(ii) $\bigoplus_{i\in I} M_i = \{(m_i)_{i\in I} : m_i \in M_i \, and \, finitely \, many \, m_i's \, are \, non-zero\}$ *denotes the direct sum of the family of R-modules* $\{M_i\}_{i\in I}$.

Further, we have $M^{(I)} = \bigoplus_{i\in I} X_i$ *and* $M^I = \Pi_{i\in I} X_i$, *where* $X_i = M$ *for every* $i \in I$.

Definition 1.0.6. (1) *A module M is said to be the direct sum of a family* $(M_i)_{i\in I}$ *(where I is an arbitrary index set) of submodules of M if*

(i) $M = \sum_{i\in I} M_i$, *and*

(ii) *Every element* $m \in M$ *can be uniquely expressed as* $m = m_{j_1} + m_{j_2} + \cdots + m_{j_n}$, *where* $m_j \in M_j$ *for each* $j \in I$.

We shall usually express this fact by $M = \bigoplus_{i \in I} M_i$.

(2) A submodule N *of a module* M *is called a direct summand of* M *if there exists a submodule* N' *of* M *such that* $M = N \oplus N'$, *and we denote it by* $N \leq^{\oplus} M$. *In this case* $M = N \oplus N'$ *implies* $M = N + N'$ *and* $N \cap N' = 0$.

Idempotents and Semi-central idempotents

Definition 1.0.7. *(i) An element* e *of a ring* R *is called an idempotent element of* R *if* $e^2 = e$.

(ii) An idempotent element $e \in R$ *is said to be right (left) semi-central if* $ea = eae$ $(ae = eae)$ *for every* $a \in R$. $\mathbb{S}_r(R)$ $(\mathbb{S}_l(R))$ *denotes the set of all right (left) semi-central elements of* R.

Proposition 1.0.8. *If* N_1 *and* N_2 *are submodules of* M *such that* $M = N_1 \oplus N_2$ *then there exists an unique idempotent endomorphism* $e \in \mathrm{End}_R(M)$ *such that* $N_1 = eM$ *and* $N_2 = (1 - e)M$.

Exact Sequences

Definition 1.0.9. *Let* M_n *be an* R*-module* $\forall\ n$ *and* φ_n *be a homomorphism from* M_n *to* M_{n-1} *for all* n. *Then a sequence* $\cdots \to M_{n+1} \xrightarrow{\varphi_{n+1}} M_n \xrightarrow{\varphi_n} M_{n-1} \to \dots$ *is called an exact sequence at* M_n *if* $\mathrm{Ker}(\varphi_n) = \mathrm{Im}(\varphi_{n+1})$, *while this sequence is called an exact sequence if it is exact at* M_n *for each* n.

Definition 1.0.10. *Let* L, M *and* N *be* R*-modules. Then an exact sequence* $0 \to L \xrightarrow{\varphi} M \xrightarrow{\psi} N \to 0$ *is called a short exact sequence.*

Proposition 1.0.11. *Let* L, M *and* N *be* R*-modules*

(i) *A sequence* $0 \to L \xrightarrow{\varphi} M$ *is exact if and only if* φ *is a monomorphism.*

(ii) *A sequence* $M \xrightarrow{\psi} N \to 0$ *is exact if and only if* ψ *is an epimorphism.*

Proposition 1.0.12. *For R-modules L, M and N, the following conditions are equivalent:*

(i) *The exact sequence* $0 \to L \xrightarrow{\varphi} M \xrightarrow{\psi} N \to 0$ *splits;*

(ii) *There exists a homomorphism* $\varphi' : M \to L$ *such that* $\varphi' o \varphi = I_L$*, where* I_L *is the identity map on L;*

(iii) *There exists a homomorphism* $\psi' : N \to M$ *such that* $\psi o \psi' = I_N$*, where* I_N *is the identity map on N.*

Proposition 1.0.13. *Let M and N be R-modules.*

(i) *Let* $\varphi : M \to N$ *and* $\psi : N \to M$ *be homomorphisms such that* $\varphi\psi = I_N$*, where* I_N *is the identity map on N. Then* $M = \mathrm{Ker}(\varphi) \oplus \mathrm{Im}(\psi)$*.*

(ii) *A monomorphism* $\psi : N \to M$ *splits if and only if* $\mathrm{Im}(\psi) \leq^{\oplus} M$*.*

(iii) *An epimorphism* $\varphi : M \to N$ *splits if and only if* $\mathrm{Ker}(\varphi) \leq^{\oplus} M$

Essential and Small Submodules, Uniform Modules, Closed Submodules, Closure of Submodules, Torsion-free Modules and Non-singular Modules

Definition 1.0.14. *A submodule N of a module M is called an essential submodule denoted as* $N \leq^{\mathrm{e}} M$ *if* $N \cap L \neq 0$ *for each non-zero submodule L of M. In this case, M is called an essential extension of N.*

Proposition 1.0.15. *The following statement hold for R-modules:*

(i) *Let L, M, N be R-modules such that $L \leq M \leq N$. Then $L \leq^e N$ if and only if $L \leq^e M$ and $M \leq^e N$.*

(ii) *Let N and N' be submodules of a module M. Then $N \cap N' \leq^e M$ if and only if $N \leq^e M$ and $N' \leq^e M$.*

(iii) *Let $\{M_i\}_{i\in I}$ be a family of R-modules and $N_i \leq M_i$ for each $i \in I$, where I be an arbitrary index set. Then $N_i \leq^e M_i$ if and only if $\bigoplus_{i\in I} N_i \leq^e \bigoplus_{i\in I} M_i$*

(iv) *If $N \leq^e M$ then for any $L \leq M$, $N \cap L \leq^e L$*

Definition 1.0.16. *A module M is said to be uniform if every non-zero submodule of M is essential in M.*

Definition 1.0.17. *A submodule N of M is called small (or superfluous) in M abbreviated as $N << M$, if whenever $N + L = M$, where L is a submodule of M, then $L = M$.*

Definition 1.0.18. *A submodule C of a module M is said to be a closed submodule of M if it has no non-zero proper essential extension in M, i.e., whenever L is a submodule of M such that $C \leq^e L$, then $C = L$*

Definition 1.0.19. *Let M be a right R-module and N be a submodule of M. The closure of N in M is denoted by $Cl_M(N)$ (or in short $Cl(N)$) and defined as $Cl_M(N) = \{m \in M : (N;m) \leq^e R\}$, where $(N;m) = \{r \in R : mr \in N\}$*

Definition 1.0.20. *A submodule C of M is said to be a complement of a submodule $N \leq M$ if it is maximal in the collection H of all submodules of M with the property $H \cap N = 0$. A submodule L is a complement in M if there exists a submodule N of M such that L is a complement of N.*

Proposition 1.0.21. *[22, 1.10] Let K, L and N be submodules of a module M with $K \subseteq L$, then*

(i) There exists a closed submodule H of M such that N is essential in H.

(ii) K is closed in M if and only if whenever P is essential in M such that $K \subseteq P$, then P/K is essential in M/K.

(iii) If L is closed in M, then L/K is closed in M/K.

(iv) If K is closed in L and L is closed in M, then K is closed in M.

Definition 1.0.22. *Let M be a right R-module with endomorphism $S = End_R(M)$, $I \subseteq S$ and $X \subseteq M$. Then the right annihilator of $X \subseteq M$ in R is defined by $\mathrm{Ann}^r_R(X) = \{r \in R : xr = 0, \forall\, x \in X\}$ and the left annihilator of $X \subseteq M$ in S is denoted by $\mathrm{Ann}^l_S(X) = \{\varphi \in S : \varphi(m) = 0, \forall\, m \in X\}$. Further, the right annihilator of $I \subseteq S$ in M is denoted by $\mathrm{Ann}^r_M(I) = \{m \in M : \varphi(m) = 0, \forall\, \varphi \in I\}$*

Definition 1.0.23. *Let R be a commutative integral domain and M be a right R-module. Then the set $\mathrm{T}(M) = \{m \in M : mr = 0 \; for\; some\; 0 \neq r \in R\}$ is a submodule of M, called the torsion submodule of M. If $\mathrm{T}(M) = M$, then M is called a torsion module and if $\mathrm{T}(M) = 0$ then M is called a torsion-free module.*

Definition 1.0.24. *For an R-module M, $\mathrm{Z}(M) = \{m \in M : Ann^r_R(m) \leq^e R\}$ is called the singular submodule of M. M is called a non-singular module if $\mathrm{Z}(M) = 0$ and M is called a singular module if $\mathrm{Z}(M) = M$*

Fully Invariant Submodules

Definition 1.0.25. *A submodule N of a module M is called a fully invariant submodule of M, denoted as $N \trianglelefteq M$, if $\varphi(N) \subseteq N$ for every $\varphi \in End_R(M)$.*

Lemma 1.0.26. *Let L and N be submodules of M such that $L \leq N \leq M$. Then $L \trianglelefteq N \trianglelefteq M$ implies $L \trianglelefteq M$.*

Lemma 1.0.27. *[48, Lemma 1.10] Let $M = M_1 \oplus M_2$ be the direct sum of R-modules M_1 and M_2. If N is a fully invariant submodule M, then $N = N_1 \oplus N_2$, where each N_i is a fully invariant submodule of M_i and $N_i = N \cap M_i$ for $i = 1, 2$.*

Finitely Generated Modules, Free Modules, Finitely Related Modules, Finitely Presented Modules, Cogenerated Modules, Cyclic Modules, Coherent Modules

Definition 1.0.28. *A module M is said to be finitely generated if there exist $m_1, m_2, \ldots, m_n \in M$ such that $M = \sum_{i=1}^{n} m_i R$. The set $\{m_1, m_2, \ldots, m_n\}$ is called a set of generators of M. A module generated by a single element is called a cyclic module. Further, a submodule is called cyclic if it is generated by a single element.*

Definition 1.0.29. *A right R-module M is called a free module if it has a basis, i.e., there exists a subset $B \subseteq M$ such that each element $m \in M$ can be uniquely expressed as a finite sum, $m = \sum_{i=1}^{n} m_i r_i$ for some $r_1, r_2, \ldots, r_n \in R$ and $m_1, m_2, \ldots, m_n \in B$.*

Definition 1.0.30. *A module N is said to be finitely related [32] if there exists an exact sequence $0 \to L \to M \to N \to 0$ of right R-modules, where M is a free module (of arbitrary rank) and L is finitely generated.*

Definition 1.0.31. *A module N is said to be finitely presented [32] if there exists an exact sequence $0 \to L \to M \to N \to 0$ of right R-modules, where M is a free module (of finite rank) and L is finitely generated (or equivalently, there exists an exact sequence $R^m \to R^n \to N \to 0$ with $m, n \in \mathbb{N}$).*

Definition 1.0.32. *Let M and V be R-modules. Then M is said to be (finitely) cogenerated by V if there exists a monomorphism $\psi : M \to V^{(I)}$ for some (finite) arbitrary index set I [5]. Equivalently, a module M is called finitely cogenerated if for every set I of submodules of M such that $\cap I = 0$, there exists a finite collection F of submodules of I such that $\cap F = 0$.*

Definition 1.0.33. *A finitely generated R-module M is said to be coherent if every finitely generated submodule of M is finitely presented. A ring R is called right (left) coherent if R_R is a right (left) coherent R-module.*

Artinian and Noetherian Modules and Rings, Serial and Uniserial Modules, Hopfian and co-Hopfian Modules

Definition 1.0.34. *A module M is called Noetherian if it satisfies the ascending chain condition on its submodules, i.e., if every ascending chain $M_1 \leq M_2 \leq \cdots \leq M_n \leq \dots$ of submodules of M becomes stationary after finitely many steps. A ring R is called right (left) Noetherian if the right (left) R-module R_R $({}_RR)$ is Noetherian.*

Theorem 1.0.35. *For a module M, the following conditions are equivalent:*

(i) M is Noetherian;

(ii) Every submodule of M is finitely generated;

(iii) Every non-empty set A of submodules of M has a maximal element.

Definition 1.0.36. *A module M is called Artinian if it satisfies the descending chain condition on its submodules, i.e., if every descending chain $M_1 \geq M_2 \geq \cdots \geq M_n \geq \dots$ of submodules of M becomes stationary after finitely many steps. A ring R is called right (left) Artinian if the right (left) R-module R_R $({}_RR)$ is Artinian.*

Definition 1.0.37. *(i) A module M is called uniserial if for any two submodules N_1 and N_2 of M either $N_1 \subseteq N_2$ or $N_2 \subseteq N_1$.*

(ii) A module M is called serial if it decomposes into a direct sum of uniserial submodules.

(iii) A ring R is called right (left) uniserial if it is a right (left) uniserial module over itself. Further, a ring R is called a right (left) serial if it is a right (left) serial module over itself

Definition 1.0.38. *(i) A module M is said to be Hopfian if any surjective endomorphism of M is an isomorphism.*

(ii) A module M is said to be co-Hopfian if any injective endomorphism of M is an isomorphism.

Pure Submodules, Flat Modules, Pure Split Modules and PDS ring

Definition 1.0.39. *A short exact sequence $0 \to N_1 \xrightarrow{\phi} N_2 \to N_3 \to 0$ of right R-modules is said to be pure exact if $0 \to N_1 \otimes F \to N_2 \otimes F \to N_3 \otimes F \to 0$ is an exact sequence (of abelian groups) for any left R-module F [32].*

According to P.M. Cohn [17], a submodule N of a right R-module M is said to be a pure submodule of M, abbreviated by $N \leq^p M$, if and only if $0 \to N \otimes L \to M \otimes L$ is exact for every left R-module L. Further, a right (left) ideal I of a ring R is said to be pure if I is a pure submodule of R_R $({}_RR)$.

Definition 1.0.40. *A right R-module M is said to be flat if $0 \to M \otimes N_1 \to M \otimes N_2$ is exact whenever $0 \to N_1 \to N_2$ is exact for left R-modules N_1 and N_2.*

Proposition 1.0.41. *(i) [32, Proposition 4.29] A ring R is Noetherian if and only if all finitely generated right R-modules are finitely presented.*

(ii) [32, Theorem 4.30] Let P be a finitely related R-module. Then P is flat if and only if it is projective.

Lemma 1.0.42. *[24, Proposition 8.1]. The following conditions hold:*

(i) Let N be a submodule of a right R module M. If M/N is flat, then N is a pure submodule of M. Moreover, for a flat right R module M, N is a pure submodule of M if and only if M/N is flat.

(ii) If N is a submodule of M such that every finitely generated submodule of N is a pure submodule of M, then N is a pure submodule of M.

Lemma 1.0.43. *[24, Proposition 7.2]. Suppose $L \subseteq N \subseteq M$ be right R modules. Then*

(i) If $L \leq^p N$ and $N \leq^p M$, then $L \leq^p M$.

(ii) If $L \leq^p M$, then $L \leq^p N$.

(iii) If $L \leq^p N$, then $N/L \leq^p M/L$.

(iv) If $L \leq^p M$ and $N/L \leq^p M/L$, then $N \leq^p M$.

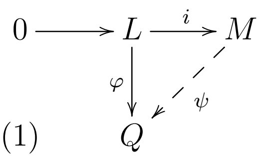
Definition 1.0.44. *A module M is called pure split if every pure submodule of M is a direct summand of M.*

Definition 1.0.45. *A ring R is called a right (left) PDS ring [24] every pure submodule of an R-module is a direct summand of M.*

Lemma 1.0.46. *If R is a Noetherian ring and M is a finitely generated R-module, then each pure submodule of M is a direct summand of M.*

Injective Modules, Injective Envelopes, FP-injective and Intrinsically Injective Modules, Cohereditary Modules

Definition 1.0.47. *(i) Let Q and M be R-modules. Then Q is said to be an M-injective module if for every submodule L of M with a monomorphism $i : L \to M$ and for any homomorphism $\varphi : L \to Q$, there exists a homomorphism*



(1)

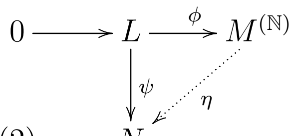
$\psi : M \to Q$ such that the diagram (1) is commutative, i.e., $\varphi = \psi i$.

(ii) A module Q is called injective if Q is injective for every R-module M.

(iii) A module Q is called quasi-injective if Q is Q-injective.

Definition 1.0.48. *An R-module H is called the injective hull (envelope) of an R-module M if H is the minimal injective module containing M. The injective hull of a module M is denoted by $E(M)$. In general, every module has an injective hull.*

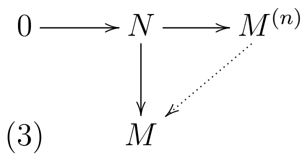
Definition 1.0.49. *Let M and N be R-modules. N is called weakly M-injective module, if for every diagram in Mod-R*



(2)

with exact row and L finitely generated, can be extended commutatively by a homomorphism $\eta : M^{(\mathbb{N})} \to N$, i.e. $\psi = \eta\phi$. If $M = R$, then weakly R-injective modules are also called FP-injective (see [60, 16.9]).

Definition 1.0.50. *An R-module M is said to be intrinsically injective module, if every diagram with exact row*



where $n \in \mathbb{N}$ and N a factor module of M, can be extended commutatively by some homomorphism from M^n to M (see [59]).

Theorem 1.0.51. *(Bass-Papp Theorem, [49, Theorem 4.1]) A ring R is right Noetherian if and only if every direct sum of injective right R-modules is injective.*

C1, C2, C3, C4-Modules, Continuous Modules, Extending (CS) Modules

Consider the following conditions for a right R-module M introduced by Jeremy [28], Mohammed and Muller [43], Ding et al. [20].

C1 : Every submodule of M is essential in a direct summand of M.

C2 : Every submodule of M that is isomorphic to a direct summand of M is itself a direct summand of M.

C3 : If L and N are direct summands of M with $L \cap N = 0$. Then $L \oplus N$ is also a direct summand of M.

C4 : Let N_1 and N_2 be submodules of M such that $M = N_1 \oplus N_2$. If $\psi : N_1 \to N_2$ be any homomorphism such that $\mathrm{Ker}(\psi) \leq^{\oplus} N_1$, then $\mathrm{Im}(\psi) \leq^{\oplus} N_2$.

Definition 1.0.52. *(i) A module with $C1$-condition is called a CS or an extending module. A module with Ci-conditions is called a Ci-module for $i = 1, 2, 3, 4$.*

(ii) A module with $C1$ and $C2$ conditions is called a continuous module.

(iii) A module with $C1$ and $C3$ conditions is called a quasi-continuous module.

Proposition 1.0.53. *[43, Proposition 2.1] Any quasi-injective module satisfies $C1$ and $C2$ conditions.*

Remark 1.0.54. *The following implications are true for a module ([43]),*

$Injective \Rightarrow Quasi{-}injective \Rightarrow Continuous \Rightarrow Quasi{-}continuous \Rightarrow Extending$

But the converse of these implications need not be true, in general (see [38]).

Cohereditary Modules, V-rings, SSI-rings

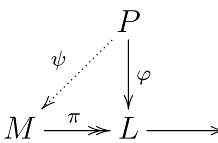
Definition 1.0.55. *A right R-module M is called a cohereditary module [61] if every factor module of M is injective.*

Definition 1.0.56. *(i) A ring R is called a right* V*-ring [18] if every simple right R-module is injective.*

(ii) A ring R is called a right SSI*-ring [11] if every semisimple right R-module is injective.*

Projective Modules, Hereditary and Semi-Hereditary Rings

Definition 1.0.57. *(i) Let P and M be R-modules. Then P is called M-projective if for every submodule L of M with an epimorphism $\pi : M \to L$ and for any*



(4)

homomorphism $\varphi : P \to L$ there exists a homomorphism $\psi : P \to M$ such that the diagram (4) is commutative, i.e., $\varphi = \pi\psi$.

(ii) A module P is called projective if P is M-projective for every R-module M.

(iii) A module P is called quasi-projective [31] if P is P-projective.

Definition 1.0.58. *(i) A ring R is called right (left) hereditary if each right (left) ideal of R is projective as an R-module.*

(ii) A ring R is called right (left) semi-hereditary if each finitely generated right (left) ideal of R is projective as an R-module.

A left hereditary (left semi-hereditary) ring is defined similarly.

Theorem 1.0.59. *[12, Theorem 5.4] The following statements are equivalents are equivalent for a ring R:*

(i) R is right hereditary;

(ii) Every submodule of a projective right R-module is projective;

(iii) Every quotient module of an injective right R-module is injective.

D1 (Lifting) Modules, D2-Modules, D3-Modules and D4-Modules

Consider the following conditions defined in [43] and [21] for an R-module M.

D1: For every submodule N of M, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq N$ and $N \cap M_2 << M$.

D2: If N is a submodule of M and M/N is isomorphic to a direct summand of M, then N is also a direct summand of M.

D3: If L and N are direct summands of M with $M = L + N$, then $L \cap N$ is a direct summand of M.

D4: If M_1 and M_2 are submodules of M with $M = M_1 \oplus M_2$ and $\varphi : M_1 \to M_2$ is a homomorphism with $\mathrm{Im}(\varphi) \leq^{\oplus} M_2$, then $\mathrm{Ker}(\varphi) \leq^{\oplus} M_1$.

Definition 1.0.60. *(i) A module with $D1$-condition is known as lifting module.*

(ii) A module with Di-condition is called a Di-module for every $i = 1, 2, 3, 4$.

Semisimple Modules and Rings, Socle of Modules, von Neumann Regular Rings

Definition 1.0.61. *Let M be a right R-module. Then*

(i) M is called a simple module if it contains no non-trivial proper submodule.

(ii) M is called an indecomposable module if it can not be written as a direct sum of two proper direct summands of M.

Definition 1.0.62. *The sum of all simple submodules of a right R-module M is called the right socle of M, and it is denoted by* $\mathrm{Soc}(M)$.

Definition 1.0.63. *A non-zero module M is called semisimple if it is expressible as a sum of simple submodules, while a ring R is called a right (left) semisimple if the right (left) R-module R_R ($_RR$) is a semisimple module.*

Proposition 1.0.64. *An R-module M is semisimple if and only if $Soc(M) = M$.*

Proposition 1.0.65. *The following conditions are equivalent for a ring R:*

(i) R is semisimple;

(ii) Every R-module is semisimple;

(iii) Every R-module is injective;

(iv) *Every R-module is projective;*

(v) *Every ideal of R is a direct summand.*

Note 1.0.66. *A ring R is called right (left) semisimple if every right (left) ideal of R is a direct summand of R.*

Definition 1.0.67. *A ring R is called von Neumann regular if for each $a \in R$, there exists $b \in R$ such that $a = aba$.*

Definition 1.0.68. *A module M is called endoregular [40] if the endomorphism ring of M is a von Neumann regular.*

Proposition 1.0.69. *[60, 3.10] The following conditions are equivalent for a ring R:*

(i) *R is von Neumann regular;*

(ii) *Every principal right ideal is a direct summand;*

(iii) *Every finitely generated right ideal is a direct summand.*

SSSP (SSP) Modules, SSIP (SIP) Modules, FI-SSP (FI-SSSP) modules

Definition 1.0.70. *An R-module M is said to have summand sum property (SSP) if the sum of any two direct summands of M is a direct summand of M, while M is said to have strong summand sum property ($SSSP$) if the sum of arbitrary direct summands of M is a direct summand of M. A module M is called an SSP ($SSSP$) module if M has SSP ($SSSP$).*

Definition 1.0.71. *An R-module M is said to have summand intersection property (SIP) if the intersection of any two direct summands of M is a direct summand of M, while M is said to have strong summand intersection property ($SSIP$) if the intersection of arbitrary direct summands of M is a direct summand of M. A module M is called an SIP ($SSIP$) module if M has SIP ($SSIP$).*

Definition 1.0.72. *An R-module M is said to have fully invariant summand sum property (FI-SSP) if the sum of any two fully invariant direct summands of M is a direct summand of M, while M is said to have fully invariant strong summand sum property (FI-$SSSP$) if the sum of every fully invariant direct summands of M is a direct summand of M. A module M is called an FI-SSP (FI-$SSSP$) module if M has FI-SSP (FI-$SSSP$).*

Baer Rings and their Generalizations

Definition 1.0.73. *A ring R is said to be right (left) Baer [30] if the right (left) annihilator of any right (left) ideal is generated by an idempotent element of R. Equivalently, a ring R is right Baer if for every right (left) ideal I of R there exists an idempotent element $a^2 = a \in R$ such that $Ann^r_R(I) = \{r \in R : xr = 0, \forall x \in I\} = aR$ ($Ann^l_R(I) = \{r \in R :\ rx = 0,\ \forall\, x \in I\} = Ra$).*

Definition 1.0.74. *A ring R is called quasi-Baer [16] (principally quasi-Baer [9]), if the right annihilator of every ideal (principal ideal) in R is generated by an idempotent element of R, i.e., for every ideal (principal ideal) there exists an idempotent element $a^2 = a \in R$ such that $Ann^r_R(I) = aR$.*

Definition 1.0.75. *(i) A ring R is called a PP ring if every principal ideal of R is projective.*

(ii) *A ring R is said to be a right (left) Rickart ring if the right (left) annihilator of every element of R is generated by an idempotent element of R.*

Note 1.0.76. *In 1960, Hattori [27] introduced the notion of a right PP ring. It was later shown that the right PP rings are precisely the right Rickart rings.*

Baer Modules and their Generalizations

Definition 1.0.77. *Let M be an R-module and $S = \mathrm{End}_R(M)$ be the endomorphism ring of M. Then M is called a Baer module [48] if for any $N \leq M$, there exists an idempotent element $e^2 = e$ such that $Ann^l_S(N) = \{\varphi \in S : \varphi(N) = 0\} = Se$*

Definition 1.0.78. *A module M is called a quasi-Baer module [48] if for any fully invariant submodule $N \trianglelefteq M$, there exists an idempotent element $e^2 = e$ such that $Ann^l_S(N) = \{\varphi \in S : \varphi(N) = 0\} = Se$*

Definition 1.0.79. *A module M is said to be a principally quasi-Baer (in short PQ-Baer) module [33] if the left annihilator in S of any cyclic submodule N of M there exists an idempotent element $e^2 = e \in S$ such that $Ann^l_S(N) = Se$.*

Definition 1.0.80. *A module M is said to be a purely Baer [6] if the right annihilator of any left ideal of S in M is a pure submodule of M.*

Definition 1.0.81. *A module M is said to be an essentially Baer module [44] if for every left ideal I of S, there exists a direct summand $N \leq^{\oplus} M$ such $Ann^r_M(I) \leq^e N$.*

Definition 1.0.82. (i) *Let M and N be modules. M is said to be N-Rickart [36] (in short M is N-Rickart) module if for every homomorphism $\psi : M \to N$, $\mathrm{Ker}(\psi)$ is a direct summand of M.*

(ii) *A module M is called Rickart [36] if for every $\varphi \in S = \mathrm{End}_R(M)$, $\mathrm{Ker}(\varphi)$ is a direct summand of M.*

Definition 1.0.83. *(i) A module M is said to be Σ-Rickart module [34] if every direct sum of copies of M is a Rickart module.*

(ii) A module M is said to be a $finite$ Σ-Rickart module [35] if every finite direct sum of copies of M is a Rickart module.

Dual-Baer Modules and their Generalizations

Definition 1.0.84. *A module M is called dual-Baer [56] if for every submodule N of M, there exists an idempotent element $e^2 = e \in S = \text{End}_R(M)$ such that $\text{D}_\text{S}(N) = \{\varphi \in S : Im(\varphi) \subseteq N\} = Se$.*

Definition 1.0.85. *A module M is called quasi-dual-Baer [4] if for every fully invariant submodule N of M, there exists an idempotent element $e^2 = e \in S$ such that $\text{D}_\text{S}(N) = \{\varphi \in S : Im(\varphi) \subseteq N\} = Se$. Equivalently, M is called a quasi-dual-Baer module if for ideal I of S , $E_M(I) = \sum_{\varphi \in I} \text{Im}(\varphi) = eM$ for some $e^2 = e \in S$.*

Definition 1.0.86. *Let M and N be modules. M is called N-dual-Rickart [37] (in short M is N-dual-Rickart) module if for every homomorphism $\psi : M \to N$, $\text{Im}(\psi)$ is a direct summand of N.*

Definition 1.0.87. *A module M is said to be dual-Rickart [37] if for every $\varphi \in S = \text{End}_R(M)$, $\text{Im}(\varphi)$ is a direct summand of M.*